CHAPTER 1
PRELIMINARIES

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Introduction

The purpose of this introductory chapter is to justify and highlight the problem posed, define the topic and explain the aim and scope of the work presented in the thesis on the basis of the literature cited. Significant contributions from the investigation are also highlighted. In this chapter we recall some standard notations, basic definitions and concepts those will be used throughout the thesis.

1.1 Historical background

Sequence space is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field K of real or complex numbers. The set of all such functions is naturally identified with the set of all possible infinite sequences with elements in K, and can be turned into a vector space under the operations of pointwise addition of functions and point wise scalar multiplication. All sequence spaces are linear subspaces of this space. Sequence spaces are typically equipped with a norm, or at least the structure of a topological vector space. The most important sequence spaces in analysis are the ℓ_p spaces, consisting of the $p-$power summable sequences, with the $p-$norm. These are special cases of L_p spaces for the counting measure on the set of natural numbers. Other important classes of sequences like convergent , null and bounded sequences form sequence spaces, respectively denoted by c, c_0 and ℓ_∞ with the supremum norm $\|x\|_\infty = sup_k |x_k|$, $k \in N$. These spaces have studied by Altay et. al. [16], Aydin and Başar [41], Şengönül and Başar [110] and many others.

In several branches of analysis, the study of sequence spaces occupies a very prominent position. The theory of sequence spaces is a part of functional analysis, motivated by problems in Fourier series, power series and systems of equations with infinitely many variables and have application in different areas of mathematical analysis such as function spaces, vector sequence spaces, vector lattices etc. Apart from this, it is a powerful tool for obtaining positive results concerning Schauder bases and their associated types. Also it has made remarkable advances in recent times in enveloping summability theory via unified techniques effecting transformations from one sequence space into another. A sequence space can also be equipped with the topology of pointwise convergence, under which it becomes a special kind of Frechet space called FK-space [115].

1.2 Objectives

The main objectives of the proposed thesis are as follows

1. To introduce some $I-$ convergent generalized difference sequence spaces associated with multiplier sequences defined by a sequence of modulli.

2. To introduce some newly defined sequence spaces using regular matrix of Fibonacci numbers.

3. To introduce the difference sequence spaces of interval numbers $c_0^i(\Delta)$, $c^i(\Delta)$ and $\ell_\infty^i(\Delta)$ and study some of their algebraic and topological properties.

4. To introduce the statistically convergent sequence spaces of interval numbers and study some of their algebraic and topological properties.

5. To introduce the $I-$ convergent sequence spaces of interval numbers and investigate some of their algebraic and topological properties.

6. To draw some relations among the newly proposed spaces with the existing spaces of interval numbers.

Now we give a brief idea of some basic concepts which are relevant for carrying out the prescribed research work.

1.3 Literature Review

The notion of statistical convergence of real numbers was introduced by Fast [66] and Schoenberg [79] independently as a generalization of ordinary convergence. Fast [66] introduced the idea of statistical convergence of real or complex numbers and Schoenberg [79] studied statistical convergence as a summability method and listed some of the properties of statistical convergence. From the point of view of sequence spaces, this concept has been generalized and developed by Fridy [80], Šalát [134], Connor [85], Connor et. al. [87] , Et and Nuray [99] and many others. Later on Kostyrko et. al. [114] generalizes the concept of statistical convergence which he called $I-$convergence.

The idea of difference sequence spaces was first introduced by Kizmaz [68] and this concept was generalized by Et and Çolak [101], Tripathy and Chandra [36] and many others. The notion of paranormed sequences was first introduced by Simons [129] . It was further investigated by Maddox [72], Tripathy and Chandra [36] and many others. The notion of multiplier sequences $\Lambda = (\lambda_k)$ was studied by Goes and Goes [61] at the initial stage. It was further investigated by Kamthan [115], Tripathy and Mahanta [37] and many others. The notion of modulus function was introduced by Nakano [69]. It was further investigated with applications to sequences by Tripathy and Chandra [36] and many others. The approach of constructing new sequence spaces by means of the matrix domain have been employed by Altay, Başar and Mursaleen [16], Altay and Başar [15], Malkowsky [52], Ng and Lee [118]. Following Altay and Başar [15], Şengönül [108] introduced the Zweier sequence spaces Z and Z_0.

