CHAPTER 6

IDEAL CONVERGENT SEQUENCES OF INTERVAL NUMBERS

Ideal Convergent Sequences of Interval Numbers

In this chapter we have introduced some $I-$ convergent, $I-$ null and $I-$ bounded sequence spaces of interval numbers. We have studied some algebraic and topological properties of these spaces. We also investigated some inclusion relations related to these spaces.

The concept of ideal convergence of interval numbers has been introduced by Hazarika [18].

Definition 6.1: [18] A sequence $x = (\bar{x}_k)$ of interval numbers is said to be $I-$convergent to an interval number $\bar{x}_0$ if $\{k \in N : d(\bar{x}_k, \bar{x}_0) \geq \epsilon\} \in I$ for every $\epsilon > 0$.

Definition 6.2: A sequence of interval numbers $\overline{x} = (\overline{x}_k) = (x_{lk}, x_{rk}) \in \omega^i$ is said to be $I-$null if $\overline{x}_0 = \theta$, where $\theta = [0, 0]$. In this case we write $I - lim\overline{x}_k = \theta$, where $\theta = [0, 0]$.

Equivalently, $I - limx_{lk} = 0$ and $I - limx_{rk} = 0$.

Definition 6.3: A sequence of interval numbers $x = (\bar{x}_k)$ is said to be $I-$ Cauchy if for each $\epsilon > 0$, there exists positive integers $l \in N$, then

$$\{k \in N : d(\bar{x}_k, \bar{x}_l) \geq \epsilon\} \in I.$$

Definition 6.4: A sequence of interval numbers $\overline{x} = (\overline{x}_k) = (x_{lk},\, x_{rk}) \in \omega^i$ is said to be $I-$bounded if there exists $\overline{M} = [M_l,\, M_r] > \theta = [0,\, 0]$ such that $\left\{k \in N : |\overline{x}_k| > \overline{M}\right\} \in I$.

Equivalently, $\{k \in N : |x_{lk}| > M_l\} \in I$ and $\{k \in N : |x_{rk}| > M_r\} \in I$.

Theorem 6.1: [18] Every ideal convergent sequence of interval numbers has only one limit.

Theorem 6.2: [18] If $I - lim_k \bar{x}_k = \bar{x}_0$ and $I - lim_k \bar{y}_k = \bar{y}_0$ then

(i) $I - lim_k(\bar{x}_k + \bar{y}_k) = \bar{x}_0 + \bar{y}_0$.
(ii) $I - lim_k(a\bar{x}_k) = a\bar{x}_0$ for $a \in R$.

Theorem 6.3: [18] Let $(\bar{x}_k)$, $(\bar{y}_k)$ and $(\bar{z}_k)$ be three sequences of interval numbers such that
(i) $\bar{x}_k \leq \bar{y}_k \leq \bar{z}_k$ for all $k \in N$.
(ii) $I - lim_k \bar{x}_k = \bar{x}_0 = I - lim_k \bar{z}_k$,
then $I - lim_k \bar{y}_k = \bar{x}_0$.

Theorem 6.4: [18] If $I - lim_k \bar{x}_k = \bar{x}_0$ and $I - lim_k \bar{y}_k = \theta$, where $\theta = [0, 0]$, then $I - lim_k(\bar{x}_k + \bar{y}_k) = lim_k \bar{x}_k$.

We have introduced the following classes of sequences of interval numbers:

$c^{I(i)} = \{(\overline{x}_k) \in \omega^i : I - lim\, \overline{x}_k = \overline{x}_0 \; for\, some\; \overline{x}_0\}$.

$c_0^{I(i)} = \{(\overline{x}_k) \in \omega^i : I - lim\, \overline{x}_k = \theta\}$.

$\ell_\infty^{I(i)} = \{(\overline{x}_k) \in \omega^i : sup_k\, |\overline{x}_k| < \infty\}$.

Also we write

$m^{I(i)} = c^{I(i)} \bigcap \ell_{\infty}^{I(i)}$

and $m_0^{I(i)} = c_0^{I(i)} \bigcap \ell_{\infty}^{I(i)}$

Theorem 6.5: The class of sequences $c^{I(i)}$, $c_0^{I(i)}$, $m^{I(i)}$ and $m_0^{I(i)}$ are complete metric spaces with the metric defined by

$$\overline{d}((\overline{x}_k), (\overline{y}_k)) = sup_k\, max\ \{|x_{lk} - y_{lk}|, |x_{rk} - y_{rk}|\}.$$

Proof: It can be established using standard technique.

Theorem 6.6: The spaces $m_0^{I(i)}$ and $m^{I(i)}$ are normed interval spaces with the norm

$\|\overline{x}\| = sup_k max \{|x_{lk}|, |x_{rk}|\}.$

Proof: Let $\mu^i = m_0^{I(i)}$ $(or\, m^{I(i)})$ and $\overline{x}$, $\overline{y} \in \mu^i$.

