In this chapter, the concept of interval valued intuitionistic fuzzy rough relation (IVIFR-relation) is to be introduced as a generalization of the IFR-relation. The notions of reflexive, reflexive of order \([\alpha,\beta]\), symmetric, anti-symmetric, transitive and equivalence IVIFR-relation of order \([\alpha,\beta]\) on a set are to be defined and a few properties of them are to be investigated. Finally, we introduce IVIFR-order relation on a set and then we present an order on the referential set induced by an IVIFR-order relation. Also, we have seen, how this induced order relation justifies the definition of anti-symmetric IVIFR-relation.

4.1. Interval valued intuitionistic fuzzy rough relations

In this section, we introduce interval valued intuitionistic fuzzy rough relations on a set and study their basic properties.

**Definition 4.1.1:** Let \(A=(U,\rho)\) be an approximation space and \(X\) be a rough set in \(A\). An interval valued intuitionistic fuzzy rough relation (IVIFR-relation) on \(X\) denoted by \(\tilde{R}=(M_{\tilde{R}},N_{\tilde{R}})\) is an interval valued intuitionistic fuzzy subset of \(U \times U\), satisfying:

(i) \(M_{\tilde{R}}(x,y)=[1,1] \) and \(N_{\tilde{R}}(x,y)=[0,0]\), \(\forall (x,y) \in X \times X\)

(ii) \(M_{\tilde{R}}(x,y)=[0,0] \) and \(N_{\tilde{R}}(x,y)=[1,1]\), \(\forall (x,y) \in [U \times U \setminus X \times X]\)

(iii) \([0,0] \subseteq M_{\tilde{R}}(x,y), N_{\tilde{R}}(x,y) \subseteq [1,1]\), \(\forall (x,y) \in [\overline{X \times X} \setminus X \times X]\)

where \(X \times X = \{(x,y) \in U \times U : [x,y] \subseteq X \times X]\), \(\overline{X \times X} = \{(x,y) \in U \times U : [x,y] \cap X \times X \neq \emptyset\}\).

We denote \(IVIFR(X)\) be the set of all IVIFR-relations on \(X\).

**Example 4.1.2:** Let \(U=\{a, b, c, d\}\) and \(\rho=\{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}\) be an equivalence relation on \(U\). Also, let \(X = \{a, c\} \subseteq U\), then upper approximation and lower approximation of \(X \times X\) are given by \(\overline{X \times X} = \{(a,a),(b,b),(c,c),(c,a),(a,c),(c,a),(b,c),(c,b)\}\) and \(\underline{X \times X} = \{(c,c)\}\). Then an IVIFR-relation on \(X \subseteq U\) can be represented as in Table 10.

**Table 10: IVIFR-relation \(\tilde{R}\)**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.5],[0.2,0.3])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.4,0.5],[0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>
Definition 4.1.3: If $\bar{R}_1$ and $\bar{R}_2$ be two IVIFR-relations on $X \subseteq U$, then for every $(x, y) \in U \times U$, $\bar{R}_1$ is IVIFR-subrelation of $\bar{R}_2$, i.e. $\bar{R}_1 \subseteq \bar{R}_2$ if and only if $M_{\bar{R}_1} \subseteq M_{\bar{R}_2}$ and $N_{\bar{R}_1} \supseteq N_{\bar{R}_2}$

i.e. $\inf M_{\bar{R}_1}(x, y) \leq \inf M_{\bar{R}_2}(x, y)$, $\sup M_{\bar{R}_1}(x, y) \leq \sup M_{\bar{R}_2}(x, y)$

and $\inf N_{\bar{R}_1}(x, y) \geq \inf N_{\bar{R}_2}(x, y)$, $\sup N_{\bar{R}_1}(x, y) \geq \sup N_{\bar{R}_2}(x, y)$.

Example 4.1.4: If we consider an IVIFR-relation $\bar{R}$ as in Table 10 and an IVIFR-relation $\bar{R}_1$ as in Table 11, then $\bar{R}_1 \subseteq \bar{R}$.

Table 11: IVIFR-relation $\bar{R}_1$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.2,0.3],[0.4,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.3,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.2,0.3],[0.4,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

Definition 4.1.5: If $\bar{R}_1$ and $\bar{R}_2$ be two IVIFR-relations on $X \subseteq U$,

[i]. Union: $\bar{R}_1 \cup \bar{R}_2 \Rightarrow \left( M_{\bar{R}_1} \cup M_{\bar{R}_2}, \bigwedge N_{\bar{R}_1} \cap N_{\bar{R}_2} \right)$

where $\left( M_{\bar{R}_1} \cup M_{\bar{R}_2} \right)(x, y) = \left[ \max \left\{ \inf M_{\bar{R}_1}(x, y), \inf M_{\bar{R}_2}(x, y) \right\}, \max \left\{ \sup M_{\bar{R}_1}(x, y), \sup M_{\bar{R}_2}(x, y) \right\} \right]$

and $\left( \bigwedge N_{\bar{R}_1} \cap N_{\bar{R}_2} \right)(x, y) = \left[ \min \left\{ \inf N_{\bar{R}_1}(x, y), \inf N_{\bar{R}_2}(x, y) \right\}, \min \left\{ \sup N_{\bar{R}_1}(x, y), \sup N_{\bar{R}_2}(x, y) \right\} \right]$.

[ii]. Intersection: $\bar{R}_1 \cap \bar{R}_2 \Rightarrow \left( M_{\bar{R}_1} \cap M_{\bar{R}_2}, \bigvee N_{\bar{R}_1} \cup N_{\bar{R}_2} \right)$

where $\left( M_{\bar{R}_1} \cap M_{\bar{R}_2} \right)(x, y) = \left[ \min \left\{ \inf M_{\bar{R}_1}(x, y), \inf M_{\bar{R}_2}(x, y) \right\}, \min \left\{ \sup M_{\bar{R}_1}(x, y), \sup M_{\bar{R}_2}(x, y) \right\} \right]$.

and $\left( \bigvee N_{\bar{R}_1} \cup N_{\bar{R}_2} \right)(x, y) = \left[ \max \left\{ \inf N_{\bar{R}_1}(x, y), \inf N_{\bar{R}_2}(x, y) \right\}, \max \left\{ \sup N_{\bar{R}_1}(x, y), \sup N_{\bar{R}_2}(x, y) \right\} \right]$.
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[iii]. Algebraic product: $\tilde{R}_1 \cdot \tilde{R}_2 \Rightarrow \left( M_{\tilde{R}_1} \cdot M_{\tilde{R}_2}, N_{\tilde{R}_1} \cdot N_{\tilde{R}_2} \right)$

where $\left( M_{\tilde{R}_1} \cdot M_{\tilde{R}_2} \right)(x, y) = \left[ \inf M_{\tilde{R}_1}(x, y) \cdot \inf M_{\tilde{R}_2}(x, y), \sup M_{\tilde{R}_1}(x, y) \cdot \sup M_{\tilde{R}_2}(x, y) \right]$

and $\left( N_{\tilde{R}_1} \cdot N_{\tilde{R}_2} \right)(x, y) = \left[ \inf N_{\tilde{R}_1}(x, y) \cdot \inf N_{\tilde{R}_2}(x, y), \sup N_{\tilde{R}_1}(x, y) \cdot \sup N_{\tilde{R}_2}(x, y) \right]$