1.4 General Definitions and Notations

Throughout N, R, C denote the sets of natural, real and complex numbers respectively.

A sequence in a set S is a function whose domain is the set N of the natural numbers, and whose range is contained in the set S.

A sequence of real numbers is a function from N to R and we denote a sequence (classical) by (x_k).

A sequence (x_k) is said to be bounded if there exists a number $M > 0$ such that $|x_k| < M$, for all $k \in N$.

Let X and Y be two non-empty sets, the Cartesian product of X and Y is defined by $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

A metric space is a pair (X, d), consisting of a non-empty set X and a metric (distance function) $d : X \times X \to R$ such that for all x, y, $z \in X$, the following conditions hold:

(i) $d(x, y) \geq 0$ (positivity)
(ii)$d(x, y) = 0$ if and only if $x = y$ (nondegeneracy)
(iii) $d(x, y) = d(y, x)$ (symmetry)
(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

A sequence $x = (x_k)$ in a metric space $(X,\, d)$ is said to converge to number l if for a given $\epsilon > 0$, there exists a positive integer n_0 such that,

$d\,(x_k, l) < \epsilon$ for all $k \geq n_0$. We write $lim_{k\to\infty} x_k = l$.

A sequence $x = (x_k)$ in a metric space $(X,\, d)$ is said to be Cauchy sequence if for a given $\epsilon > 0$, there exists a positive integer n_0 such that,

$d\,(x_k,\, x_n) < \epsilon$ for all $k,\, n \geq n_0$.

A metric space $(X,\, d)$ is said to be complete if every Cauchy sequence in X converges to a point of X.

Throughout π denotes a permutation over N.

For $x = (x_k)$ a given sequence, $S\,(x)$ denotes the set of all permutation of the elements of x_k, that is $S\,(x) = \left\{\left(x_{\pi(k)}\right)\right\}$.

A sequence space E is said to be symmetric if $S\,(x) \subseteq E$, for all $x \in E$.

A sequence space E is said to be solid (or normal) if $(y_k) \in E$ whenever $(x_k) \in E$ and $|y_k| \leq |x_k|$ for all $k \in N$.

Let $K = \{k_1 < k_2 < k_3 ...\} \subseteq N$ and E be a sequence space.

A $K-$ step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_n) \in E\}$.

A canonical pre-image of a sequence $\left(x_{k_n} \in \lambda_K^E\right)$ is a sequence (y_n) defined as follows:

$$(y_n) = \begin{cases} x_n & \text{if } n \in K \\ 0 & \text{otherwise} \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E, i.e., y is in canonical pre-image λ_K^E if and only if y is canonical pre-image of some $x \in \lambda_K^E$.

A sequence space E is said to be monotone if E contains the canonical preimage of all its step spaces.

A sequence space E is said to be sequence algebra if $(x_k y_k) \in E$ whenever (x_k), $(y_k) \in E$.

A sequence space E is said to be convergence free if $(y_k) \in E$, whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

A sequence space E is normal implies that it is monotone. (see [116])

A BK- space $(X, \|.\|)$ is a linear space whose elements are complex sequences $x = (x_k)$ and which is also a Banach space in which the co-ordinate maps are continuous, that is $\left|x_k^{(m)} - x_k\right| \to 0$ whenever $\left|x^{(m)} - x\right| \to 0$ as $m \to \infty$ and $x^{(m)} = (x_k^m)$, $x = (x_k)$.

For $p \geq 1$, the $p-$norm of $x \in C^n$ is defined as

$$\|x\|_p = (\Sigma_{i=1}^n |x_i|^p)^{\frac{1}{p}}.$$

It can be proven that the following properties of the Euclidean norm are in fact valid for all p- norms:

i) $\|x\|_p \geq 0$

ii) $\|x\|_p = 0 \Longleftrightarrow \ x = 0$

iii) $\|\alpha x\|_p = |\alpha| \ \|x\|_p$

iv) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all scalars α.

