Chapter-6

Interval valued intuitionistic fuzzy soft topological spaces

In this chapter, the concept of interval valued intuitionistic fuzzy soft topological space (briefly, IVIFS-topological space) is introduced. We define the notion of neighbourhood, interior, exterior, closure of an IVIFS-set and basis and subspace for IVIFS-topological space. Also, we introduce an investment decision making procedure for solving investment decision making formed by IVIFS-set using weighted reduct intuitionistic fuzzy soft set (WRIFS-set) introduced by Qin et al. [42] in 2011.

6.1. Interval valued intuitionistic fuzzy soft topological spaces

Let U be a universe, E be the set of parameters, P(U) be the set of all subsets of U, IVIFS(U) be the set of all interval valued intuitionistic fuzzy sets in U and IVIFSS(U; E) be the family of all interval valued intuitionistic fuzzy soft sets over U via parameters in E. Simsekler and Yuksel [46] introduced fuzzy soft topology over a fuzzy soft set with a fixed parameter set \( A \subseteq E \). Li and Cui [29] defined the topological structured of intuitionistic fuzzy soft sets taking the whole parameter set E. In this section we introduce the concept of interval valued intuitionistic fuzzy soft topological spaces with a fixed parameter set \( A \subseteq E \), which is the extension of fuzzy soft topological spaces introduced by Simsekler and Yuksel [46] as well as intuitionistic fuzzy soft topological spaces introduced by Li and Cui [29].

**Definition 6.1.1:** Let \((\xi, E)\) be an element of IVIFSS(U; E), \(P(\xi, E)\) be the collection of all IVIFS-subsets of \((\xi, E)\). A subfamily \( \tau \) of \( P(\xi, E) \) is called an interval valued intuitionistic fuzzy soft topology (in short IVIFS-topology) on \((\xi, E)\) if the following axioms are satisfied:

\[
\begin{align*}
[O_1] & \quad (\phi, E), (\xi, E) \in \tau, \\
[O_2] & \quad \{ (f^k, E) | k \in K \} \subseteq \tau \Rightarrow \bigcup_{k \in K} (f^k, E) \in \tau \\
[O_3] & \quad \text{If } (f, E), (g, E) \in \tau \text{, then } (f, E) \cap (g, E) \in \tau
\end{align*}
\]

Then the pair \(( (\xi, E), \tau) \) is called an interval valued intuitionistic fuzzy soft topological space (briefly, IVIFS-topological space). The members of \( \tau \) are called \( \tau \)-open IVIFS sets (or simply open sets). (where \( \phi_{\xi} : A \rightarrow \text{IVIFS}(U) \) is defined as \( \phi_{\xi}(e) = \{ (x, [0, 0], [1, 1]) : x \in U \}, \forall e \in A. \)
Example 6.1.2: Let $U = \{u^1, u^2, u^3\}$, $E = \{e_1, e_2, e_3, e_4\}$, $A = \{e_1, e_2, e_3\}$ and

\[
(\xi, E) = \left\{ e_1 = \left\{ u^1_{[1,1][0,0]}, u^2_{[0.7,0.8][0,0]}, u^3_{[1,1][0,0]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0.4,0.5][0,0]}, u^2_{[1,1][0,0]}, u^3_{[0.4,0.5][0.2,0.3]} \right\}, \\
e_3 = \left\{ u^1_{[0.1,1][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0.4,0.5][0.1,0.2]} \right\}, \\
(\phi_x, E) = \left\{ e_1 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \\
e_3 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \\
(f^1_x, E) = \left\{ e_1 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \\
e_3 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \\
(f^2_x, E) = \left\{ e_1 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \\
e_3 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \\
(f^3_x, E) = (f^1_x, E) \cap (f^2_x, E) = \left\{ e_1 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \\
e_3 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}, \\
(f^4_x, E) = (f^1_x, E) \cup (f^2_x, E) = \left\{ e_1 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \right. \\
e_2 = \left\{ u^1_{[0,0.5][0.2,0.3]}, u^2_{[0,0.5][0.2,0.3]}, u^3_{[0,0][1,1]} \right\}, \\
e_3 = \left\{ u^1_{[0,0][1,1]}, u^2_{[0,0][1,1]}, u^3_{[0,0][1,1]} \right\}. \\
\]

Then the subfamily $\tau = \{ (\phi_x, E), (\xi, E), (f^1_x, E), (f^2_x, E), (f^3_x, E), (f^4_x, E) \}$ of $P(\xi, E)$ is an IVIFS-topology on $(\xi, E)$ since it satisfies the necessary three axioms $[O_1], [O_2]$ and $[O_3]$ and $((\xi, E), \tau)$ is an IVIFS-topological space. But the subfamily $\tau_2 = \{ (\phi_x, E), (\xi, E), (f^1_x, E), (f^2_x, E) \}$ of $P(\xi, E)$ is not an IVIFS-topology on $(\xi, E)$ since the union $(f^1_x, E) \cup (f^2_x, E) = (f^3_x, E)$ which does not belong to $\tau_2$. 


Definition 6.1.3: As every IVIFS-topology on \((\xi, E)\) must contain the sets \((\phi_{\xi}, E)\) and \((\xi, E)\), so the family \(\varnothing = \{(\phi_{\xi}, E), (\xi, E)\}\), forms an IVIFS-topology on \((\xi, E)\). This topology is called indiscrete IVIFS-topology and the pair \(((\xi, E), \varnothing)\) is called an indiscrete interval valued intuitionistic fuzzy soft topological space (or simply indiscrete IVIFS-topological space).

Definition 6.1.4: Let \(D\) denote the family of all IVIFS-subsets of \((\xi, E)\). Then we observe that \(D\) satisfies all the axioms for topology on \((\xi, E)\). This topology is called discrete IVIFS-topology and the pair \(((\xi, E), D)\) is called discrete IVIFS-topological space.

Theorem 6.1.5: Let \(\{\tau_i : i \in I\}\) be any collection of IVIFS-topologies on \((\xi, E)\). Then their intersection \(\bigcap_{i \in I} \tau_i\) is also a topology on \((\xi, E)\).

Proof: [\(O_1\).] Since \((\phi_{\xi}, E), (\xi, E)\in \tau_i\) for each \(i \in I\), hence \((\phi_{\xi}, E), (\xi, E)\in \bigcap_{i \in I} \tau_i\).

[\(O_2\).] Let \(\{(f_k^i, E) | k \in K\}\) be an arbitrary family of IVIFS-sets where \((f_k^i, E)\in \bigcap_{i \in I} \tau_i\) for each \(k \in K\). Then for each \(i \in I\), \(k \in K\), \((f_k^i, E)\in \tau_i\) and since for each \(i \in I\), \(\tau_i\) is a topology, therefore \(\bigcup_{k \in K} (f_k^i, E)\in \tau_i\), for each \(i \in I\). Hence \(\bigcup_{k \in K} (f_k^i, E)\in \bigcap_{i \in I} \tau_i\).

[\(O_3\).] Let \((f_i, E)\) and \((g_i, E)\in \bigcap_{i \in I} \tau_i\), then \((f_i, E)\) and \((g_i, E)\in \tau_i\), for each \(i \in I\) and since \(\tau_i\) for each \(i \in I\) is a topology, therefore \((f_i, E)\cap (g_i, E)\in \tau_i\), for each \(i \in I\). Hence \((f_i, E)\cap (g_i, E)\in \bigcap_{i \in I} \tau_i\).

Thus \(\bigcap_{i \in I} \tau_i\) satisfies all the axioms of topology. Hence \(\bigcap_{i \in I} \tau_i\) forms a topology. But union of topologies need not be a topology; we can show this with following example.

Remark 6.1.6: The union of two IVIFS-topologies may not be an IVIFS-topology. If we consider the example 6.1.2, then the subfamilies \(\tau_3 = \{(\phi_{\xi}, E), (\xi, E), (f_i^j, E)\}\) and \(\tau_5 = \{(\phi_{\xi}, E), (\xi, E), (f_i^j, E)\}\) are the topologies in \((\xi, E)\). But their union \(\tau_3 \cup \tau_5 = \{(\phi_{\xi}, E), (\xi, E), (f_i^j, E), (f_i^j, E)\} = \tau_5\) which is not a topology on \((\xi, E)\).
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Interval valued intuitionistic fuzzy soft topological spaces

**Definition 6.1.7:** Let \((\xi, E, \tau)\) be an IVIFS-topological space over \((\xi, E)\).

An IVIFS-subset \((f_A, E)\) of \((\xi, E)\) is called interval valued intuitionistic fuzzy soft closed (briefly, IVIFS-closed) if its complement \((f_A, E)^c\) is a member of \(\tau\).