[iv]. Algebraic sum: $\tilde{R}_1 \oplus \tilde{R}_2 \Rightarrow \left( M_{\tilde{R}_1} \oplus M_{\tilde{R}_2}, N_{\tilde{R}_1} \oplus N_{\tilde{R}_2} \right)$,

where $\left( M_{\tilde{R}_1} \oplus M_{\tilde{R}_2} \right)(x, y) = \left[ \inf M_{\tilde{R}_1}(x, y) + \inf M_{\tilde{R}_2}(x, y) - \inf M_{\tilde{R}_1}(x, y) \cdot \inf M_{\tilde{R}_2}(x, y), \right.$

$\left. \sup M_{\tilde{R}_1}(x, y) + \sup M_{\tilde{R}_2}(x, y) - \sup M_{\tilde{R}_1}(x, y) \cdot \sup M_{\tilde{R}_2}(x, y) \right]$

$\left( N_{\tilde{R}_1} \oplus N_{\tilde{R}_2} \right)(x, y) = \left[ \inf N_{\tilde{R}_1}(x, y) + \inf N_{\tilde{R}_2}(x, y), \sup N_{\tilde{R}_1}(x, y) + \sup N_{\tilde{R}_2}(x, y) \right]$

[v]. Arithmetic mean: $\tilde{R}_1 \oplus \tilde{R}_2 \Rightarrow \left( M_{\tilde{R}_1} \oplus M_{\tilde{R}_2}, N_{\tilde{R}_1} \oplus N_{\tilde{R}_2} \right)$,

where $\left( M_{\tilde{R}_1} \oplus M_{\tilde{R}_2} \right)(x, y) = \left[ \frac{1}{2} \left( \inf M_{\tilde{R}_1}(x, y) + \inf M_{\tilde{R}_2}(x, y) \right), \frac{1}{2} \left( \sup M_{\tilde{R}_1}(x, y) + \sup M_{\tilde{R}_2}(x, y) \right) \right]$

and $\left( N_{\tilde{R}_1} \oplus N_{\tilde{R}_2} \right)(x, y) = \left[ \frac{1}{2} \left( \inf N_{\tilde{R}_1}(x, y) + \inf N_{\tilde{R}_2}(x, y) \right), \frac{1}{2} \left( \sup N_{\tilde{R}_1}(x, y) + \sup N_{\tilde{R}_2}(x, y) \right) \right]$

[vi]. Geometric mean: $\tilde{R}_1 \# \tilde{R}_2 \Rightarrow \left( M_{\tilde{R}_1} \# M_{\tilde{R}_2}, N_{\tilde{R}_1} \# N_{\tilde{R}_2} \right)$,

where $\left( M_{\tilde{R}_1} \# M_{\tilde{R}_2} \right)(x, y) = \left[ \sqrt[\inf M_{\tilde{R}_1}(x, y) \cdot \inf M_{\tilde{R}_2}(x, y) \cdot \sqrt[\sup M_{\tilde{R}_1}(x, y) \cdot \sup M_{\tilde{R}_2}(x, y)] \right]$

and $\left( N_{\tilde{R}_1} \# N_{\tilde{R}_2} \right)(x, y) = \left[ \sqrt[\inf N_{\tilde{R}_1}(x, y) \cdot \inf N_{\tilde{R}_2}(x, y) \cdot \sqrt[\sup N_{\tilde{R}_1}(x, y) \cdot \sup N_{\tilde{R}_2}(x, y)] \right]$

[vii]. $\Box \tilde{R} = \left( M_{\tilde{R}}(x, y), \left[ \inf N_{\tilde{R}}(x, y), 1 - \sup M_{\tilde{R}}(x, y) \right] \right)$

[viii]. $\Delta \tilde{R} = \left( \left[ \inf M_{\tilde{R}}(x, y), 1 - \sup N_{\tilde{R}}(x, y) \right], N_{\tilde{R}}(x, y) \right)$
Example 4.1.6: Let us consider an approximation space $A = (U, \rho)$ as in example 4.1.2 and let $X = \{a,c\} \subseteq U$. We consider two IVIFR-relations $\tilde{R}_1$ and $\tilde{R}_2$ as in Table 12 and Table 13. Then their union, intersection and algebraic product can be represented as in Table 14, Table 15 and Table 16.

**Table 12: IVIFR-relation $\tilde{R}_1$**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.5], [0.2,0.3])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([0.4,0.5], [0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([0.3,0.5], [0.2,0.4])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
<td>([1,1], [0,0])</td>
</tr>
</tbody>
</table>

**Table 13: IVIFR-relation $\tilde{R}_2$**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.1,0.3], [0.3,0.6])</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
</tr>
<tr>
<td>b</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.3,0.5], [0.2,0.4])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
</tr>
<tr>
<td>c</td>
<td>([0.3,0.6], [0.3,0.4])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([1,1], [0,0])</td>
</tr>
</tbody>
</table>

**Table 14: IVIFR-relation $\tilde{R}_1 \cup \tilde{R}_2$**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.5], [0.2,0.3])</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.4,0.5], [0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.3,0.5], [0.2,0.4])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5], [0.2,0.5])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
<td>([1,1], [0,0])</td>
</tr>
</tbody>
</table>

**Table 15: IVIFR-relation $\tilde{R}_1 \cap \tilde{R}_2$**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.1,0.3], [0.3,0.6])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([0.3,0.6], [0.3,0.4])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([0.3,0.5], [0.2,0.4])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
</tr>
<tr>
<td>c</td>
<td>([0.3,0.6], [0.3,0.4])</td>
<td>([0.1,0.5], [0.2,0.5])</td>
<td>([1,1], [0,0])</td>
</tr>
</tbody>
</table>

**Table 16: IVIFR-relation $\tilde{R}_1 \bar{\triangle} \tilde{R}_2$**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.03,0.15], [0.06,0.18])</td>
<td>([0.04,0.25], [0.04,0.25])</td>
<td>([0.12,0.3], [0.06,0.2])</td>
</tr>
<tr>
<td>b</td>
<td>([0.04,0.25], [0.04,0.25])</td>
<td>([0.09,0.25], [0.04,0.16])</td>
<td>([0.03,0.3], [0.06,0.2])</td>
</tr>
<tr>
<td>c</td>
<td>([0.12,0.3], [0.06,0.2])</td>
<td>([0.03,0.3], [0.06,0.2])</td>
<td>([1,1], [0,0])</td>
</tr>
</tbody>
</table>
Also, their algebraic sum, arithmetic mean, geometric mean, $\cap \bar{R}_1$ and $\Delta \bar{R}_1$ can be represented in the following tables.