In practice, only three of the $p-$ norms are used, and they are

$\|x\|_1 = \Sigma_{i=1}^n |x_i|$ (the grid norm)

$\|x\|_2 = \left(\Sigma_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$ (the euclidean norm)

$\|x\|_\infty = lim_{p\to\infty} \|x\|_p = lim_{p\to\infty} (\Sigma_{i=1}^n |x_i|^p)^{\frac{1}{p}} = max_i |x_i|$ (the max norm).

1.5 Statistical Convergence of Sequences

The notion of limit is one of the central notions in mathematical analysis. The concept of ordinary limit has been generalized by mathematicians in various ways. One of the natural generalizations of limit is to define an operator extending the usual limit which assigns a value to some non-convergent sequences too. For example if we want to define an extended limit in such a way that the sequence $(1, 0, 1, 0, ...)$ has a limit, one would expect that this limit to be $\frac{1}{2}$. The notion of statistical convergence was introduced by Fast [66] and Schoenberg [79] independently as a generalization of the ordinary convergence. Over the years and under different names statistical convergence has been discussed in the theory of fourier series, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy and Orhan [82], Šalát [134] and many others. Moreover, statistical convergence is closely related to the concept of convergence in probability. The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, approximation theory, fuzzy set theory and so forth.

The idea of statistical convergence [66] depends on the density of subsets of the set N of natural numbers. Let N be the set of natural numbers.

If $A \subseteq N = \{1, 2, 3, \ldots, n, \ldots\}$, then χ_A denotes the characteristic function of the set A

i.e., $\chi_A(k) = \begin{cases} 1 & \text{if } k \in A \\ 0 & \text{if } k \in N \setminus A \end{cases}$ and $\delta_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$

Then the number $\underline{\delta}(A) = \liminf\{\delta_n(A)\}$ and $\overline{\delta}(A) = \limsup\{\delta_n(A)\}$ are called the lower and upper asymptotic density of A respectively.

If $\underline{\delta}(A) = \overline{\delta}(A) = \delta(A)$ then $\delta(A)$ is called the asymptotic density of A.

We see that asymptotic density is limit of frequencies of numbers in the set $\{0, 1, 2, ...\}$, therefore it is (when it exists) intuitively correct measure of size of subsets of integers. It is clear that any finite subset of N has natural density zero and $\delta(A^c) = 1 - \delta(A)$.

Definition 1.5.1: A sequence $x = (x_n)$ is said to be statistically convergent [66, 79] to a number $l \in R$ if for each $\varepsilon > 0$, $\delta(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in N : |x_n - l| \geq \varepsilon\}$.

If a sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero, we can say that every ordinary convergent sequence is statistically convergent. Asymptotic density is (in some context) appropriate way to describe whether a subset of natural numbers is small or large.

Definition 1.5.2: $c \in R$ is called a statistical cluster point of a sequence $x = (x_i)$ provided that the natural density of the set

$$\{i \in N : \|x_i - c\| < \epsilon\}$$

is different from zero for every $\epsilon > 0$.

A sequence $x = (x_i)$ is said to be statistically bounded if there exists a positive real number M such that

$$\delta(\{i \in N : \|x_i\| \geq M\}) = 0.$$

For a sequence $x = (x_i)$ of real numbers, the notions of statistical limit superior and statistical limit inferior are defined as follows

$$\text{st-lim sup } x := \begin{cases} B_x, & B_x \neq \phi \\ -\infty, & otherwise \end{cases}$$

$$\text{st-lim inf } x := \begin{cases} A_x, & A_x \neq \phi \\ +\infty, & otherwise \end{cases}$$

where $A_x := \{a \in R : \delta(\{i \in N : x_i < a\}) \neq 0\}$

and $B_x := \{b \in R : \delta(\{i \in N : x_i > a\}) \neq 0\}$.