**Example 6.1.8:** Let us consider example 6.1.2, then the IVIFS-closed sets in \((\xi, E, \tau)\) are

\[
\phi_{\xi_e}, E)^c = (U, E) = \left\{ e_1 = \{u_1^{1111110000}, u_2^{1111110000}, u_3^{1111110000}\},
\quad e_2 = \{u_1^{1111110000}, u_2^{1111110000}, u_3^{1111110000}\},
\quad e_3 = \{u_1^{1111110000}, u_2^{1111110000}, u_3^{1111110000}\}\right\},
\]

\[
(\xi, E)^c = \left\{ e_1 = \{u_1^{0000111111}, u_2^{0000111111}, u_3^{0000111111}\},
\quad e_2 = \{u_1^{0000111111}, u_2^{0000111111}, u_3^{0000111111}\},
\quad e_3 = \{u_1^{0000111111}, u_2^{0000111111}, u_3^{0000111111}\}\right\},
\]

\[
(f_1^1, E)^c = \left\{ e_1 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_2 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_3 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\}\right\},
\]

\[
(f_2^2, E)^c = \left\{ e_1 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_2 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_3 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\}\right\},
\]

\[
(f_3^3, E)^c = \left\{ e_1 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_2 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_3 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\}\right\},
\]

\[
(f_4^4, E)^c = \left\{ e_1 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_2 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\},
\quad e_3 = \{u_1^{1111000002}, u_2^{1111000002}, u_3^{1111000002}\}\right\}.
\]
**Theorem 6.1.9:** Let \( ((\xi, E), \tau) \) be an IVIFS-topological space over \((\xi_0, E)\). Then,

1. \((\xi_0, E)^c\) and \((\xi, E)^c\) are IVIFS-closed sets,

2. The arbitrary intersection of IVIFS-closed sets is IVIFS-closed,

3. The union of two IVIFS-closed sets is an IVIFS-closed set.

**Proof:**

1. Since \((\phi, E), (\xi_0, E) \in \tau, (\phi, E)^c\) and \((\xi, E)^c\) are closed

2. Let \( \{(f_k)(E) \mid k \in K, (f_k)(E)^c\} \) be an arbitrary family of IVIFS closed sets in \( ((\xi, E), \tau) \) and let \( (f_k)(E) = \cap_{k \in K} (f_k)(E) \).

Now, since \((f_k)(E)^c = (\cap_{k \in K} (f_k)(E))^c = \cup_{k \in K} (f_k)(E)^c\) and \((f_k)(E)^c \in \tau\), for each \( k \in K \), so \( \cup_{k \in K} (f_k)(E)^c \in \tau \). Hence \((f_k)(E)^c \in \tau\). Thus \((f_k)(E)\) is IVIFS closed set.

3. Let \( \{(f'_i)(E) \mid i = 1,2,3,\ldots,n\} \) be a finite family of IVIFS closed sets in \( ((\xi, E), \tau) \) and let \( (g_i)(E) = \cup_{i=1}^n (f'_i)(E) \).

Now, since \( (g_i)(E)^c = (\cup_{i=1}^n (f'_i)(E))^c = \cap_{i=1}^n (f'_i)(E)^c\) and \((f'_i)(E)^c \in \tau\).

So \( \cap_{i=1}^n (f'_i)(E)^c \in \tau \). Hence \((g_i)(E)^c \in \tau\). Thus \((g_i)(E)\) is an IVIFS-closed set.

**Remark 6.1.10:** The intersection of an arbitrary family of IVIFS-open set may not be an IVIFS-open and the union of an arbitrary family of IVIFS-closed set may not be an IVIFS-closed. It can be shown in following:

Let us consider \( U = \{u^1, u^2, u^3\}, E = \{e_1, e_2, e_3, e_4\}, A = \{e_1, e_2, e_3\} \) and let

\[
(\xi, E) = \{e_1 = \{u^1_{\{0,0|0,1\}}, u^2_{\{0,0|1,0\}}, u^3_{\{0,0|1,1\}}\},

\quad e_2 = \{u^1_{\{0,0|0,1\}}, u^2_{\{0,0|1,0\}}, u^3_{\{0,0|1,1\}}\},

\quad e_3 = \{u^1_{\{0,0|0,1\}}, u^2_{\{0,0|1,0\}}, u^3_{\{0,0,1,1\}}\}\}
\]

\[
(\phi, E) = \{e_1 = \{u^1_{\{0,0|1,1\}}, u^2_{\{0,0|0,1\}}, u^3_{\{0,0|1,1\}}\},

\quad e_2 = \{u^1_{\{0,0|1,1\}}, u^2_{\{0,0|0,1\}}, u^3_{\{0,0,1,1\}}\},

\quad e_3 = \{u^1_{\{0,0,1,1\}}, u^2_{\{0,0,1,1\}}, u^3_{\{0,0,1,1\}}\}\}
\]
For each \( n \in \mathbb{N} \), we define
\[
\left( f^a_n, E \right) = \left\{ e_1 = \left\{ u^1_{\left[ 1,1 \right]}, u^2_{\left[ 0,0 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_2 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_3 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\} \right\}
\]

Let us consider the subfamily \( \tau \) of \( P(\xi_n, E) \), such that \( (\phi_x, E) \), \( (\xi_n, E) \) \( \in \tau \) and \( (f^a_n, E) \) \( \in \tau \), for \( n = 1, 2, 3, \ldots \).

Then we observe that \( \tau \) is an IVIFS-topology on \( (\xi_n, E) \). But
\[
\bigcap_{n=1}^{\infty} \left( f^a_n, E \right) = \left\{ e_1 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_2 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_3 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\} \right\} \notin \tau.
\]

The IVIFS-closed sets in the IVIFS-topological space \( (\xi_n, E, \tau) \) are:
\[
(\phi_x, E)^c, (\xi_n, E)^c, (f^a_n, E)^c \quad (\text{for } n=1,2,3,\ldots).
\]

But
\[
\bigcup_{n=1}^{\infty} \left( f^a_n, E \right)^c = \left\{ e_1 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_2 = \left\{ u^1_{\left[ 0,0 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\}, e_3 = \left\{ u^1_{\left[ 1,1 \right]}, u^2_{\left[ 1,1 \right]}, u^3_{\left[ 0,0 \right]} \right\} \right\}
\]
is not an IVIFS-closed set in IVIFS-topological space \( (\xi_n, E, \tau) \), since
\[
\left( \bigcup_{n=1}^{\infty} \left( f^a_n, E \right)^c \right)^c \notin \tau.
\]

**Definition 6.1.11:** Let \( (\xi_n, E, \tau_1) \) and \( (\xi_n, E, \tau_2) \) be two IVIFS-topological spaces. If each \( (f^a_n, E) \in \tau_1 \Rightarrow (f^a_n, E) \in \tau_2 \), then \( \tau_2 \) is called IVIFS-finer topology than \( \tau_1 \) and \( \tau_1 \) is called IVIFS-coarser topology than \( \tau_2 \).

**Example 6.1.12:** If we consider the topologies \( \tau_1 = \left\{ (\phi_x, E), (\xi_n, E), (f^a_n, E), (f^a_n, E), (f^a_n, E), (f^a_n, E) \right\} \) as in the example: 6.1.2 and \( \tau_2 = \left\{ (\phi_x, E), (\xi_n, E), (f^a_n, E) \right\} \) on \( (\xi_n, E) \), then \( \tau_1 \) is IVIFS-finer topology than \( \tau_2 \) and \( \tau_2 \) IVIFS-coarser topology than \( \tau_1 \).
Definition 6.1.13: Let \((\xi, E), \tau\) be an IVIFS-topological space on \((\xi, E)\) and \(B\) be a subfamily of \(\tau\). If every element of \(\tau\) can be express as the arbitrary interval valued intuitionistic fuzzy soft union of some element of \(B\), then \(B\) is called an interval valued intuitionistic fuzzy soft basis for the IVIFS-topology \(\tau\).