### Table 17: IVIFR-relation $\bar{R}_1 \oplus \bar{R}_2$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.37,0.65],[0.06,0.18])</td>
<td>([0.46,0.75],[0.04,0.25])</td>
<td>([0.58,0.8],[0.06,0.2])</td>
</tr>
<tr>
<td>b</td>
<td>([0.46,0.75],[0.04,0.25])</td>
<td>([0.51,0.75],[0.04,0.16])</td>
<td>([0.37,0.8],[0.06,0.2])</td>
</tr>
<tr>
<td>c</td>
<td>([0.58,0.8],[0.06,0.2])</td>
<td>([0.37,0.8],[0.06,0.2])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Table 18: IVIFR-relation $\bar{R}_1 \otimes \bar{R}_2$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.2,0.4],[0.25,0.45])</td>
<td>([0.25,0.5],[0.2,0.5])</td>
<td>([0.35,0.55],[0.25,0.45])</td>
</tr>
<tr>
<td>b</td>
<td>([0.25,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.2,0.55],[0.25,0.45])</td>
</tr>
<tr>
<td>c</td>
<td>([0.35,0.55],[0.25,0.45])</td>
<td>([0.2,0.55],[0.25,0.45])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Table 19: IVIFR-relation $\bar{R}_1 \# \bar{R}_2$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.17,0.39],[0.24,0.42])</td>
<td>([0.2,0.5],[0.2,0.5])</td>
<td>([0.35,0.55],[0.25,0.45])</td>
</tr>
<tr>
<td>b</td>
<td>([0.2,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.17,0.55],[0.25,0.45])</td>
</tr>
<tr>
<td>c</td>
<td>([0.35,0.55],[0.25,0.45])</td>
<td>([0.17,0.55],[0.25,0.45])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Table 20: IVIFR-relation $\Box \bar{R}_1$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.5],[0.2,0.5])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.4,0.5],[0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Table 21: IVIFR-relation $\Delta \bar{R}_1$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.8],[0.3,0.2])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.4,0.5],[0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.2,0.4])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Proposition 4.1.7:
Let $\bar{R}_1$ and $\bar{R}_2$ be two IVIFR-relations on $X \subseteq U$. Then $\bar{R}_1 \cup \bar{R}_2$ and $\bar{R}_1 \cap \bar{R}_2$ are also IVIFR-relations on $X \subseteq U$. 

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Proof: Let \( R_1 \) and \( R_2 \) be two IVIFR-relations on \( X \subseteq U \) and let
\[ M' = M_{R_1} \cup M_{R_2} \text{ and } N' = N_{R_1} \cap N_{R_2}. \]
Then
\[
M'(x, y) = \max \left\{ \inf M_{R_1}(x, y), \inf M_{R_2}(x, y) \right\}, \max \left\{ \sup M_{R_1}(x, y), \sup M_{R_2}(x, y) \right\}
\]
and
\[
N'(x, y) = \min \left\{ \inf N_{R_1}(x, y), \inf N_{R_2}(x, y) \right\}, \min \left\{ \sup N_{R_1}(x, y), \sup N_{R_2}(x, y) \right\}
\]
Since \( R_1 \) and \( R_2 \) are two IVIFR-relations,
\[
M_{R_1}(x, y) = [1,1] = M_{R_2}(x, y), \quad \forall (x, y) \in X \times X
\]
\[\Rightarrow \] \[\Rightarrow M'(x, y) = [1,1], \quad \forall (x, y) \in X \times X \]
and
\[
N_{R_1}(x, y) = [0,0] = N_{R_2}(x, y), \quad \forall (x, y) \in X \times X
\]
\[\Rightarrow \] \[\Rightarrow N'(x, y) = [0,0], \quad \forall (x, y) \in X \times X \]
Also, since \( R_1 \) and \( R_2 \) are two IVIFR-relations,
\[
M_{R_1}(x, y) = [0,0] = M_{R_2}(x, y), \quad \forall (x, y) \in [U \times U \setminus X \times X]
\]
\[\Rightarrow \] \[\Rightarrow M'(x, y) = [0,0], \quad \forall (x, y) \in [U \times U \setminus X \times X] \]
and
\[
N_{R_1}(x, y) = [1,1] = N_{R_2}(x, y), \quad \forall (x, y) \in [U \times U \setminus X \times X]
\]
\[\Rightarrow \] \[\Rightarrow N'(x, y) = [1,1], \quad \forall (x, y) \in [U \times U \setminus X \times X] \]
Again, since \([0,0] \subseteq M_{\tilde{R}_1}(x,y), M_{\tilde{R}_2}(x,y) \subseteq [1,1], \forall (x,y) \in [X \times X \setminus X \times X] \)

\[\Rightarrow [0,0] \subseteq \left[ \max \left\{ \inf M_{\tilde{R}_1}(x,y), \inf M_{\tilde{R}_2}(x,y) \right\}, \right. \]

\[\left. \quad \max \left\{ \sup M_{\tilde{R}_1}(x,y), \sup M_{\tilde{R}_2}(x,y) \right\} \right] \subseteq [1,1], \]

\[\Rightarrow [0,0] \subseteq M_{\tilde{R}}(x,y) \subseteq [1,1], \forall (x,y) \in [X \times X \setminus X \times X] \]

and \([0,0] \subseteq N_{\tilde{R}_1}(x,y), N_{\tilde{R}_2}(x,y) \subseteq [1,1], \forall (x,y) \in [X \times X \setminus X \times X] \)

\[\Rightarrow [0,0] \subseteq \left[ \min \left\{ \inf N_{\tilde{R}_1}(x,y), \inf N_{\tilde{R}_2}(x,y) \right\}, \right. \]

\[\left. \quad \min \left\{ \sup N_{\tilde{R}_1}(x,y), \sup N_{\tilde{R}_2}(x,y) \right\} \right] \subseteq [1,1], \]

\[\Rightarrow [0,0] \subseteq N_{\tilde{R}}(x,y) \subseteq [1,1], \forall (x,y) \in [X \times X \setminus X \times X] \].

Thus \(\tilde{R}_1 \cup \tilde{R}_2\) is IVIFR-relation on \(X \subseteq U\).

Similarly, we can prove that intersection of two IVIFR-relations on \(X \subseteq U\) is also IVIFR-relation on \(X \subseteq U\).

**Proposition 4.1.8:** Let \(\tilde{R}_1\) and \(\tilde{R}_2\) be two IVIFR-relations on \(X \subseteq U\). Then \(\tilde{R}_1 \cap \tilde{R}_2\) and \(\tilde{R}_1 \oplus \tilde{R}_2\) are also IVIFR-relations on \(X \subseteq U\).

**Proof:** Obvious.

**Proposition 4.1.9:** Let \(\tilde{R}_1\) and \(\tilde{R}_2\) be two IVIFR-relations on \(X \subseteq U\). Then \(\tilde{R}_1 \odot \tilde{R}_2\) and \(\tilde{R}_1 \# \tilde{R}_2\) are also IVIFR-relations on \(X \subseteq U\).

**Proof:** Obvious.

**Proposition 4.1.10:** Let \(\tilde{R}\) be an IVIFR-relation. Then \(\square \tilde{R}\) and \(\triangle \tilde{R}\) are also IVIFR-relations.

**Proof:** Obvious.

### 4.2. Various types of IVIFR-relations

**Definition 4.2.1:** An IVIFR-relation \(\tilde{R}\) on \(X \subseteq U\) is said to be reflexive IVIFR-relation if

\[M_{\tilde{R}}(x,x) = [1,1] \quad \text{and} \quad N_{\tilde{R}}(x,x) = [0,0], \forall x \in U.\]
Definition 4.2.2: An IVIFR-relation $\tilde{R}$ on $X \subseteq U$ is said to be reflexive IVIFR-relation of order $[\alpha, \beta]$ if

$$M_{\tilde{R}}(x, x) \supseteq [\alpha, \beta] \text{ and } N_{\tilde{R}}(x, x) \subseteq [\alpha, \beta], \forall x \in U.$$ 

If $\alpha = \beta$ then the above relation is called reflexive IVIFR-relation of order $\alpha$.