Example 1.5.1: Let us consider the sequence $l = \{a_i : i = 1, 2, 3, ...\}$ whose terms are

$$a_i = \begin{cases} i & \text{when } i = n^2 \, for \, all \, n = 1, 2, 3, ... \\ \frac{1}{i} & \text{otherwise} \end{cases}$$

Then, it is easy to see that the sequence l is divergent in the ordinary sense, while 0 is the statistical limit of l since $\delta(K) = 0$ where $K = \{n^2 \, for \, all \, n = 1, 2, 3, ...\}$.

Not all properties of convergent sequences are true for statistical convergence. For instance, it is known that a subsequence of a convergent sequence is convergent. However, for statistical convergence this is not true.

1.6 Ideal Convergence of Sequences

In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Kostyrko et. al. [114] presented a new generalization of statistical convergence and called it $I-$convergence. They used the notion of an ideal I of subsets of the set N of natural numbers to define such a concept.

Definition 1.6.1: Let X be a non-empty set. A family of sets $I \subset 2^X$ is said to be an ideal if

$i)$ I is additive, i.e., $A, B \in I \Rightarrow A \cup B \in I$ and

$ii)$ hereditary i.e., $A \in I,\ B \subset A \Rightarrow B \in I$.

An ideal $I \subset 2^X$ is called non-trivial if $I \neq 2^X$.

A non-trivial ideal I is called admissible iff $I \supset \{\{x\} : x \in X\}$.

A non-trivial ideal I is maximal if there does not exist any non-trivial ideal $J \neq I$, containing I as a subset.

A sequence $x = (x_n)$ is said to be $I-$convergent to a number $l \in R$ w.r.t. ideal I if for each $\varepsilon > 0$, we have $A(\varepsilon) = \{n \in N : |x_n - l| \geq \varepsilon\} \in I$.

The element l is called the $I-$ limit of the sequence $x = (x_n)$.

Let (X, d) be a fixed metric space and I denotes a non-trivial ideal of subsets of N.

Definition 1.6.2: [114] A sequence (x_n) of elements of X is said to be $I-$ convergent to $\xi \in X$ ($\xi = I - lim_{n\to\infty} x_n$) if and only if for each $\epsilon > 0$ the set

$A(\epsilon) = \{n \in N \,:\, d(x_n, \xi) \geq \epsilon\}$ belongs to I.

The element ξ is called the $I-$ limit of the sequence $x = (x_n)$.

Example 1.6.1: [114] (a) Take I for the class I_f of all finite subsets of N. Then I_f is a non-trivial admissible ideal and I_f- convergence ciuncides with the usual convergence with respect to the metric d in X.

(b) Denote by I_δ the class of all $A \subset N$ with $\delta(A) = 0$, then I_δ is non-trivial admissible ideals, $I_\delta-$ convergence coincides with the statistical convergence.

Remark 1.6.1: [114] Note that if I is an admissible ideal, then the usual convergence in X implies $I-$ convergence in X. The following properties are the most familiar axioms of convergence (see [117]):

(S) Every constant sequence $\{\xi, \xi, ..., \xi, ...\}$ converges to ξ.

(H) The limit of any convergent sequence is uniquely determined.

(F) If a sequence $x = (x_n)$ has the limit ξ, then each of its subsequences has the same limit.

(U) If each subsequence of the sequence $x = (x_n)$ has a subsequence which converges to ξ, then x converges to ξ.

(i) The $I-$ convergence satisfies (S), (H) and (U).

(ii) If $I-$ contains an infinite set, then I- convergence does not satisfy (F).

Remark 1.6.2: [114] If I is an admissible ideal which does not contains any infinite set, then $I-$ convergence coincides with the usual convergence and obviously fulfills (F).

Remark 1.6.3: [114] If $I \subset 2^N$ is a maximal ideal then for each $A \subset N$, we have either $A \in I$ or $N - A \in I$.

Lemma 1.6.1: [114] Let $I \subset 2^N$ be a maximal admissible ideal. Then each bounded sequence $x = (x_n)$ of real numbers is $I-$ convergent, i.e., there exists $\xi \in R$ such that $\xi = I - lim_{n\to\infty} x_n$.