Example 6.1.14: In the example: 6.1.2 for the topology \(\tau = \{(\phi, E), (\xi, E), (f^1_3, E), (f^2_4, E), (f^3_4, E)\}\) the subfamily \(B = \{(\phi, E), (\xi, E), (f^1_3, E), (f^2_4, E), (f^3_4, E)\}\) of \(P(\xi, E)\) is a basis for the topology \(\tau\).

6.2. Neighbourhood, interior, exterior and closure

In this section, we introduce neighbourhood of an IVIFS-set and the neighbourhood systems for future discussion.

Definition 6.2.1: Let \(\tau\) be the IVIFS-topology on \((\xi, E) \in IVIFS(U; E)\) and \((f_\lambda, E)\) be an IVIFS-set in \(P(\xi, E)\). An IVIFS-set \((f, E)\) in \(P(\xi, E)\) is a neighbourhood of an IVIFS-set \((g, E)\) if and only if there exists a \(\tau\)-open IVIFS-set \((h, E)\), i.e. \((h, E) \in \tau\), such that \((g, E) \subseteq (h, E) \subseteq (f, E)\).

Example 6.2.2: In an IVIFS-topology \(\tau = \{(\phi, E), (\xi, E), \{e_1 = \{u^1_{(0,0.5],[0,1]}, u^2_{(0.4,0.5],[0.5,0.6]}, u^3_{(0.4,0.5],[0,1])}\}\}\), where \((\xi, E) = \{e_1 = \{u^1_{(0,0.5],[0,0.5],[0.5,0.6])}, u^2_{(0.4,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), \(e_2 = \{u^1_{(0.4,0.5],[0.5,0.6])}, u^2_{(0.4,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), \(e_3 = \{u^1_{(0,0.5],[0,0.5],[0.5,0.6])}, u^2_{(0,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), and \((\phi, E) = \{e_1 = \{u^1_{(0,0.5],[0,0.5],[0.5,0.6])}, u^2_{(0,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), \(e_2 = \{u^1_{(0,0.5],[0,0.5],[0.5,0.6])}, u^2_{(0,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), \(e_3 = \{u^1_{(0,0.5],[0,0.5],[0.5,0.6])}, u^2_{(0,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,1])}\}\), the IVIFS-set \((f, E) = \{e_1 = \{u^1_{(0.5,0.6],[0,2,0.3])}, u^2_{(0,0.5],[0.5,0.6])}, u^3_{(0.4,0.5],[0,0.1])}\}\) is a neighbourhood of the IVIFS-set.
(g_\Delta, E) = \left\{ e_1 = \{ u_1^{0.3,0.41(0.3,0.41)} \}, u_2^{0.1,0.2(0.6,0.7)} \}, u_3^{0.4,0.51(0.3,0.41)} \right\}, \text{ because there exists a } \tau \text{-open IVIFS-set }

(h_\Delta, E) = \left\{ e_1 = \{ u_1^{0.4,0.51(0.4,0.5)} \}, u_2^{0.3,0.4(0.5,0.6)} \}, u_3^{0.3,0.4(0.1,0.2)} \right\} \in \tau, \text{ such that }

(g_\Delta, E) \subseteq (h_\Delta, E) \subseteq (f_\Delta, E).

**Theorem 6.2.3:** An IVIFS-set \((f_\Delta, E)\) in \(P(\xi_\Delta, E)\) is an open IVIFS-set if and only if \((f_\Delta, E)\) is a neighbourhood of each IVIFS-set \((g_\Delta, E)\) contained in \((f_\Delta, E)\).

**Proof:** Let \((f_\Delta, E)\) be an open IVIFS-set and \((g_\Delta, E)\) be any IVIFS-set contained in \((f_\Delta, E)\). Since we have \((g_\Delta, E) \subseteq (f_\Delta, E) \subseteq (f_\Delta, E)\), it follows that \((g_\Delta, E)\) is a neighbourhood of \((g_\Delta, E)\).

Conversely, let \((f_\Delta, E)\) be a neighbourhood for every IVIFS-sets contained it. Since \((f_\Delta, E) \subseteq (f_\Delta, E)\), there exist an open IVIFS-set \((h_\Delta, E)\), such that \((f_\Delta, E) \subseteq (h_\Delta, E) \subseteq (f_\Delta, E)\). Hence \((f_\Delta, E) = (h_\Delta, E)\) and \((f_\Delta, E)\) is open.

**Definition 6.2.4:** Let \(((\xi_\Delta, E), \tau)\) be an IVIFS-topological space on \((\xi_\Delta, E)\) and \((f_\Delta, E)\) be a IVIFS-set in \(P(\xi_\Delta, E)\). A collection \(B_{(f_\Delta, E)} \subseteq P(\xi_\Delta, E)\) of all subsets, containing the IVIFS-set \((f_\Delta, E)\) is called a neighbourhood basis of \((f_\Delta, E)\) if

1. Every element of \(B_{(f_\Delta, E)}\) is a neighbourhood of \((f_\Delta, E)\);
2. Every neighbourhood of \((f_\Delta, E)\) contains an element of \(B_{(f_\Delta, E)}\), as a subset.

Now, we introduce interior IVIFS-set, interior of an IVIFS-set, exterior IVIFS-set, exterior of an IVIFS-set.

**Definition 6.2.5:** Let \(((\xi_\Delta, E), \tau)\) be an IVIES-topological space on \((\xi_\Delta, E)\) and \((f_\Delta, E)\), \((g_\Delta, E)\) be IVIFS-sets in \(P(\xi_\Delta, E)\), such that \((g_\Delta, E) \subseteq (f_\Delta, E)\). Then \((g_\Delta, E)\) is called an interior IVIFS-set of \((f_\Delta, E)\) if and only if \((f_\Delta, E)\) is a neighbourhood of \((g_\Delta, E)\).
Definition 6.2.6: Let $((\xi, E), \tau)$ be an IVIFS-topological space on $(\xi, E)$ and $(f_A, E)$ be an IVIFS-set in $P(\xi, E)$. Then the union of all interior IVIFS-set of $(f_A, E)$ is called the interior of $(f_A, E)$ and is denoted by $\text{int}(f_A, E)$ and defined by

$$\text{int}(f_A, E) = \bigcup \{(g_A, E) | (f_A, E) \text{ is a neighbourhood of } (g_A, E)\}.$$ 

Or equivalently

$$\text{int}(f_A, E) = \bigcup \{(g_A, E) | (g_A, E) \text{ is an IVIFS-open set contained in } (f_A, E)\}.$$ 

Example 6.2.7: Let us consider the IVIFS-topology

$$\tau = \{\{\phi, E\}, (\xi, E), (f_A, E), (f^2_A, E), (f^3_A, E), (f^4_A, E)\}$$ 

as in the example 6.1.2 and let

$$(f_A, E) = \{e_1 = \{u^1_{[0.0,1],[0.0,1]} \cup u^2_{[0.0,0.7],[0.0,3.0]} \cup u^3_{[0.1,1],[0.0]}\},$$

$$e_2 = \{u^1_{[0.0,1],[0.0,1]} \cup u^2_{[0.1,1],[0.0]} \cup u^3_{[0.0,1],[1.1]}\},$$

$$e_3 = \{u^1_{[0.0,1],[0.0]} \cup u^2_{[0.0,1],[0.0]} \cup u^3_{[0.0,1],[1.1]}\}$$

be an IVIFS-set, then

$$\text{int}(f_A, E) = \bigcup \{(g_A, E) | (g_A, E) \text{ is an IVIFS-open set contained in } (f_A, E)\}$$

$$= (f^2_A, E) \bigcup (f^3_A, E)$$

$$= (f^2_A, E) = \{e_1 = \{u^1_{[0.0,1],[0.0,1]} \cup u^2_{[0.0,0.7],[0.0,3.0]} \cup u^3_{[0.1,1],[0.0]}\},$$

$$e_2 = \{u^1_{[0.0,1],[0.0,1]} \cup u^2_{[0.1,1],[0.0]} \cup u^3_{[0.0,1],[1.1]}\},$$

$$e_3 = \{u^1_{[0.0,1],[0.0]} \cup u^2_{[0.0,1],[0.0]} \cup u^3_{[0.0,1],[1.1]}\}.$$ 

Since $(f^2_A, E) \subseteq (f_A, E)$ and $(f^3_A, E) \subseteq (f_A, E)$.

Theorem 6.2.8: Let $((\xi, E), \tau)$ be an IVIFS-topological space on $(\xi, E)$ and $(f_A, E)$ be an IVIFS-set in $P(\xi, E)$. Then

3. $\text{int}(f_A, E)$ is an open and $\text{int}(f_A, E)$ is the largest open IVIFS-set contained in $(f_A, E)$.