Example 4.2.3: If we consider an IVIFR-relation $\tilde{R}$ as in example 4.1.2, then $\tilde{R}$ is a reflexive IVIFR-relation of order $[0.3, 0.5]$, since $M_{\tilde{R}}(x, x) \supseteq [0.3, 0.5]$ and $N_{\tilde{R}}(x, x) \subseteq [0.3, 0.5], \forall x \in U$.

Definition 4.2.4: An IVIFR-relation $\tilde{R}$ on $X \subseteq U$ is said to be weakly reflexive IVIFR-relation if

$$M_{\tilde{R}}(x, x) \supseteq M_{\tilde{R}}(x, y) \text{ and } N_{\tilde{R}}(x, x) \subseteq N_{\tilde{R}}(x, y), \forall x, y \in U.$$ 

Example 4.2.5: Let us consider an approximation space $A = (U, \rho)$ as in example 4.1.2 and let $X = \{a, c\} \subseteq U$. Then the relation $\tilde{R}$ as in Table 22 is weakly reflexive IVIFR-relation.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$([0.4,0.5],[0.2,0.3])$</td>
<td>$([0.1,0.5],[0.2,0.5])$</td>
<td>$([0.3,0.5],[0.2,0.3])$</td>
</tr>
<tr>
<td>b</td>
<td>$([0.1,0.5],[0.2,0.5])$</td>
<td>$([0.5,0.6],[0.2,0.3])$</td>
<td>$([0.3,0.6],[0.3,0.4])$</td>
</tr>
<tr>
<td>c</td>
<td>$([0.3,0.5],[0.2,0.3])$</td>
<td>$([0.3,0.6],[0.3,0.4])$</td>
<td>$([1,1],[0,0])$</td>
</tr>
</tbody>
</table>

Definition 4.2.6: An IVIFR-relation $\tilde{R}$ on $X \subseteq U$ is said to be w- reflexive IVIFR-relation if $M_{\tilde{R}}(x, x) \supseteq M(x)$ and $N_{\tilde{R}}(x, x) \subseteq N(x), \forall x \in U$, where $M : U \rightarrow \text{INT}[0,1]$ and $N : U \rightarrow \text{INT}[0,1]$ is the pre-defined membership function and non membership function on $U$.

Proposition 4.2.7: Let $\tilde{R}_1$ and $\tilde{R}_2$ be two reflexive IVIFR-relations on $X \subseteq U$. Then $\tilde{R}_1 \cup \tilde{R}_2$ and $\tilde{R}_1 \cap \tilde{R}_2$ are also reflexive IVIFR-relations on $X \subseteq U$.

Proof: Obvious.
Proposition 4.2.8: Let $\tilde{R}_1$ and $\tilde{R}_2$ be two reflexive IVIF-relations on $X \subseteq U$. Then $\tilde{R}_1 \circ \tilde{R}_2$ and $\tilde{R}_1 \oplus \tilde{R}_2$ are also reflexive IVIF-relations on $X \subseteq U$.

Proof: Obvious.

Proposition 4.2.9: Let $\tilde{R}$ be a reflexive IVIF-relation on $X \subseteq U$, then $\cap \tilde{R}$ and $\Delta \tilde{R}$ are also reflexive IVIF-relations on $X \subseteq U$.

Proof: Obvious.

Definition 4.2.10: Let $\tilde{R}_1$ and $\tilde{R}_2$ be two IVIF-relations on $X \subseteq U$. Then the composition of two IVIF-relations is denoted by $\tilde{R}_1 \circ \tilde{R}_2$ and defined by

$$\tilde{R}_1 \circ \tilde{R}_2 = \left( M_{\tilde{R}_1} \circ M_{\tilde{R}_2}, N_{\tilde{R}_1} \cap N_{\tilde{R}_2} \right),$$

where

$$\left( M_{\tilde{R}_1} \circ M_{\tilde{R}_2} \right)(x,y) = \bigcup_{u \in U} \left( M_{\tilde{R}_1}(x,u) \cap M_{\tilde{R}_2}(u,y) \right),$$

$$\left( N_{\tilde{R}_1} \cap N_{\tilde{R}_2} \right)(x,y) = \bigcap_{u \in U} \left( N_{\tilde{R}_1}(x,u) \cup N_{\tilde{R}_2}(u,y) \right), \forall (x,y) \in U \times U.$$

Example 4.2.11: Let us consider an approximation space $A = (U, \rho)$ as in example 4.1.2 and let $X = \{a, c\} \subseteq U$. If we consider two IVIF-relations $\tilde{R}_1$ and $\tilde{R}_2$ as in Table 23 and Table 24, then their composition can be represented as in Table 25.

### Table 23: IVIF-relation $\tilde{R}_1$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.5],[0.2,0.3])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.4,0.5],[0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

### Table 24: IVIF-relation $\tilde{R}_2$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
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<th>c</th>
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<tbody>
<tr>
<td>a</td>
<td>([0.1,0.3],[0.3,0.6])</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
</tr>
<tr>
<td>b</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
</tr>
<tr>
<td>c</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>
Table 25: IVIFR-relation $\bar{R}_1 \Theta \bar{R}_2$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>([0.3,0.6], [0.2,0.5])</td>
<td>([0.3,0.5], [0.2,0.5])</td>
<td>([0.4,0.5], [0.2,0.5])</td>
</tr>
<tr>
<td>b</td>
<td>([0.3,0.6], [0.2,0.4])</td>
<td>([0.3,0.5], [0.2,0.4])</td>
<td>([0.3,0.6], [0.2,0.4])</td>
</tr>
<tr>
<td>c</td>
<td>([0.3,0.6], [0.3,0.4])</td>
<td>([0.4,0.5], [0.2,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

Property:

(i) Commutative: $\bar{R}_1 \Theta \bar{R}_2 = \bar{R}_2 \Theta \bar{R}_1$

(ii) Associative: $\bar{R}_1 \Theta (\bar{R}_2 \Theta \bar{R}_3) = (\bar{R}_1 \Theta \bar{R}_2) \Theta \bar{R}_3$

**Proposition 4.2.12**: Let $\bar{R}_1$ and $\bar{R}_2$ be two IVIFR-relations on $X \subseteq U$. Then $\bar{R}_1 \Theta \bar{R}_2$ is also an IVIFR-relation on $X \subseteq U$.

**Proof**: Let $\bar{R}_1$ and $\bar{R}_2$ be two IVIFR-relations on $X \subseteq U$ and let $M' = M_{\bar{R}_1} \circ M_{\bar{R}_2}$ and $N' = N_{\bar{R}_1} \sqcap N_{\bar{R}_2}$.