Definition 1.6.3: [114] Let (X, d) be a metric space, $x = (x_n)$ be a sequence of elements of X.

a) An element $\xi \in X$ is said to be an $I-$ limit point of x provided that there is a set $M = \{m_1 < m_2 < ... <\} \subset N$ such that $M \notin I$ and

$lim_{k\to\infty} x_{m_k} = \xi$.

b) An element $\xi \in X$ is said to be an $I-$ cluster point of x if and only if for each $\epsilon > 0$ we have $\{n \in N : d(x_n, \xi) < \epsilon\} \notin I$.

1.7 Difference Sequences

The idea of difference sequence spaces was introduced by Kizmaz [68] in 1981 and later on studied further in different aspects by Et [96], Et and Çolak [101], Tripathy [24, 25] , Bektaş et. al.[42], Tripathy and Esi [28] and others.

Let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ respectively with complex terms, normed by

$$\|x\|_\infty = sup_k |x_k|,\ k \in N.$$

Then Kizmaz [68] defined the sequence spaces

$$\ell_\infty(\Delta) = \{x = (x_k) : \Delta x \in \ell_\infty\},$$

$$c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$

$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, and showed that these are Banach spaes with norm

$$\|x\| = |x_1| + \|\Delta x\|_\infty$$

The idea of Kizmaz [68] was applied for introducing different types of difference sequence spaces and for studying their different algebraic and topological properties by Tripathy [24, 25] and many others.

In 1993, Et [96] defined the sequence spaces

$$\ell_\infty(\Delta^2) = \{x = (x_k) : \Delta^2 x \in l_\infty\},$$

$$c(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c\},$$

$$c_0(\Delta^2) = \{x = (x_k) : \Delta^2 x \in c_0\},$$

where $\Delta^2 x = (\Delta^2 x_k) = (\Delta x_k - \Delta x_{k+1})$, and showed these are Banach spaces with norm

$$\|x\|_1 = |x_1| + |x_2| + \|\Delta^2 x\|_\infty$$

In 1995, Et and Çolak [101] defined the sequence spaces

$$\ell_\infty(\Delta^m) = \{x = (x_k) : \Delta^m x \in l_\infty\},$$

$$c(\Delta^m) = \{x = (x_k) : \Delta^m x \in c\},$$

$$c_0(\Delta^m) = \{x = (x_k) : \Delta^m x \in c_0\},$$

where $m \in N$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) =$

$(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ and so that

$$\Delta^m x_k = \Sigma_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+v}$$

and showed that these sequence spaces are Banach spaces with norm

$$\|x\|_{\Delta} = \Sigma_{i=1}^{m} |x_i| + \|\Delta^m x\|_{\infty}.$$

Further the inclusions $c_0(\Delta^m) \subset c_0(\Delta^{m+1})$, $c(\Delta^m) \subset c(\Delta^{m+1})$, $\ell_\infty(\Delta^m) \subset \ell_\infty(\Delta^{m+1})$, and so $c_0(\Delta^m) \subset c(\Delta^m) \subset \ell_\infty(\Delta^m)$ are satisfied and strict.

This concept was generalized by Tripathy and Esi [28] as follows:

Let $m \geq 0$, be a fixed integer, then

$Z(\Delta_m) = \{(x_k) \in w : (\Delta_m x_k) \in Z\}$, for $Z = c, c_0, \ell_\infty$, where $\Delta_m x = (\Delta_m x_k) = x_k - x_{k+m}$, and $\Delta_0 x_k = x_k$ for all $k \in N$.

They showed that the above spaces are Banach spaces, normed by

$$\|(x_k)\|_{\Delta_m} = \Sigma_{i=1}^{m} |x_i| + sup_k \|\Delta_m x_k\|_{\infty}.$$

The concept of difference statistical convergence or Δ^m- statistical convergence of sequences was studied by Et and Nuray [100] in 2001, the definitions and few results are given below.