4. The IVIFS set $(f_A, E)$ is open if and only if $(f_A, E) = \text{int}(f_A, E)$.
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Proof:

1. Since \( \text{int}(f, A) = \bigcup \{(g, A) \mid (f, A) \text{ is a neighbourhood of } (g, A)\} \), we have \( (f, A) \) is itself an interior IVIFS-set of \( (f, A) \). So there exists an open IVIFS-set \( (h, A) \), such that \( \text{int}(f, A) \subseteq (h, A) \subseteq (f, A) \). But \( (h, A) \) is an interior IVIFS-set of \( (f, A) \), hence \( (h, A) \subseteq \text{int}(f, A) \). Hence \( (h, A) = \text{int}(f, A) \). Thus \( \text{int}(f, A) \) is open and \( \text{int}(f, A) \) is the largest open IVIFS set contained in \( (f, A) \).

2. Let \( (f, A) \) be an open IVIFS set. Since \( \text{int}(f, A) \) is an interior IVIFS-set of \( (f, A) \) we have \( (f, A) = \text{int}(f, A) \). Conversely, if \( (f, A) = \text{int}(f, A) \) then \( (f, A) \) is obviously open.

Proposition 6.2.9: For any two IVIFS-sets \( (f, A) \) and \( (g, A) \) in an IVIFS-topological space \( ((\xi, A), \tau) \) on \( (\xi, A) \), we have

(i) \( (g, A) \subseteq (f, A) \Rightarrow \text{int}(g, A) \subseteq \text{int}(f, A) \)

(ii) \( \text{int}(\phi, A) = \{\phi, A\} \) and \( \text{int}(\xi, A) = (\xi, A) \)

(iii) \( \text{int}(\text{int}(f, A)) = \text{int}(f, A) \)

(iv) \( \text{int}(\{g, A\} \cap (f, A)) = \text{int}(g, A) \cap \text{int}(f, A) \)

(v) \( \text{int}(\{g, A\} \cup (f, A)) \supseteq \text{int}(g, A) \cup \text{int}(f, A) \)

Proof: (i) Since \( (g, A) \subseteq (f, A) \), implies all the IVIFS-open set contained in \( (g, A) \) also contained in \( (f, A) \). Therefore

\[ \{(g^*, A) \mid (g^*, A) \text{ is an IVIFS-open set contained in } (g, A)\} \]

\[ \subseteq \{(f^*, A) \mid (f^*, A) \text{ is an IVIFS-open set contained in } (f, A)\} \].

This implies

\[ \bigcup \{(g^*, A) \mid (g^*, A) \text{ is an IVIFS-open set contained in } (g, A)\} \]

\[ \subseteq \bigcup \{(f^*, A) \mid (f^*, A) \text{ is an IVIFS-open set contained in } (f, A)\} \].

So \( \text{int}(g, A) \subseteq \text{int}(f, A) \).
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(ii) \( \text{int}\left(\text{int}(f_A, E)\right) = \bigcup\{(g_A, E) \mid (g_A, E) \text{ is an IVIFS-open set contained in } \text{int}(f_A, E)\} \)

and since \( \text{int}(f_A, E) \) is the largest open IVIFS set contained in \( \text{int}(f_A, E) \), therefore \( \text{int}\left(\text{int}(f_A, E)\right) = \text{int}(f_A, E) \).

(iii) Since \( \text{int}\left((g_A, E)\right) \subseteq (g_A, E) \) and \( \text{int}\left((f_A, E)\right) \subseteq (f_A, E) \), we have, \( (g_A, E) \cap \text{int}(f_A, E) \subseteq (g_A, E) \cap (f_A, E) \). Hence \( (g_A, E) \cap \text{int}(f_A, E) \subseteq \text{int}\left((g_A, E) \cap (f_A, E)\right) \).\(........................(1)\)

Again, since \( (g_A, E) \cap (f_A, E) \subseteq (g_A, E) \) and \( (g_A, E) \cap (f_A, E) \subseteq (f_A, E) \), we have \( \text{int}\left((g_A, E) \cap (f_A, E)\right) \subseteq (g_A, E) \) and \( \text{int}\left((g_A, E) \cap (f_A, E)\right) \subseteq (f_A, E) \).

So \( \text{int}\left((g_A, E) \cap (f_A, E)\right) \subseteq \text{int}(g_A, E) \cap \text{int}(f_A, E) \).\(........................(2)\)

Using (1) and (2), we get \( \text{int}\left((g_A, E) \cap (f_A, E)\right) = \text{int}(g_A, E) \cap \text{int}(f_A, E) \).

(iv) Since \( (g_A, E) \subseteq (g_A, E) \cup (f_A, E) \) and \( (f_A, E) \subseteq (g_A, E) \cup (f_A, E) \), so \( \text{int}(g_A, E) \subseteq \text{int}\left((g_A, E) \cup (f_A, E)\right) \) and \( \text{int}(f_A, E) \subseteq \text{int}\left((g_A, E) \cup (f_A, E)\right) \). Hence \( (g_A, E) \cup \text{int}(f_A, E) \subseteq \text{int}\left((g_A, E) \cup (f_A, E)\right) \).

**Definition 6.2.10:** Let \((\xi_A, E), \tau\) be an IVIFS-topological space on \((\xi_A, E)\) and let \((f_A, E), (g_A, E)\) be two IVIFS-sets in \(P(\xi_A, E)\). Then \((g_A, E)\) is called an exterior IVIFS-set of \((f_A, E)\) if and only if \((g_A, E)\) is an interior IVIFS-set of the complement of \((f_A, E)\).

**Definition 6.2.11:** Let \((\xi_A, E), \tau\) be an interval valued intuitionistic fuzzy soft topological space on \((\xi_A, E)\) and \((f_A, E)\) be an IVIFS-set in \(P(\xi_A, E)\). Then the union of all exterior IVIFS-set of \((f_A, E)\) is called the exterior of \((f_A, E)\) and is denoted by \(\text{ext}(f_A, E)\) and is defined by

\[
\text{ext}(f_A, E) = \bigcup\{(g_A, E) \mid (f_A, E)^\complement \text{ is a neighbourhood of } (g_A, E)\}.
\]

Clearly from definition

\[
\text{ext}(f_A, E) = \text{int}\left((f_A, E)^\complement\right).
\]
Proposition 6.2.12: For any two IVIFS-sets \((f, A)\) and \((g, A)\) in an IVIFS-topological space \(\left((\xi, E), \tau\right)\) on \((\xi, E)\), we have

(i) \(\text{ext}(f, A)\) is open and is the largest open set contained in \((f, A)^c\).

(ii) \((f, A)^c\) is open if and only if \((f, A)^c = \text{ext}(f, A)\).

(iii) \((g, A) \subseteq (f, A) \Rightarrow \text{ext}(f, A) \subseteq \text{ext}(g, A)\).

(iv) \(\text{ext}((g, A) \cap (f, A)) \supseteq \text{ext}(g, A) \cup \text{ext}(f, A)\)

(v) \(\text{ext}((g, A) \cup (f, A)) = \text{ext}(g, A) \cap \text{ext}(f, A)\)

Proof: Straight forward.

Definition 6.2.13: Let \(\left((\xi, E), \tau\right)\) be an IVIFS-topological space on \((\xi, E)\) and \((f, A)\) be an IVIFS-set in \(P(\xi, E)\). Then the intersection of all closed IVIFS-sets containing \((f, A)\) is called the closure of \((f, A)\) and is denoted by \(c\left(f, A\right)\) and defined by

\[c\left(f, A\right) = \bigcap\{(g, A) \mid (g, A)\text{ is an IVIFS-closed set containing } (f, A)\}\]

Observe first that \(c\left(f, A\right)\) is an IVIFS-closed set, since it is the intersection of IVIFS-closed sets. Furthermore, \(c\left(f, A\right)\) is the smallest IVIFS-closed set containing \((f, A)\).