$N_{1.}$ Since $\|\overline{x}\|_{\mu^i} = sup_k\, max \{|x_{lk}|, |x_{rk}|\}$, then we have $\| \overline{x} \|_{\mu^i} > 0 \,\forall \overline{x} \in \mu^i - \{\theta\}$.

N_2. $\| \overline{x} \|_{\mu^i} = 0 \Leftrightarrow sup_k\, max \{| x_{lk} |, | x_{rk} |\} = 0 \Leftrightarrow \overline{x} = \theta$, where $\theta = [0, 0]$.

N_3. $\| \overline{x} + \overline{y} \|_{\mu^i} = sup_k\, max \{| x_{lk} + y_{lk} |, | x_{rk} + y_{rk} |\}$

$\leq sup_k\, max \{| x_{lk} | + | y_{lk} |, | x_{rk} | + | y_{rk} |\}$

$\leq sup_k \, max \{| x_{lk} |, | x_{rk} |\} + sup_k \, max \{| y_{lk} |, | y_{rk} |\}$

$= \| \overline{x} \|_{\mu^i} + \| \overline{y} \|_{\mu^i}$

N_4. $\| \alpha\overline{x} \|_{\mu^i} = sup_k \, max \{| \alpha x_{lk} |, | \alpha x_{rk} |\}$

$= | \alpha | \, sup_k \, max \{| x_{lk} |, | x_{rk} |\}$

$= | \alpha | \| \overline{x} \|_{\mu^i}$

Hence, $\| \overline{x} \|_{\mu^i}$ is a norm on μ^i.

Theorem 6.7: The spaces $c_0^{I(i)}$ and $m_0^{I(i)}$ are solid.

Proof: We shall prove the result for $c_0^{I(i)}$. For $m_0^{I(i)}$, the result can be proved similarly.

Let, $\overline{x} = (\overline{x}_k) \in c_0^{I(i)}$. Then

$I - lim \, \overline{x}_k = \theta$............................(6.1.1)

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in N$. Then the result follows from (6.1.1) and the following inequality

$|\alpha_k \overline{x}_k| \leq |\alpha_k| \, |\overline{x}_k| \leq |\overline{x}_k|$for all $k \in N$.

Theorem 6.8: The space $c^{I(i)}$, $c_0^{I(i)}$ are sequence algebra.

Proof: We shall prove that $c_0^{I(i)}$ is a sequence algebra. Then

$I - lim\,\overline{x}_k = \theta$

and $I - lim\,\overline{y}_k = \theta$.

Then we have $I - lim\,(\overline{x}_k.\overline{y}_k) = \theta$.

Thus, $(\overline{x}_k.\overline{y}_k) \in c_0^{I(i)}$ is a sequence algebra.

Result 6.1: The spaces $c^{I(i)}$, $c_0^{I(i)}$, $m_0^{I(i)}$ and $m^{I(i)}$ are not convergence free in general.

Proof: Let $I = I_f$. Here we give a counter example.

Example 6.1: Let, $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ be two sequences of interval numbers.

Now let, $\overline{x}_k = \left[\frac{1}{k},\ \frac{1}{k^2}\right]$

and $\overline{y}_k = \left[k,\ \frac{1}{k} + 1\right]$ for all $k \in N$.

Then $(\overline{x}_k) \in c^{I(i)}, c_0^{I(i)}, m_0^{I(i)}$ and $m^{I(i)}$ but $(\overline{y}_k) \notin c^{I(i)}, c_0^{I(i)}, m_0^{I(i)}$ and $m^{I(i)}$.

Hence the spaces $c^{I(i)}$, $c_0^{I(i)}$, $m_0^{I(i)}$ and $m^{I(i)}$ are not convergence free in general.

Theorem 6.9: The inclusions $c_0^{I(i)} \subset c^{I(i)} \subset l_\infty^{I(i)}$ hold and are strict.

Proof: Let, $\overline{x} = (\overline{x}_k) \in c^{I(i)}$. Then, $I - lim\overline{x}_k = \overline{x}_0$

We have,

$$|\overline{x}_k| \leq \frac{1}{2}|\overline{x}_k - \overline{x}_0| + |\overline{x}_0|.$$

Taking supremum over k on both sides we get $(\overline{x}_k) \in l_\infty^{I(i)}$. The inclusion $c_0^{I(i)} \subset c^{I(i)}$ is obvious.

That the inclusion is proper follows from the following example.

Example 6.2: (a) Let $I = I_\delta$. Consider the sequence $\overline{x} = (\overline{x}_k)$ is defined by $\overline{x}_k = [1,\ 2]$ for all $k \in N$. Then $(\overline{x}_k) \in c^{I(i)}$, but $(\overline{x}_k) \notin c_0^{I(i)}$.

(b) Consider the sequence $\overline{y} = (\overline{y}_k)$ is defined as follows:

$$\overline{y}_k = \begin{cases} [2,\ -3]\,, & k\, is\, even \\ [0,\ 0]\,, & \text{otherwise} \end{cases}$$

Then, $(\overline{y}_k) \in l_\infty^{I(i)}$, but $(\overline{x}_k) \notin c^{I(i)}$.

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