Definition 1.7.1: [100] Let ω be a linear space of complex sequences. Then $x = (x_k) \in \omega$ is said to be Δ^m-statistically convergent to a complex number l, if for every $\epsilon > 0$, we have

$$lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |\Delta^m x_k - l| \geq \epsilon\}| = 0$$

and we write $st - lim\Delta^m x_k = l$.

Definition 1.7.2: [100] The sequence $x = (x_k) \in \omega$ is said to be Δ^m-statistically Cauchy, if for every $\epsilon > 0$ there exists a positive integer N such that

$$lim_{n\to\infty}\frac{1}{n}\left|\{k \leq n : |\Delta^m x_k - \Delta^m x_N| \geq \epsilon\}\right| = 0.$$

Theorem 1.7.1: [100] Every Δ^m- statistically convergent sequence is a Δ^m-statistically Cauchy sequence.

Theorem 1.7.2: [100] If x is a sequence for which there is a Δ^m- statistically convergent sequence y such that $\Delta^m x_k = \Delta^m y_k$, for all most all k, then x is Δ^m- statistically convergent sequence.

Let $\Delta^m(m_0)$ and $\Delta^m(m)$ denote the sets of Δ^m-bounded statistically convergent sequences of real numbers and Δ^m- bounded sequences of real numbers respectively.

Theorem 1.7.3: [100] The set $\Delta^m(m_0)$ is a closed linear space of the linear normed space $\Delta^m(m)$

1.8 Para-normed Sequence Spaces

At the elementary stage paranormed sequence spaces were introduced by Nakano [70] and Simons [129]. Afterwards it was furthermore studied by Maddox [74], Lascarides [43], Lascarides and Maddox [44], Tripathy and Sen [34] and many others.

A linear topological space X over the real field R is said to be a paranorm space if there is a sub additive function $g : X \to R$ such that

$i)$ $g(\theta) = 0$, (ii) scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and

$g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α in R, (iii) $g(-x) = g(x)$, (iv) $g(x+y) \leq g(x) + g(y)$, for all x in X, where θ is the zero vector in the linear space X.

For $k \in N$ Simons introduced the sequence spaces $\ell(p_k)$ and $m(p_k)$ be defined as follows:

$$\ell(p_k) = \{(x_k) : (x_k) \in \omega, \, \Sigma_k |x_k|^{p_k} < \infty\}$$

$$\ell(q_k) = \{(y_k) : (y_k) \in \omega, \, sup_k \Sigma_k |x_k|^{p_k} < \infty\}$$

These sequence spaces are linear spaces.

1.9 Modulus Function

The notion of modulus function was introduced by Nakano [70].

A modulus function f is a mapping from $[0, \infty)$ into $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$

(ii) $f(x+y) \leq f(x) + f(y)$

(iii) f is increasing

(iv) f is continuous from the right at 0

Hence f is continuous everywhere in $[0, \infty)$.

Ruckle [142] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \Sigma_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is FK space, and Ruckle [142] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences. The space $X(f)$ is closely related to the space ℓ_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Ruckle [143] proved that, for any modulus f, $X(f) \subset \ell_1$ and $X(f)^{\alpha} = \ell_{\infty}$, where $X(f)^{\alpha} = \{y = (y_k) \in \omega : \Sigma_{k=1}^{\infty} f(|y_k x_k|) < \infty\}$. The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \Sigma_{k=1}^{\infty} f(|x_k|) < \infty$ (see [143]).

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in [17]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Therefore symmetric sequence spaces, which are locally convex have been frequently studied by Garling [46, 47] , Köthe [60] and Ruckle [141, 142] . After then Kolk [49, 50] gave an extension of $X(f)$ by considering a sequence of modulli $F = (f_k)$ and defined the sequence space $X(f) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$.

1.10 Interval Numbers

Interval arithmetic was first suggested by Dwyer [119] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [122] in 1959 and Moore and Yang [122] in 1962.

Recently the sequence of interval numbers and usual convergence of sequences of interval numbers are studied by Chiao [92]. Later on, Şengönül and Eryılmaz [109] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. In the recent days, Esi [1, 2 3] introduced and studied strongly almost λ- convergence and statistically almost λ- convergence of interval numbers and lacunary sequence spaces of interval numbers respectively.