Example 6.2.14: Let us consider an IVIFS-topology \(\tau_1 = \{\phi, (\xi, E), (f, A), (f^1, A), (f^2, A), (f^3, A), (f^4, A)\}\) as in the example 6.1.2 and let

\[\left(\xi, E\right) = \left\{e_1 = \{u^1_{[0.2,0.3],[0.6,0.7]}, u^2_{[0.0,0.1],[0.4,0.5]}, u^3_{[0.0,1],[1.1]}\}, e_2 = \{u^1_{[0.1,0.2],[0.5,0.6]}, u^2_{[0.0,0.4],[0.5]}, u^3_{[1.1],[0.0]}\}, e_3 = \{u^1_{[0.0],[1.1]}, u^2_{[0.0],[1.1]}, u^3_{[0.0],[1.1]}\}, e_4 = \{u^1_{[0.0],[1.1]}, u^2_{[0.0],[1.1]}, u^3_{[0.0],[1.1]}\}\right\}\]

be an IVIFS-set, then
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\[ \text{cl}(f_A, E) = \bigcap \{(g_A, E) \mid (g_A, E) \text{ is an IVIFS-closed set containing } (f_A, E) \} \]

\[ = (f_A^1, E)^c \cap (f_A^3, E)^c \]

\[ = (f_A^1, E)^c = \{ e_1 = \{ u_{[0.2,0.3],[0.5,0.6],u_{[0.2,0.3],[0.4,0.5]},u_{[0.0,0.1],[0.1,1]}) \}, \]

\[ e_2 = \{ u_{[0.2,0.3],[0.4,0.5]},u_{[0.0,0.1],[0.4,0.5]},u_{[0.1,1],[0.0]) \}, \]

\[ e_3 = \{ u_{[0.1,1],[0.0]},u_{[0.1,1],[0.0]},u_{[0.1,1],[0.0]) \} \}. \]

Since \((f_A, E) \subseteq (f_A^1, E)^c\) and \((f_A, E) \subseteq (f_A^3, E)^c\).

**Proposition 6.2.15:** For any two IVIFS-sets \((f_A, E)\) and \((g_A, E)\) in an
IVIFS-topological space \(\{(\xi_A, E), \tau\}\) on \((\xi_A, E)\), we have

(i) \(\text{cl}(f_A, E)\) is the smallest IVIFS-closed set containing \((f_A, E)\).

(ii) \((f_A, E)\) is IVIFS-closed if and only if \((f_A, E) = \text{cl}(f_A, E)\).

(iii) \((g_A, E) \subseteq (f_A, E) \Rightarrow \text{cl}(g_A, E) \subseteq \text{cl}(f_A, E)\)

(iv) \(\text{cl}(\text{cl}(f_A, E)) = \text{cl}(f_A, E)\)

(v) \(\text{cl}(\phi_A, E) = (\phi_A, E)\) and \(\text{cl}(\xi_A, E) = (\xi_A, E)\)

(vi) \(\text{cl}(\text{cl}(g_A, E) \cup (f_A, E)) = \text{cl}(g_A, E) \cup \text{cl}(f_A, E)\)

(vii) \(\text{cl}(\text{cl}(g_A, E) \cap (f_A, E)) \subseteq \text{cl}(g_A, E) \cap \text{cl}(f_A, E)\)

**Proof:** (i) Since \((g_A, E) \subseteq (f_A, E)\), implies all the closed sets containing
\((f_A, E)\) also contained \((g_A, E)\). Therefore

\[ \bigcap \{(g_A^*, E) \mid (g_A^*, E) \text{ is an IVIFS-closed set containing } (g_A, E) \} \]

\[ \subseteq \bigcap \{(f_A^*, E) \mid (f_A^*, E) \text{ is an IVIFS-closed set containing } (f_A, E) \}. \]

So \(\text{cl}(g_A, E) \subseteq \text{cl}(f_A, E)\)

(ii) \(\text{cl}(\text{cl}(f_A, E)) = \bigcup \{(g_A, E) \mid (g_A, E) \text{ is an IVIFS-closed set containing } \text{cl}(f_A, E)\}\)

and since \(\text{cl}(f_A, E)\) is the largest closed IVIFS-set containing \(\text{cl}(f_A, E)\),
therefore \(\text{cl}(\text{cl}(f_A, E)) = \text{cl}(f_A, E)\).
(iii) Since \( cl((g_A, E)) \supseteq (g_A, E) \) and \( cl((f_A, E)) \supseteq (f_A, E) \), we have \( cl(g_A, E) \cup cl(f_A, E) \supseteq (g_A, E) \cup (f_A, E) \). This implies \( cl(g_A, E) \cup cl(f_A, E) \supseteq cl((g_A, E) \cup (f_A, E)) \) .........................(1)

Again, since \( (g_A, E) \cup (f_A, E) \supseteq (g_A, E) \) and \( (g_A, E) \cup (f_A, E) \supseteq (f_A, E) \), so \( cl((g_A, E) \cup (f_A, E)) \supseteq cl(g_A, E) \) and \( cl((g_A, E) \cup (f_A, E)) \supseteq cl(f_A, E) \). Therefore \( cl((g_A, E) \cup (f_A, E)) \supseteq cl(g_A, E) \cup cl(f_A, E) \) .........................(2)

Using (1) and (2), we get \( cl((g_A, E) \cup (f_A, E)) = cl(g_A, E) \cup cl(f_A, E) \)

(iv) Since \( (g_A, E) \supseteq (g_A, E) \cap (f_A, E) \) and \( (f_A, E) \supseteq (g_A, E) \cap (f_A, E) \), so \( cl((g_A, E) \cap (f_A, E)) \supseteq cl((g_A, E) \cap (f_A, E)) \) and \( cl((f_A, E) \supseteq cl((g_A, E) \cap (f_A, E)) \)).

Hence \( cl(g_A, E) \cap cl(f_A, E) \supseteq cl((g_A, E) \cap (f_A, E)) \).

6.3. Interval valued intuitionistic fuzzy soft subspace topology

**Theorem 6.3.1:** Let \( ((\xi_A, E), \tau) \) be an IVIFS-topological space on \( (\xi_A, E) \) and \( (f_A, E) \) be an IVIFS-set in \( P(\xi_A, E) \). Then the collection \( \tau_{(f_A, E)} = \{(f_A, E) \cap (g_A, E) | (g_A, E) \in \tau \} \) is an IVIFS-topology on the IVIFS-set \( (f_A, E) \).

**Proof:**
(i) Since \( (\phi_{\xi_A}, E), (\xi_A, E) \in \tau \), \( (f_A, E) = (f_A, E) \cap (\xi_A, E) \) and \( (\phi_{f_A}, E) = (f_A, E) \cap (\phi_{\xi_A}, E) \), therefore \( (\phi_{f_A}, E), (f_A, E) \in \tau_{(f_A, E)} \).

(ii) Let \( \{(f_A^i, E) | i = 1,2,3,...,n \} \) be a finite subfamily of IVIFS-open sets in \( \tau_{(f_A, E)} \), then for each \( i = 1,2,3,...,n \), there exist \( (g_A^i, E) \in \tau \) such that \( (f_A^i, E) = (f_A, E) \cap (g_A^i, E) \). Now \( \cap_{i=1}^n (f_A^i, E) = \cap_{i=1}^n (f_A, E) \cap (g_A^i, E) = (f_A, E) \cap \bigcap_{i=1}^n (g_A^i, E) \) and since \( \bigcap_{i=1}^n (g_A^i, E) \in \tau \), so \( \cap_{i=1}^n (f_A^i, E) \in \tau_{(f_A, E)} \).

(iii) Let \( \{(f_A^k, E) | k \in K \} \) be an arbitrary family of IVIFS-open sets in \( \tau_{(f_A, E)} \), then for each \( k \in K \), there exist \( (g_A^k, E) \in \tau \), such that \( (f_A^k, E) = (f_A, E) \cap (g_A^k, E) \).

Now \( \cup_{k \in K} (f_A^k, E) = \cup_{k \in K} (f_A, E) \cap (g_A^k, E) = (f_A, E) \cap \bigcup_{k \in K} (g_A^k, E) \) and since \( \bigcup_{k \in K} (g_A^k, E) \in \tau \), so \( \cup_{k \in K} (f_A^k, E) \in \tau_{(f_A, E)} \).
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**Definition 6.3.2:** Let \(((\xi, E), \tau)\) be an IVIFS-topological space on \((\xi, E)\) and \((f, E)\) be an IVIFS-set in \(P(\xi, E)\). Then the IVIFS-topology \(\tau_{(f, E)} = \{((f, E)\cap (g, E))\mid (g, E) \in \tau\}\) is called interval valued intuitionistic fuzzy soft subspace topology (briefly, IVIFS-topological subspace) and \(((f, E), \tau_{(f, E)})\) is called interval valued intuitionistic fuzzy soft subspace of \(((\xi, E), \tau)\).