Then $M'(x,y) = (M_{\bar{R}_1} \circ M_{\bar{R}_2})(x,y) = \bigcup_{u \in U}(M_{\bar{R}_1}(x,u) \cap M_{\bar{R}_2}(u,y))$

and $N'(x,y) = (N_{\bar{R}_1} \sqcap N_{\bar{R}_2})(x,y) = \bigcap_{u \in U}(N_{\bar{R}_1}(x,u) \cup N_{\bar{R}_2}(u,y))$

Since $\bar{R}_1$ and $\bar{R}_2$ are two IVIFR-relations, therefore $\forall (x,y) \in X \times X$,

$M_{\bar{R}_1}(x,y) = [1,1] = M_{\bar{R}_2}(x,y)$ and $N_{\bar{R}_1}(x,y) = [0,0] = N_{\bar{R}_2}(x,y)$

$\Rightarrow \bigcup_{u \in U}(M_{\bar{R}_1}(x,u) \cap M_{\bar{R}_2}(u,y)) = [1,1]$ and $\bigcap_{u \in U}(N_{\bar{R}_1}(x,u) \cup N_{\bar{R}_2}(u,y)) = [0,0]$

$\Rightarrow M'(x,y) = [1,1]$ and $N'(x,y) = [0,0]$

Also, since $\bar{R}_1$ and $\bar{R}_2$ are two IVIFR-relations, therefore $\forall (x,y) \in [U \times U \setminus X \times X]$,

$M_{\bar{R}_1}(x,y) = [0,0] = M_{\bar{R}_2}(x,y)$ and $N_{\bar{R}_1}(x,y) = [1,1] = N_{\bar{R}_2}(x,y)$

$\Rightarrow \bigcup_{u \in U}(M_{\bar{R}_1}(x,u) \cap M_{\bar{R}_2}(u,y)) = [0,0]$ and $\bigcap_{u \in U}(N_{\bar{R}_1}(x,u) \cup N_{\bar{R}_2}(u,y)) = [1,1]$

$\Rightarrow M'(x,y) = [0,0]$ and $N'(x,y) = [1,1]$,
Again, since $\forall (x, y) \in [X \times X \setminus X \times X]$, 
$[0, 0] \subset M_{R_1}(x, y), M_{R_2}(x, y) \subset [1, 1]$ and 
$[0, 0] \subset N_{R_1}(x, y), N_{R_2}(x, y) \subset [1, 1]$ 
$\Rightarrow [0, 0] \subset \bigcup_{u \in U} \left( M_{R_1}(x, u) \cap M_{R_2}(u, y) \right) \subset [1, 1]$ and 
$[0, 0] \subset \bigcap_{u \in U} \left( N_{R_1}(x, u) \cup N_{R_2}(u, y) \right) \subset [1, 1]$ 
$\Rightarrow [0, 0] \subset M'(x, y) \subset [1, 1]$ and 
$[0, 0] \subset N'(x, y) \subset [1, 1]$ 
Thus $\tilde{R}_1 \Theta \tilde{R}_2$ is also IVIFR-relation on $X \subseteq U$.

**Proposition 4.2.13:** If $\tilde{R}_1$ and $\tilde{R}_2$ be two reflexive IVIFR-relations on 
$X \subseteq U$. Then $\tilde{R}_1 \Theta \tilde{R}_2$ is also reflexive IVIFR-relation on $X \subseteq U$.

**Proof:** Since $\forall x \in U$ 
$\left( M_{\tilde{R}_1} \circ M_{\tilde{R}_2} \right)(x, x) = \bigcup_{u \in U} \left( M_{\tilde{R}_1}(x, u) \cap M_{\tilde{R}_2}(u, x) \right)$
$\quad = \left( M_{\tilde{R}_1}(x, x) \cap M_{\tilde{R}_2}(x, x) \right) \cup \left( \bigcup_{u \neq x} \left( M_{\tilde{R}_1}(x, u) \cap M_{\tilde{R}_2}(u, x) \right) \right)$
$\quad = [1, 1]$, \[as \tilde{R}_1 \text{ and } \tilde{R}_2 \text{ are two reflexive IVIFR-relations}\]
and 
$\left( N_{\tilde{R}_1} \cap N_{\tilde{R}_2} \right)(x, x) = \bigcap_{u \in U} \left( N_{\tilde{R}_1}(x, u) \cup N_{\tilde{R}_2}(u, x) \right)$
$\quad = \left( N_{\tilde{R}_1}(x, x) \cup N_{\tilde{R}_2}(x, x) \right) \cap \left( \bigcap_{u \neq x} \left( N_{\tilde{R}_1}(x, u) \cup N_{\tilde{R}_2}(u, x) \right) \right)$
$\quad = [0, 0]$, \[as \tilde{R}_1 \text{ and } \tilde{R}_2 \text{ are two reflexive IVIFR-relations}\]
Thus $\tilde{R}_1 \Theta \tilde{R}_2$ is reflexive IVIFR-relation on $X \subseteq U$.

**Definition 4.2.14:** An IVIFR-relation $\tilde{R}$ on $X \subseteq U$ is said to be symmetric 
IVIFR-relation if $\forall x, y \in U$, $M_{\tilde{R}}(x, y) = M_{\tilde{R}}(y, x)$ and 
$N_{\tilde{R}}(x, y) = N_{\tilde{R}}(y, x)$.

**Example 4.2.15:** Let us consider an approximation space $A=(U, \rho)$ as in 
example 4.1.2 and let $X=\{a,c\} \subseteq U$. Then the relation $\tilde{R}_1$ as in Table 26 is a 
symmetric IVIFR-relation.
Intuitionistic fuzzy rough relations and its generalization, topological structures formed by soft multisets and interval valued intuitionistic fuzzy soft sets

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IVIFR-relations & different types of operations …

Table 26: Symmetric IVIFR-relation $\tilde{R}$

<table>
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<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(0.3,0.5),(0.2,0.3)$</td>
<td>$(0.1,0.5),(0.2,0.5)$</td>
<td>$(0.4,0.5),(0.2,0.5)$</td>
</tr>
<tr>
<td>b</td>
<td>$(0.1,0.5),(0.2,0.5)$</td>
<td>$(0.3,0.5),(0.2,0.4)$</td>
<td>$(0.3,0.6),(0.3,0.4)$</td>
</tr>
<tr>
<td>c</td>
<td>$(0.4,0.5),(0.2,0.5)$</td>
<td>$(0.3,0.6),(0.3,0.4)$</td>
<td>$(1,1),(0,0)$</td>
</tr>
</tbody>
</table>

Definition 4.2.16: An IVIFR-relation $\tilde{R}$ on $X \subseteq U$ is said to be antisymmetric IVIFR-relation if $\forall x, y \in U$, $M_{\tilde{R}}(x,y) = M_{\tilde{R}}(y,x)$ and $V_{\tilde{R}}(x,y) = V_{\tilde{R}}(y,x) \Rightarrow x = y$.

Example 4.2.17: Let us consider an approximation space $A = (U, \rho)$, where $U = \{a, b, c\}$ and $\rho$ be the universal equivalence relation on $U$. Let $X = \{a,b,c\} \subseteq U$, then the upper approximation of $X \times X$ is given by $X \times X = U \times U$ and the lower approximation is $X \times X = \emptyset$. Then the IVIFR-relation $\tilde{R}$ as in Table 27 is anti-symmetric IVIFR-relation.

Table 27: Anti-symmetric IVIFR-relation $\tilde{R}$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(0.1,0.5),(0.1,0.2)$</td>
<td>$(0.0,0.1),(0.2,0.3)$</td>
<td>$(0.2,0.4),(0.4,0.7)$</td>
</tr>
<tr>
<td>b</td>
<td>$(0.2,0.3),(0.6,0.7)$</td>
<td>$(0.2,0.3),(0.4,0.6)$</td>
<td>$(0.2,0.5),(0.3,0.5)$</td>
</tr>
<tr>
<td>c</td>
<td>$(0.1,0.3),(0.1,0.5)$</td>
<td>$(0.2,0.4),(0.4,0.6)$</td>
<td>$(0.3,0.5),(0.2,0.4)$</td>
</tr>
</tbody>
</table>

Definition 4.2.18: An IVIFR-relation $\tilde{R}$ on $X$ is said to be transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.