The set of all real valued closed intervals is denoted by $R(I)$. Any elements of $R(I)$ is called interval number and denoted by $\overline{x} = [x_l, x_r]$. The absolute value (magnitude or interval norm) of an interval number is defined by $|\overline{x}| = max\ \{|x_l|, |x_r|\}$.

For $x_1,\ x_2 \in R(I)$, we have $\overline{x}_1 = \overline{x}_2 \Leftrightarrow x_{l1} = x_{l2},\ x_{r1} = x_{r2},\ \overline{x}_1 + \overline{x}_2 = \{x \in R : x_{l1} + x_{l2} \leq x \leq x_{r1} + x_{r2}\}$, and if $\alpha \geq 0$, then $\alpha\overline{x} = \{x \in R : \alpha x_{l1} \leq x \leq \alpha x_{r1}\}$ and if $\alpha < 0$, then $\alpha\overline{x} = \{x \in R : \alpha x_{r1} \leq x \leq \alpha x_{l1}\}$,

$$\begin{aligned}\overline{x}_1.\overline{x}_2 = \{x \in R : min\{x_{l1}.x_{l2}, x_{l1}.x_{r2}, x_{r1}.x_{l2}, x_{r1}.x_{r2}\}\\ \leq x \leq max\{x_{l1}.x_{l2}, x_{l1}.x_{r2}, x_{r1}.x_{l2}, x_{r1}.x_{r2}\}\}.\end{aligned}$$

The set of all interval numbers $R(I)$ is a complete metric space defined by

$$d(\overline{x}_1, \overline{x}_2) = max\ \{|x_{l1} - x_{l2}|, |x_{r1} - x_{r2}|\}.$$

In the special case $\overline{x}_1 = [a, a]$ and $\overline{x}_2 = [b, b]$, we obtain usual metric on R with

$$d(\overline{x}_1, \overline{x}_2) = |a - b|.$$

Let us define transformation f from N to $R(I)$ by $k \rightarrow f(k) = (\overline{x}_k)$. Then $(\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $(\overline{x}_k)$. ω^i denotes the set of all sequences of interval numbers with real terms and c_0^i, c^i and ℓ_∞^i denote the set of all null, convergent and bounded sequences of intervals with real terms.

The set of all sequences of intervals ω^i is a quasivector space as the following rules are clearly satisfied: $(\overline{x}_k) + (\overline{y}_k) = (\overline{y}_k) + (\overline{x}_k)$; $(\overline{x}_k) + ((\overline{y}_k) + (\overline{z}_k)) = ((\overline{x}_k) + (\overline{y}_k)) + (\overline{z}_k)$; $(\overline{x}_k) + (\overline{y}_k) = (\overline{x}_k) + (\overline{z}_k)$ implies $(\overline{y}_k) = (\overline{z}_k)$; $\alpha\,((\overline{x}_k) + (\overline{y}_k)) = \alpha\,(\overline{x}_k) + \alpha\,(\overline{y}_k)$; $(\alpha + \beta)\,(\overline{x}_k) = \alpha\,(\overline{x}_k) + \beta\,(\overline{y}_k)$, (where $\alpha\beta \geq 0$); $\alpha(\beta\,(\overline{x}_k)) = (\alpha\beta)\,(\overline{x}_k)$; $(\overline{x}_k) = [1,\, 1]\,(\overline{x}_k)$. Then zero element of ω^i is the sequece $\theta = (\theta_k) = ([0,\, 0])$ all terms of which are zero interval.

Definition 1.10.1: [92] A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_0$ if for each $\epsilon > 0$ there exists a positive number k_0 such that $d(\overline{x}_k, \overline{x}_0) < \epsilon$ for all $k \geq k_0$ and we denote it by $lim_k\ \overline{x}_k = \overline{x}_0$. Thus $lim_k\ \overline{x}_k = \overline{x}_0 \Leftrightarrow lim_k\ x_{lk} = x_{l0}$ and $lim_k\ x_{rk} = x_{r0}$.