**Example 6.3.3:** Let us consider the IVIFS-topology \(\tau_{\xi} = \{\phi_{\xi}, E)\}, (\xi, E), (f, E), (f^2, E), (f^3, E), (f^4, E)\}\) as in the example 6.1.2 and an IVIFS-set
\[
(f, E) = \left\{e_1 = \left\{u^1_{\text{[0.2,0.3],[0.0,0.1]}}, u^2_{\text{[0.5,0.6],[0.1,0.2]}}, u^3_{\text{[0.0,0.3],[0.6,0.7]}}\right\},
  e_2 = \left\{u^1_{\text{[0.3,0.4],[0.1,0.2]}}, u^2_{\text{[0.4,0.5],[0.2,0.3]}}, u^3_{\text{[0.0,0.5],[0.2,0.3]}}\right\},
  e_3 = \left\{u^1_{\text{[0.0],[1.1]}}, u^2_{\text{[0.0],[1.1]}}, u^3_{\text{[0.0],[1.1]}}\right\}\} \subseteq P(\xi, E).
\]
Let
\[
(\phi_{f, E}) = (f, E) \cap (\xi, E) = \left\{e_1 = \left\{u^1_{\text{[0.0,0.1],[1.1]}}, u^2_{\text{[0.0,0.1],[1.1]}}, u^3_{\text{[0.0,0.1],[1.1]}}\right\},
  e_2 = \left\{u^1_{\text{[0.0,0.1],[1.1]}}, u^2_{\text{[0.0,0.1],[1.1]}}, u^3_{\text{[0.0,0.1],[1.1]}}\right\},
  e_3 = \left\{u^1_{\text{[0.0,0.1],[1.1]}}, u^2_{\text{[0.0,0.1],[1.1]}}, u^3_{\text{[0.0,0.1],[1.1]}}\right\}\}.
\]
\[
(g^1, E) = (f, E) \cap (f^1, E) = \left\{e_1 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_2 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_3 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\}\}.
\]
\[
(g^2, E) = (f, E) \cap (f^2, E) = \left\{e_1 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_2 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_3 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\}\}.
\]
\[
(g^3, E) = (f, E) \cap (f^3, E) = \left\{e_1 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_2 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\},
  e_3 = \left\{u^1_{\text{[0.0,0.1],[0.2,0.3]}}, u^2_{\text{[0.0,0.1],[0.2,0.3]}}, u^3_{\text{[0.0,0.1],[0.2,0.3]}}\right\}\}.
\]
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Interval valued intuitionistic fuzzy soft topological spaces

\[(g^4_4, E) = (f_4, E) \cap (f^4_4, E) = \{e_i = \{u^1_{[0.2,0.3],[0.1,0.2]}, u^2_{[0.5,0.6],[0.1,0.2]}, u^3_{[0.2,0.3],[0.6,0.7]}\},\]
\[e_2 = \{u^1_{[0.3,0.4],[0.1,0.2]}, u^2_{[0.4,0.5],[0.2,0.3]}, u^3_{[0.0],[0.0],[1.1]}\},\]
\[e_3 = \{u^1_{[0.0],[1.1]}, u^2_{[0.0],[1.1]}, u^3_{[0.0],[1.1]}\}\].

Then \(\tau_{(f_4, E)} = \{((\phi_4, E), (f_4, E), (g^4_4, E), (g^2_4, E), (g^3_4, E), (g^4_4, E))\}\) is an IVIFS-subspace topology for \(\tau\) and \(\{(f_4, E), \tau_{(f_4, E)}\}\) is called IVIFS-subspace of \((\xi_4, E), \tau\)\

**Theorem 6.3.4:** Let \((\xi_4, E), \tau\) be an IVIFS-topological space on \((\xi_4, E), B\) be an IVIFS-basis for \(\tau\) and \((f_4, E)\) be an IVIFS-set in \(P(\xi_4, E)\). Then the family \(B_{(f_4, E)} = \{(f_4, E) \cap (g_4, E) | (g_4, E) \in B\}\) is an IVIFS-basis for subspace topology \(\tau_{(f_4, E)}\).

**Proof:** Let \((h_4, E) \in \tau_{(f_4, E)}\), then there exists an IVIFS-set \((g_4, E) \in \tau\), such that \((h_4, E) = (f_4, E) \cap (g_4, E)\). Since \(B\) is a base for \(\tau\), so there exists subcollection \(\{A^i \in B\} \) of \(B\), such that \((g_4, E) = \bigcup_{i \in I} (A^i, E)\). Therefore \((h_4, E) = (f_4, E) \cap (g_4, E) = (f_4, E) \cap \bigcup_{i \in I} (A^i, E) = \bigcup_{i \in I} ((f_4, E) \cap (A^i, E))\). Since \((f_4, E) \cap (A^i, E) \in B_{(f_4, E)}\), which implies that \(B_{(f_4, E)}\) is an IVIFS-basis for the IVIFS-subspace topology \(\tau_{(f_4, E)}\).

**Theorem 6.3.5:** Let \((\xi_4, E), \tau\) be an IVIFS-topological subspace of \((\eta_4, E), \tau^*\) and let \((\eta_4, E), \tau^*\) be an IVIFS-topological subspace of \((\xi_4, E), \tau^{**}\). Then \((\xi_4, E), \tau\) is also an IVIFS-topological subspace of \((\xi_4, E), \tau^{**}\).

**Proof:** Since \((\xi_4, E) \subseteq (\eta_4, E) \subseteq (\xi_4, E), ((\xi_4, E), \tau)\) is an IVIFS-topological subspace of \((\xi_4, E), \tau^{**}\), if and only if \(\tau^{**} = \tau\). Let \((f_4, E) \in \tau\), now since \((\xi_4, E), \tau\) be an IVIFS-topological subspace of \((\eta_4, E), \tau^*\), i.e. \(\tau^* = \tau\), so there exist \((f_4, E) \in \tau^*\), such that \((f_4, E) = (\xi_4, E) \cap (f_4, E)\). But \((\eta_4, E), \tau^*\) be an IVIFS-topological subspace of \((\xi_4, E), \tau^{**}\), so there exist \((f_4, E) \in \tau^{**}\), such that \((f_4, E) = (\eta_4, E) \cap (f_4, E)\).
Thus \((f_\delta, E) = (\xi_\delta, E) \cap (f_\eta, E) = (\xi_\delta, E) \cap (f_\tau, E) = (\xi_\delta, E) \cap (f_\tau, E) = (\xi_\delta, E) \cap (f_\tau, E)\),
since \((\xi_\delta, E) \subseteq (\eta_\delta, E)\); so \((f_\delta, E) \in \tau_{(\xi_\delta, E)}\). Accordingly, \(\tau \subseteq \tau_{(\xi_\delta, E)}\).

Now assume \((g_\delta, E) \in \tau_{(\xi_\delta, E)}\), i.e. there exist \((h_\delta, E) \in \tau_{(\xi_\delta, E)}\) such that
\((g_\delta, E) = (\xi_\delta, E) \cap (h_\delta, E)\). But
\((\eta_\delta, E) \cap (h_\delta, E) \in \tau_{(\xi_\delta, E)}\), so
\((\xi_\delta, E) \cap (\eta_\delta, E) \cap (\eta_\delta, E) \in \tau_{(\xi_\delta, E)} = \tau\). Since
\((\xi_\delta, E) \cap (\eta_\delta, E) \cap (\eta_\delta, E) = (\xi_\delta, E) \cap (h_\delta, E) = (g_\delta, E)\), we have \((g_\delta, E) \in \tau\). Accordingly, \(\tau_{(\xi_\delta, E)} \subseteq \tau\) and thus the theorem is proved.

6.4. Application of IVIFS-set in investment decision making

In our real life application, there are large numbers of problems that assurance logical, rational and logical decisions that suit best for the achievement of desirable objective. In this research works, we have proposed an investment decision making procedure for solving investment problems based on an IVIFS-set, using different decision criteria, different information regarding investment decision making and information of varying attribute. Also, we have presented an algorithm for investment decision making based on an IVIFS-set and try to apply IVIFS-set to investment problems using weighted reduct intuitionistic fuzzy soft set (weighted RIFS-set) initiated by Qin et al. [42].