Example 4.2.19: Let us consider an approximation space $A = (U, \rho)$ as in example 4.1.2 and let $X = \{a,b,c\} \subseteq U$. Then the relation $\tilde{R}$ as in Table 28 is a transitive IVIFR-relation.

Table 28: Transitive IVIFR-relation $\tilde{R}$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$(0.3,0.4),(0.1,0.2)$</td>
<td>$(0.2,0.3),(0.2,0.3)$</td>
<td>$(0.0,0.1),(0.8,0.9)$</td>
</tr>
<tr>
<td>b</td>
<td>$(0.7,0.8),(0.0,0.1)$</td>
<td>$(0.3,0.4),(0.0,0.1)$</td>
<td>$(0.0,0.1),(0.8,0.9)$</td>
</tr>
<tr>
<td>c</td>
<td>$(0.0,0.1),(0.8,0.9)$</td>
<td>$(0.0,0.1),(0.8,0.9)$</td>
<td>$(1,1),(0,0)$</td>
</tr>
</tbody>
</table>
Chapter - 4

IVIFR-relations & different types of operations...

**Proposition 4.2.20:** If $\bar{R}$ be a symmetric and transitive IVIFR-relation on $X \subseteq U$, then $\bar{R}$ is weakly reflexive IVIFR-relation.

**Proof:** Since $\bar{R}$ be a symmetric IVIFR relation
\[ M_{\bar{R}}(x,y) = M_{\bar{R}}(y,x), \quad \forall x, y \in U. \]
Again, since $\bar{R}$ be a transitive IVIFR relation
\[ M_{\bar{R}}(x,y) \supseteq \bigcup_{u \in U} \left( M_{\bar{R}}(x,u) \cap M_{\bar{R}}(u,y) \right) \]
\[ N_{\bar{R}}(x,y) \subseteq \bigcap_{u \in U} \left( N_{\bar{R}}(x,u) \cup N_{\bar{R}}(u,y) \right), \quad \forall x, y \in U. \]
Taking $y = x$, we get
\[ M_{\bar{R}}(x,x) \supseteq \bigcup_{u \in U} \left( M_{\bar{R}}(x,u) \cap M_{\bar{R}}(u,x) \right) = \bigcup_{u \in U} \left( M_{\bar{R}}(x,u) \right) \supseteq M_{\bar{R}}(x,x) \]
and
\[ N_{\bar{R}}(x,x) \subseteq \bigcap_{u \in U} \left( N_{\bar{R}}(x,u) \cup N_{\bar{R}}(u,x) \right) = \bigcap_{u \in U} \left( N_{\bar{R}}(x,u) \right) \subseteq N_{\bar{R}}(x,x), \quad \forall x, y \in U \]

**Proposition 4.2.21:** If $\bar{R}$ be a w-reflexive IVIFR-relation on $X \subseteq U$, then $\bar{R} \subseteq \bar{R} \circ \bar{R}$.

**Proof:** We see that $\forall x, y \in U$,
\[ (M_{\bar{R}} \circ M_{\bar{R}})(x,y) = \bigcup_{u \in U} \left( M_{\bar{R}}(x,u) \cap M_{\bar{R}}(u,y) \right) \]
\[ \supseteq M_{\bar{R}}(x,x) \cap M_{\bar{R}}(x,y) \]
\[ \supseteq M(x) \cap M_{\bar{R}}(x,y), \quad [\text{as } \bar{R} \text{ is w-reflexive}] \]
Again,
\[ M_{\bar{R}}(x,y) \subseteq M(x) \cap M(y) \subseteq M(x) \]
So $M_{\bar{R}}(x,y) \subseteq (M_{\bar{R}} \circ M_{\bar{R}})(x,y), \quad \forall x, y \in U$
Therefore, $M_{\bar{R}} \subseteq M_{\bar{R}} \circ M_{\bar{R}}$.

Also
\[ (N_{\bar{R}} \circ N_{\bar{R}})(x,y) = \bigcap_{u \in U} \left( N_{\bar{R}}(x,u) \cup N_{\bar{R}}(u,y) \right) \]
\[ \subseteq N_{\bar{R}}(x,x) \cup N_{\bar{R}}(x,y) \]
\[ \subseteq N(x) \cup N_{\bar{R}}(x,y), \quad [\text{as } \bar{R} \text{ is w-reflexive}] \]
Again
\[ N_R(x, y) \supseteq N(x) \cup N(y) \supseteq N(x), \]
so \( (N_R \cap N_R) (x, y) \subseteq N_R (x, y), \ \forall x, y \in U. \)

Therefore, \( N_R \cap N_R \subseteq N_R. \)

Thus \( \bar{R} \subseteq \bar{R} \cup \bar{R} \)

**Proposition 4.2.22:** If \( \bar{R}_1 \) and \( \bar{R}_2 \) be two be w-reflexive IVIFR-relations on \( X \subseteq U \), then \( \bar{R}_1 \cup \bar{R}_2 \subseteq \bar{R}_1 \Theta \bar{R}_2. \)

**Proof:** Since \( \forall x, y \in U, \)
\[
\left( M_{\bar{R}_1} 0 M_{\bar{R}_2} \right) (x, y) = \bigcup_{u \in U} \left( M_{\bar{R}_1} (x, u) \cap M_{\bar{R}_2} (u, y) \right)
\supseteq M_{\bar{R}_1} (x, x) \cap M_{\bar{R}_2} (x, y)
\supseteq M (x) \cap M_{\bar{R}_1} (x, y), \ \forall x, y \in U, \quad [as \ \bar{R}_1 \ is \ w-reflexive].
\]

Again,
\[
M_{\bar{R}_2} (x, y) \subseteq M (x) \cap M (y) \subseteq M (x)
\]
So \( \left( M_{\bar{R}_1} 0 M_{\bar{R}_2} \right) (x, y) \supseteq M_{\bar{R}_1} (x, y), \ \forall x, y \in U \)

Therefore, \( M_{\bar{R}_2} \subseteq M_{\bar{R}_1} \circ M_{\bar{R}_2}. \)

Similarly, we can show that \( M_{\bar{R}_1} \subseteq M_{\bar{R}_1} \circ M_{\bar{R}_2}. \)

Hence \( M_{\bar{R}_1} \cup M_{\bar{R}_2} \subseteq M_{\bar{R}_1} \circ M_{\bar{R}_2}. \)

Also, it can be proved that \( N_{\bar{R}_1} \cap N_{\bar{R}_2} \supseteq N_{\bar{R}_1} \cap N_{\bar{R}_2}. \)

Thus \( \bar{R}_1 \cup \bar{R}_2 \subseteq \bar{R}_1 \Theta \bar{R}_2. \)

**Definition 4.2.23:** An IVIFR-relation \( \bar{R} \) on \( X \subseteq U \) is said to be an equivalence IVIFR-relation if it is reflexive, symmetric and transitive IVIFR-relation.

**Definition 4.2.24:** An IVIFR-relation \( \bar{R} \) on \( X \subseteq U \) is said to be an equivalence IVIFR-relation of order \([\alpha, \beta]\) if it is reflexive of order \([\alpha, \beta]\), symmetric and transitive IVIFR-relation.