Definition 1.10.2: For $\bar{y} \in \omega^i$ where $\bar{y} = ([y_{lk}, y_{rk}])$, if $y_{lk} = y_{rk}$ for all $k \in N$, then the sequence $\bar{y} = (\bar{y}_k)$ is called degenerate interval sequence.

In [109] Şengönül and Eryilamz defined sequence spaces of null, convergent and bounded of the interval numbers and denoted the spaces by c_0^i, c^i and ℓ_∞^i respectively, that is

$$c_0^i = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : lim_k\,\overline{x}_k = \theta\right\},\ where\ \theta = [0,\, 0]\,.$$

$$c^i = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : lim_k\,\overline{x}_k = \overline{x}_0\right\}.$$

$$\ell_\infty^i = \left\{\overline{x} = (\overline{x}_k) \in \omega^i : sup_k\ \{|x_{lk}|\,, |x_{rk}| < \infty\}\right\}.$$

The spaces c_0^i, c^i and ℓ_∞^i are subspaces of the space ω^i.

Theorem 1.10.1: [109] $\left(c_0^i,\, \overline{d}\right), \left(c^i,\, \overline{d}\right)$ and $(\ell_\infty^i,\ \overline{d}\ )$ are complete metric spaces with the metric defined by $\overline{d}\,(\overline{x}_k, \overline{y}_k) = sup_k\, max\, \{|x_{lk} - y_{lk}|\,,\ |x_{rk} - y_{rk}|\}$.

Definition 1.10.3: [92] A norm on μ^i is a non-negative function $\|.\|_{\mu^i} = \mu^i \rightarrow R^+ \cup \{0\}$ that satisfies the following properties: $\forall \bar{x}, \bar{y} \in \mu^i$ and $\forall \alpha \in R, \forall \overline{x} \in \mu^i - \{\theta\}$,

N_1. $\|\overline{x}\|_{\mu^i} > 0$;

N_2. $\|\overline{x}\|_{\mu^i} = 0 \Leftrightarrow \overline{x} = \theta$;

N_3. $\|\overline{x} + \overline{y}\|_{\mu^i} \leq \|\overline{x}\|_{\mu^i} + \|\overline{y}\|_{\mu^i}$;

N_4. $\|\alpha\overline{x}\|_{\mu^i} = |\alpha|\|\overline{x}\|_{\mu^i}$.

Theorem 1.10.2: [109] The spaces c_0^i, c^i and ℓ_∞^i are normed interval spaces with the norm

$$\|\overline{x}\| = sup_k\, max\, \{|x_{lk}|, |x_{rk}|\}.$$

Definition 1.10.4: [109] An interval valued sequence space $\overline{E}$ is said to be solid if $\overline{y} = (\overline{y}_k) \in \overline{E}$ whenever $\|\overline{y}_k\| \leq \|\overline{x}_k\|$ for all $k \in N$ and $\overline{x} = (\overline{x}_k) \in \overline{E}$.

Definition 1.10.5: [2] An interval valued sequence space $\overline{E}$ is said to be convergence free if $\overline{y} = (\overline{y}_k) \in \overline{E}$ whenever $\overline{x} = (\overline{x}_k) \in \overline{E}$ and $\overline{x}_k = \theta$ implies $\overline{y}_k = \theta$, where $\theta = [0, 0]$.

Definition 1.10.6: [2] An interval valued sequence space $\overline{E}$ is said to be sequence algebra if $(\overline{x}_k\ \overline{y}_k) \in \overline{E}$ whenever $\overline{x} = (\overline{x}_k) \in \overline{E}$, $\overline{y} = (\overline{y}_k) \in \overline{E}$ $(k \in N)$.

Theorem 1.10.3: [109] The spaces c_0^i and c^i are solid.

Theorem 1.10.4: [109] The inclusion $\omega \subset \omega^i$ holds.

Theorem 1.10.5: [109] The inclusion $c_0^i \subset c^i$ holds.

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