**Investment decision rule on RIFS-set**

In this present section, we define an investment decision rule "\(\cdot \cdot \cdot \)" on parameterized family of a weighted RIFS-set of an IVIFS-set.

**Definition 6.4.1:** Suppose \((F, A)\) be an IVIFS-set over \((U, E)\) and \((F_\omega, A)\) be the weighted RIFS-set of \((F, A)\). Let \(\{F_\omega(a) : a \in A\}\) be the parameterized family of \((F_\omega, A)\). For any \(F_\omega(a), F_\omega(b) \in \{F_\omega(a) : a \in A\}\), we define an investment decision rule by \(F_\omega(a) \cdot F_\omega(b)\) defined as

\[
F_\omega(a) \cdot F_\omega(b) = \left\{ \begin{array}{ll}
u \\ \left( \mu_{F_\omega(a) \cdot F_\omega(b)}(u), V_{F_\omega(a) \cdot F_\omega(b)}(u) \right) \end{array} : u \in U \right\},
\]

where

\[
\mu_{F_\omega(a) \cdot F_\omega(b)}(u) = \mu_{F_\omega(a)}(u) \cdot \mu_{F_\omega(b)}(u),
\]

\[
V_{F_\omega(a) \cdot F_\omega(b)}(u) = V_{F_\omega(a)}(u) \cdot V_{F_\omega(b)}(u).
\]
6.4.2. Properties of investment decision rule

(1) \( F_\alpha(a) \ast F_\beta(b) = F_\gamma(b) \ast F_\delta(a) \)
(2) \( F_\gamma(a) \ast (F_\beta(b) \ast F_\delta(c)) = (F_\gamma(a) \ast F_\delta(b)) \ast F_\gamma(c) \)

6.4.3. Application of weighted RIFS-set in investment decision making

In this current section, we present our algorithm for investment decision making based on an IVIFS-set.

**Algorithm 3**

**Step 1.** Input the (resultant) IVIFS-set \((F, A)\)

**Step 2.** Input an opinion weighting vector \(\omega = (\alpha, \beta, \gamma, \delta)\).

**Step 3.** Compute the weighted RIFS-set \((F_\omega, A)\) of the IVIFS-set \((F, A)\) with respect to the opinion weighting vector \(\omega\) (or choose \((F_{LU}, A)\) or \((F_{UL}, A)\) or \((F_{NN}, A)\) of \((F, A)\)).

**Step 4.** Input the investment decision parameter \(a, b \in A\) preference by the investor.

**Step 5.** Apply the investment decision rule "*" on \(\{F_\omega(a), F_\omega(b)\}\).

**Step 6.** Compute \(F_\omega(a) \ast F_\omega(b)\) and present it in tabular form.

**Step 7.** The final optimal decision is to select \(u\) if the corresponding membership value \(\mu_{F_\omega(a) \ast F_\omega(b)}(u)\) is maximized and non membership value \(\nu_{F_\omega(a) \ast F_\omega(b)}(u)\) is minimized.

**Step 8.** If \(u\) has more than one value, then the investor may be chosen any one of \(u\).

**Remark 6.4.4:** In the step 8 of our algorithm, if there are too many optimal choices obtained, then the investor may go back to the step 2 as in our algorithm and replace the opinion weighting vector \(\omega\) that the investor once used to adjust the final optimal decision in the IVIFS-set based investment decision making problems.

6.4.5. Investment decision parameters and investment avenues

In this section, we try to apply an IVIFS-set to investment problems using weighted reduct intuitionistic fuzzy soft set initiated by Qin et al. [42]. We have distinguished the following investment decision parameters that affect their investment decision furthermore the different investment avenues they like better.
Investment decision parameters

- **a = Safety of funds**: It is surety of profit for capital and the affirmation of insurance to the funds contributed under the evolving conditions.
- **b = High returns**: Investor target to higher return that encourages fast development of the funds contributed.
- **c = Maximum benefit in the minimum period**: The decision of the investor is affected by the connection between the period of contributing and rate of return. Also, Investors select an investment avenue in which maximum return is possible in the minimum period.
- **d = Easy accessibility**: It refers effortlessness of strategies and customs during the process of investment.
- **e = Tax concession**: Certain investments and the returns from investments are appropriate for deduction under the income tax. Moreover an investor, who specific to profit tax concession prefers such appropriate investments.

Investment avenues

The investment avenues that are generally favored by the specimen collegians are as follows:
- **A₁**: Bank deposit
- **A₂**: Insurance
- **A₃**: Postal savings
- **A₄**: Shares and stocks
- **A₅**: Mutual fund
- **A₆**: Gold

6.4.6. Application in investment problems

For the application of IVIFS-set in investment problems, we consider $U = \{A₁, A₂, A₃, A₄, A₅, A₆\}$ be the universal set of various types of investment avenues and $E = \{a, b, c, d, e\}$ be the collection of investment decision parameters and let $A=E$.

We consider an IVIFS-set $(F, A)$ represented as in Table 32.
Table 32: IVIFS-set \((F, A)\)

<table>
<thead>
<tr>
<th>(U)</th>
<th>(F(a))</th>
<th>(F(b))</th>
<th>(F(c))</th>
<th>(F(d))</th>
<th>(F(e))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>((0.8,0.85)), ((0.05,0.1))</td>
<td>((0.4,0.5)), ((0.3,0.4))</td>
<td>((0.4,0.45)), ((0.3,0.4))</td>
<td>((0.8,0.85)), ((0.7,0.75)), ((0.05,0.1))</td>
<td>((0.1,0.2))</td>
</tr>
<tr>
<td>(A_2)</td>
<td>((0.7,0.8)), ((0.1,0.15))</td>
<td>((0.5,0.6)), ((0.3,0.35))</td>
<td>((0.15,0.2)), ((0.6,0.7))</td>
<td>((0.7,0.8)), ((0.1,0.15))</td>
<td>((0.8,0.9)), ((0.05,0.1))</td>
</tr>
<tr>
<td>(A_3)</td>
<td>((0.85,0.9)), ((0.05,0.1))</td>
<td>((0.4,0.5)), ((0.3,0.4))</td>
<td>((0.4,0.45)), ((0.3,0.4))</td>
<td>((0.7,0.8)), ((0.05,0.1))</td>
<td>((0.1,0.2))</td>
</tr>
<tr>
<td>(A_4)</td>
<td>((0.3,0.4)), ((0.4,0.5))</td>
<td>((0.5,0.6)), ((0.3,0.4))</td>
<td>((0.8,0.9)), ((0.05,0.1))</td>
<td>((0.1,0.2)), ((0.3,0.4))</td>
<td>((0.7,0.8))</td>
</tr>
<tr>
<td>(A_5)</td>
<td>((0.5,0.6)), ((0.3,0.4))</td>
<td>((0.5,0.6)), ((0.3,0.4))</td>
<td>((0.6,0.7)), ((0.05,0.1))</td>
<td>((0.5,0.6)), ((0.3,0.4))</td>
<td>((1,1))</td>
</tr>
<tr>
<td>(A_6)</td>
<td>((0.7,0.8)), ((0.1,0.2))</td>
<td>((0.7,0.8)), ((0.1,0.2))</td>
<td>((0.6,0.7)), ((0.2,0.3))</td>
<td>((0.8,0.9)), ((0.05,0.1))</td>
<td>((1,1))</td>
</tr>
</tbody>
</table>

If we deal with this problem by the pessimistic-pessimistic-RIFS-set \((F_{LU}, A)\) to solve the problem, at first we obtain the pessimistic-pessimistic-RIFS-set \((F_{LU}, A)\) as in Table 33.