**Proposition 4.2.25:** Let \( X \) is an IVIFR-set of universe \( U \), where \( M \) is the membership function and \( N \) be a non-membership function on \( U \). Also, let \( \bar{R} \) be an equivalence IVIFR-relation of order \([\alpha, \beta]\). Then for each \( x \in U \), \( \exists \) an IVIFR-subset \( \left( M_{\bar{R}}, N_{\bar{R}} \right) \) of \( X \) determined by membership function \( M_{\bar{R}} \) and non-membership function \( N_{\bar{R}} \), satisfying the following:
Chapter 4

IVIFR-relations & different types of operations ...

(i) \( M_{R_x}(x) \supseteq [\alpha, \beta] \) and \( N_{R_x}(x) \subseteq [\alpha, \beta] \)

(ii) \( M_{R_x}(y) = M_{R_x}(x) \) and \( N_{R_x}(y) = N_{R_y}(x) \)

(iii) \( M_{R_y}(y) \supseteq [\alpha_i, \alpha_2], \quad M_{R_y}(z) \supseteq [\alpha_i, \alpha_2] \Rightarrow \mu_{R_x}(z) \supseteq [\alpha_i, \alpha_2] \)

\( \text{and} \quad N_{R_y}(y) \subseteq [\alpha_i, \alpha_2], \quad N_{R_y}(z) \subseteq [\alpha_i, \alpha_2] \Rightarrow N_{R_x}(z) \subseteq [\alpha_i, \alpha_2] \)

(iv) \( M_{R_x}(y) = [0, 0] \) and \( N_{R_x}(y) = [1, 1] \Rightarrow (M_{R_x}, N_{R_x}) \cap (M_{R_y}, N_{R_y}) = \emptyset \)

Proof: We see that \( \forall x, y \in U, \)

\( M_{R}(x, y) \subseteq M(x) \cap M(y) \) and \( N_{R}(x, y) \supseteq N(x) \cup N(y). \)

Now for each \( x \in U, \) we defined

\( M_{R_x}(y) = M_{R}(x, y) \) and \( N_{R_x}(y) = N_{R}(x, y), \quad \forall x, y \in U. \n\)

We note that

\( M_{R_x}(y) = M_{R}(x, y) \subseteq M(x) \cap M(y) \)

\( \text{and} \quad N_{R_x}(y) = N_{R}(x, y) \supseteq N(x) \cup N(y), \quad \forall x, y \in U. \n\)

Thus the IVIFR-set determined by \( (M_{R_x}, N_{R_x}) \) is an IVIFR-subset of X.

(i) \( M_{R_x}(x) = M_{R}(x, x) \supseteq [\alpha, \beta] \) and \( N_{R_x}(x) = N_{R}(x, x) \subseteq [\alpha, \beta] \),

[since \( \bar{R} \) is a reflexive IVIFR-relation of order \( [\alpha, \beta] \)]

(ii) \( M_{R_y}(y) = M_{R}(y, x) = M_{R_y}(y, x) = M_{R_y}(x) \)

and \( N_{R_y}(y) = N_{R}(y, x) = N_{R_y}(y, x) = N_{R_y}(x), \)

[as \( \bar{R} \) is a symmetric IVIFR-relation].

(iii) Let us consider,

\( M_{R_x}(y) \supseteq [\alpha_i, \alpha_2], \quad M_{R_y}(z) \supseteq [\alpha_i, \alpha_2] \)

\( \Rightarrow M_{R}(x, y), \quad M_{R_y}(y, z) \supseteq [\alpha_i, \alpha_2] \)

\( \Rightarrow M_{R}(x, y) \cap M_{R_y}(y, z) \supseteq [\alpha_i, \alpha_2] \).............(1)

By transitivity of \( \bar{R} \)

\( M_{R}(x, z) \supseteq \bigcup_{u \in U} \left( M_{R}(x, u) \cap M_{R}(u, z) \right) \)

\( \supseteq M_{R}(x, u) \cap M_{R}(u, z) \supseteq [\alpha_i, \alpha_2], \) [using (1)]

\( \Rightarrow M_{R_x}(z) = M_{R}(x, z) \supseteq [\alpha_i, \alpha_2] \)
Similarly, we can be proved that
\[ N_{R_x}(y) \subseteq [\alpha_1, \alpha_2], \ N_{R_y}(z) \subseteq [\alpha_1, \alpha_2] \Rightarrow N_{R_x}(z) \subseteq [\alpha_1, \alpha_2]. \]

(iv) \( M_{R_x}(y) = [0, 0] \) and \( N_{R_x}(y) = [1, 1] \).

We need to show that \( (M_{R_x}, N_{R_x}) \cap (M_{R_y}, N_{R_y}) = \emptyset, \)
\[ i.e. \ (M_{R_x} \cap M_{R_y})(u) = [0, 0] \] and \( (N_{R_x} \cup N_{R_y})(u) = [1, 1], \ \forall u \in U. \)

If possible, let \( z \in U \), such that
\[ (M_{R_x} \cap M_{R_y})(z) \supseteq [0, 0] \]
\[ \Rightarrow M_{R_x}(z) \cap M_{R_y}(z) \supseteq [0, 0] \]
\[ \Rightarrow M_{R_x}(z), M_{R_y}(z) \supseteq [0, 0] \]
\[ \Rightarrow M_{R_x}(z), M_{R_y}(y) \supseteq [0, 0], \ [\text{using (ii)}] \]
\[ \Rightarrow M_{R_y}(y) \supseteq [0, 0], \ [\text{using (iii)}] \]

This contradicts \( M_{R_x}(y) = [0, 0] \), thus \( (M_{R_x} \cap M_{R_y})(u) = [0, 0], \ \forall u \in U. \)

Also, if possible, let \( t \in U \), such that
\[ (N_{R_x} \cup N_{R_y})(t) \subseteq [1, 1] \]
\[ \Rightarrow N_{R_x}(t) \cup N_{R_y}(t) \subseteq [1, 1] \]
\[ \Rightarrow N_{R_x}(t), N_{R_y}(t) \subseteq [1, 1] \]
\[ \Rightarrow N_{R_x}(t), N_{R_y}(y) \subseteq [1, 1], \ [\text{using (ii)}] \]
\[ \Rightarrow N_{R_y}(y) \subseteq [1, 1], \ [\text{using (iii)}] \]

This contradicts \( N_{R_x}(y) = [1, 1] \), thus \( (N_{R_x} \cup N_{R_y})(t) = [1, 1] \). Hence
\[ (M_{R_x}, N_{R_x}) \cap (M_{R_y}, N_{R_y}) = \emptyset. \]

**Definition 4.2.26:** An IVIFR-relation \( \tilde{R} \) on \( X \subseteq U \) is said to be an IVIFR-order relation if it is reflexive, anti-symmetric and transitive IVIFR-relation.

**Definition 4.2.27:** An IVIFR-relation \( \tilde{R} \) on \( X \subseteq U \) is said to be an IVIFR-order relation of order \( [\alpha, \beta] \) if it is reflexive of order \( [\alpha, \beta] \), anti-symmetric and transitive IVIFR-relation.
4.3. Order on referential set induced by an IVIF-relations

IVIF-relations can induce different relations in the universal set U. Now we are going to study one of them.

**Definition 4.3.1:** Let \( \tilde{R} \) be an IVIF-relations on \( X \subseteq U \), we define a relation \( \leq \) on \( X \) by \( x \leq y \iff M_{\tilde{R}}(y,x) \subseteq M_{\tilde{R}}(x,y) \) and \( N_{\tilde{R}}(y,x) \supseteq N_{\tilde{R}}(x,y) \), \( \forall x, y \in U \).

**Example 4.3.2:** Let us consider an approximation space \( A=(U, \rho) \) as in example 4.1.2 and let \( X=\{a,c\} \subseteq U \). We consider an IVIF-relations \( \tilde{R} \) as in Table 29.

<table>
<thead>
<tr>
<th></th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>([0.3,0.5],[0.2,0.3])</td>
<td>([0.3,0.5],[0.2,0.3])</td>
<td>([0.1,0.3],[0.4,0.6])</td>
</tr>
<tr>
<td>( b )</td>
<td>([0.1,0.5],[0.2,0.5])</td>
<td>([0.3,0.5],[0.2,0.4])</td>
<td>([0.4,0.6],[0.1,0.3])</td>
</tr>
<tr>
<td>( c )</td>
<td>([0.4,0.5],[0.2,0.5])</td>
<td>([0.3,0.6],[0.3,0.4])</td>
<td>([1,1],[0,0])</td>
</tr>
</tbody>
</table>

Then here
\( a \leq \tilde{R} b \), since \( M_{\tilde{R}}(b,a) \subseteq M_{\tilde{R}}(a,b) \) and \( N_{\tilde{R}}(b,a) \supseteq N_{\tilde{R}}(a,b) \),
\( b \leq \tilde{R} c \), since \( M_{\tilde{R}}(c,b) \subseteq M_{\tilde{R}}(b,c) \) and \( N_{\tilde{R}}(c,b) \supseteq N_{\tilde{R}}(b,c) \),
\( c \leq \tilde{R} a \), since \( M_{\tilde{R}}(a,c) \subseteq M_{\tilde{R}}(c,a) \) and \( N_{\tilde{R}}(a,c) \supseteq N_{\tilde{R}}(c,a) \).

**Remark 4.3.3:** From the above example 4.3.2, we say that \( a \leq \tilde{R} b \) and \( b \leq \tilde{R} c \) but it does not imply \( a \leq \tilde{R} c \), i.e. transitive property does not hold with respect to this relation \( \leq \).

**Proposition 4.3.4:** Let \( \tilde{R} \) be an IVIF-order relation on \( X \subseteq U \), then \( \leq \) is of ordinary order relation in \( X \).

**Proof:** (i) \( \leq \) is reflexive:
Since \( M_{\tilde{R}}(x,x) \subseteq M_{\tilde{R}}(x,x) \) and \( N_{\tilde{R}}(x,x) \supseteq N_{\tilde{R}}(x,x) \Rightarrow \leq \) is reflexive.

(ii) \( \leq \) is anti—symmetric: If \( x \leq \tilde{R} y \) and \( y \leq \tilde{R} x \), then \( M_{\tilde{R}}(y,x) = M_{\tilde{R}}(x,y) \) and \( N_{\tilde{R}}(y,x) = N_{\tilde{R}}(x,y) \), which implies that \( x=y \) [since \( \tilde{R} \) is anti-symmetric].
(iii) ≤ᵣ is transitive: Let \( x ≤ᵣ y \) with \( x ≠ y \), \( y ≤ᵣ z \) and \( y ≠ z \).

This implies \( Mᵣ (y, x) \subseteq Mᵣ (x, y) \), \( Nᵣ (y, x) \supseteq Nᵣ (x, y) \),

and \( Mᵣ (z, y) \subseteq Mᵣ (y, z) \), \( Nᵣ (z, y) \supseteq Nᵣ (y, z) \).

At first we have to proved that they cannot occur at the same time \( Mᵣ (z, x) \supseteq Mᵣ (x, y) \) and \( Mᵣ (z, x) \supseteq Mᵣ (y, z) \).

If possible, let they occur at the same time
\[
Mᵣ (x, y) \subseteq Mᵣ (z, x) \cap Mᵣ (x, y)
\]
\[
\subseteq \bigcup_u \left( Mᵣ (z, u) \cap Mᵣ (u, y) \right) = Mᵣ (z, y).
\]
\[
Mᵣ (y, z) = Mᵣ (y, z) \cap Mᵣ (z, x)
\]
\[
\subseteq \bigcup_u \left( Mᵣ (y, u) \cap Mᵣ (u, x) \right) = Mᵣ (y, x).
\]

Therefore \( Mᵣ (y, x) \subseteq Mᵣ (x, y) \subseteq Mᵣ (z, y) \subseteq Mᵣ (y, x) \subseteq Mᵣ (y, x) \subseteq Mᵣ (z, y) \).

Therefore \( Mᵣ (y, x) = Mᵣ (x, y) = Mᵣ (z, y) = Mᵣ (y, z) = Mᵣ (y, x) = Mᵣ (z, y) \).

That is to say \( Mᵣ (y, x) = Mᵣ (x, y) = Mᵣ (z, y) = Mᵣ (y, z) \) and as \( \tilde{R} \) is anti-symmetric IVIFR-relation, we get \( x = y \) and \( y = z \) in opposition to the hypothesis, from where it is deduced that only one of the following possibilities can occur:

(a) \( Mᵣ (z, x) \subset Mᵣ (x, y) \) or (b) \( Mᵣ (z, x) \subset Mᵣ (y, z) \).

From (a) it is deduced that
\[
Mᵣ (z, x) \subseteq Mᵣ (z, x) \cap Mᵣ (x, y)
\]
\[
\subseteq \bigcup_u \left( Mᵣ (z, u) \cap Mᵣ (u, y) \right) \subseteq Mᵣ (y, z).
\]

So
\[
Mᵣ (z, x) \subseteq Mᵣ (x, y) \cap Mᵣ (y, z)
\]
\[
\subseteq \bigcup_u \left( Mᵣ (x, u) \cap Mᵣ (u, z) \right) \subseteq Mᵣ (x, z).
\]

From (b) it is deduced that
$M_R(z, x) \subseteq M_R(y, z) \cap M_R(z, x) \subseteq \bigcup_u (M_R(y, u) \cap M_R(u, x)) \subseteq M_R(y, x) \subseteq M_R(x, y)$.

Therefore

$M_R(z, x) \subseteq M_R(x, y) \cap M_R(y, z) \subseteq \bigcup_u (M_R(x, u) \cap M_R(u, z)) \subseteq M_R(x, z)$.

Similarly, it can be proved that $N_R(z, x) \supseteq N_R(x, z)$. This implies that $x \leq_R z$.

Thus $\leq_R$ is ordinary order relation in $X$.

**Conclusion**

As a generalization of interval-valued intuitionistic fuzzy set theory [7] makes descriptions of the objective world more realistic, practical and accurate in some cases, making it very promising. In this chapter, the concept of IVIFR-relation is to be introduced as a generalization of the IFR-relation. The notions of reflexive, reflexive of order $[\alpha, \beta]$, symmetric, anti-symmetric, transitive and equivalence IVIFR-relation of order $[\alpha, \beta]$ on a set are to be defined and a few properties of them are to be investigated. Finally, we introduced an order relation on the referential set induced by an IVIFR-order relation and some basic properties of this concept are studied. Also, we have seen how this induced order relation justifies the definition of anti-symmetric IVIFR-relation.