Table 33: Pessimistic-Pessimistic- RIFS-set \((F_{LU}, A)\) of \((F, A)\)

<table>
<thead>
<tr>
<th>(U)</th>
<th>(F_{LU}(a))</th>
<th>(F_{LU}(b))</th>
<th>(F_{LU}(c))</th>
<th>(F_{LU}(d))</th>
<th>(F_{LU}(e))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_1)</td>
<td>((0.8,0.1))</td>
<td>((0.4,0.4))</td>
<td>((0.4,0.4))</td>
<td>((0.8,0.1))</td>
<td>((0.7,0.2))</td>
</tr>
<tr>
<td>(A_2)</td>
<td>((0.7,0.15))</td>
<td>((0.5,0.35))</td>
<td>((0.15,0.7))</td>
<td>((0.7,0.15))</td>
<td>((0.8,0.1))</td>
</tr>
<tr>
<td>(A_3)</td>
<td>((0.85,0.1))</td>
<td>((0.4,0.4))</td>
<td>((0.4,0.5))</td>
<td>((0.7,0.2))</td>
<td>((0.7,0.1))</td>
</tr>
<tr>
<td>(A_4)</td>
<td>((0.3,0.5))</td>
<td>((0.5,0.4))</td>
<td>((0.8,0.1))</td>
<td>((0.5,0.4))</td>
<td>((0.1,0.8))</td>
</tr>
<tr>
<td>(A_5)</td>
<td>((0.5,0.4))</td>
<td>((0.5,0.4))</td>
<td>((0.6,0.3))</td>
<td>((0.5,0.4))</td>
<td>((0.1))</td>
</tr>
<tr>
<td>(A_6)</td>
<td>((0.7,0.2))</td>
<td>((0.7,0.2))</td>
<td>((0.6,0.3))</td>
<td>((0.8,0.1))</td>
<td>((0.1))</td>
</tr>
</tbody>
</table>

Now, we have established an investment decision making model using investment decision rule by considering a collection of investment decision parameters favored by an investor to distinguish the investment avenue get to that suits best for the requirements of the investor.

**Case 1:** Let us consider an investor Mr. X prefers the investment decision parameters as safety of funds (a) and high returns (b).

Then the problem can be explained using investment decision rule \(F_{LU}(\text{safety of funds}) \times F_{LU}(\text{high returns})\) by considering \(F_{LU}(\text{safety of funds})\) and \(F_{LU}(\text{high returns})\) of the investment avenues, which ensures that the safety of funds (a) and high returns (b) is formed for the investor Mr. X.
Table 34: Use of investment decision rule on safety of funds and high returns

<table>
<thead>
<tr>
<th>U</th>
<th>$F_{LU}(a)$</th>
<th>$F_{LU}(b)$</th>
<th>$F_{LU}(a)*F_{LU}(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.8,0.1)</td>
<td>(0.4,0.4)</td>
<td>(0.32,0.04)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.7,0.15)</td>
<td>(0.5,0.35)</td>
<td>(0.35,0.0525)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.85,0.1)</td>
<td>(0.4,0.4)</td>
<td>(0.34,0.04)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.3,0.5)</td>
<td>(0.5,0.4)</td>
<td>(0.15,0.2)</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(0.5,0.4)</td>
<td>(0.5,0.3)</td>
<td>(0.25,0.12)</td>
</tr>
<tr>
<td>$A_6$</td>
<td>(0.7,0.2)</td>
<td>(0.7,0.2)</td>
<td>(0.49,0.04)</td>
</tr>
</tbody>
</table>

From the Table 34, we see that $A_6$ (gold) has the largest membership value 0.49 and the smallest non membership value 0.04; hence gold is the best suits for the requirement of Mr. X.

In the same way, the selection of an investment avenue by any investor can be accessed at depending on any collection of investment decision parameters favored by such investor. Some illustrations are given as follows:

**Case 2:** If an investor Mr. Y prefers the investment decision parameters as high returns ($b$), maximum profit in minimum period ($c$) and tax concession ($e$), then the problem can be solved using investment decision rule $F_{LU}(b)*F_{LU}(c)*F_{LU}(e)$ as in Table 35.

Table 35: Use of investment decision rule on high returns, maximum profit in minimum period and tax concession

<table>
<thead>
<tr>
<th>U</th>
<th>$F_{LU}(b)$</th>
<th>$F_{LU}(c)$</th>
<th>$F_{LU}(e)$</th>
<th>$F_{LU}(b)*F_{LU}(c)*F_{LU}(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>(0.4,0.4)</td>
<td>(0.4,0.4)</td>
<td>(0.7,0.2)</td>
<td>(0.112,0.032)</td>
</tr>
<tr>
<td>$A_2$</td>
<td>(0.5,0.35)</td>
<td>(0.15,0.7)</td>
<td>(0.8,0.1)</td>
<td>(0.06,0.0245)</td>
</tr>
<tr>
<td>$A_3$</td>
<td>(0.4,0.4)</td>
<td>(0.4,0.5)</td>
<td>(0.7,0.1)</td>
<td>(0.112,0.02)</td>
</tr>
<tr>
<td>$A_4$</td>
<td>(0.5,0.4)</td>
<td>(0.8,0.1)</td>
<td>(0.1,0.8)</td>
<td>(0.04,0.032)</td>
</tr>
<tr>
<td>$A_5$</td>
<td>(0.5,0.3)</td>
<td>(0.6,0.3)</td>
<td>(0.1)</td>
<td>(0.009)</td>
</tr>
<tr>
<td>$A_6$</td>
<td>(0.7,0.2)</td>
<td>(0.6,0.3)</td>
<td>(0.1)</td>
<td>(0.06)</td>
</tr>
</tbody>
</table>

From the Table 35, we see that the $A_1$ (bank deposit) and $A_3$ (postal savings) both has the largest membership value 0.112, but $A_3$ has the smallest non membership value 0.02, hence postal savings is the best suits for the requirement of Mr. Y.

**Case 3:** If an investor, Mr. Z prefers the investment decision parameters as safety of funds ($a$), maximum profit in minimum period ($c$), easy accessibility ($d$) and tax concession ($e$), then the problem can be solved using investment decision rule $F_{LU}(a)*F_{LU}(c)*F_{LU}(d)*F_{LU}(e)$ as in Table 36.
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Table 36: Use of investment decision rule on safety of funds, maximum profit in minimum period, easy accessibility and tax concession

<table>
<thead>
<tr>
<th>U</th>
<th>( F_{LU}(a) )</th>
<th>( F_{LU}(c) )</th>
<th>( F_{LU}(d) )</th>
<th>( F_{LU}(e) )</th>
<th>( \frac{F_{LU}(a) \cdot F_{LU}(c)^{\alpha} \cdot F_{LU}(d) \cdot F_{LU}(e)^{\beta}}{F_{LU}(a) \cdot F_{LU}(c)^{\alpha} \cdot F_{LU}(d) \cdot F_{LU}(e)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>(0.8,0.1)</td>
<td>(0.4,0.4)</td>
<td>(0.8,0.1)</td>
<td>(0.7,0.2)</td>
<td>(0.1792,0.0008)</td>
</tr>
<tr>
<td>A_2</td>
<td>(0.7,0.15)</td>
<td>(0.15,0.7)</td>
<td>(0.7,0.15)</td>
<td>(0.8,0.1)</td>
<td>(0.0588,0.00157)</td>
</tr>
<tr>
<td>A_3</td>
<td>(0.85,0.1)</td>
<td>(0.4,0.5)</td>
<td>(0.7,0.2)</td>
<td>(0.7,0.1)</td>
<td>(0.165,0.001)</td>
</tr>
<tr>
<td>A_4</td>
<td>(0.3,0.5)</td>
<td>(0.8,0.1)</td>
<td>(0.5,0.4)</td>
<td>(0.1,0.8)</td>
<td>(0.092,0.016)</td>
</tr>
<tr>
<td>A_5</td>
<td>(0.5,0.4)</td>
<td>(0.6,0.3)</td>
<td>(0.5,0.4)</td>
<td>(0.1)</td>
<td>(0.048)</td>
</tr>
<tr>
<td>A_6</td>
<td>(0.7,0.2)</td>
<td>(0.6,0.3)</td>
<td>(0.8,0.1)</td>
<td>(0.1)</td>
<td>(0.006)</td>
</tr>
</tbody>
</table>

From the Table 36, we see that the \( A_1 \) (bank deposit) has the largest membership value 0.1792 and smallest non-membership value 0.0008 and hence bank deposit is best suits for the requirement of Mr. Z.

Conclusion

In our real life application, there are large numbers of problems that assurance logical, rational and logical decisions that suit best for the achievement of desirable objective. In this chapter, we have introduced the concept of interval valued intuitionistic fuzzy soft topological spaces together with some basic concepts over a fixed parameter set, which is the extension of fuzzy soft topological spaces introduced by Simsekler and Yuksel [46] as well as intuitionistic fuzzy soft topological spaces introduced by Li and Cui [29]. Also, we have proposed an investment decision making procedure for solving investment problems based on an IVIFS-set, using different decision criteria, different information regarding investment decision making and information of varying attribute. Also, we have presented an algorithm for investment decision making based on an IVIFS-set by choosing certain opinion weighting vectors and try to apply IVIFS-set to investment problems.