Chapter 3

On Some Generalized Geometric Difference Sequence Spaces

3.1 Introduction

After introducing geometric sequence spaces in Chapter 2, here we find the generalized geometric difference sequence spaces and deduce their dual spaces. Also we discuss their properties.

Let $\ell_\infty$, $c$ and $c_0$ be the Banach spaces of bounded, convergent and null sequences, respectively, normed by $||x||_\infty = \sup_k |x_k|$. It is easy to prove that

$$\omega(G) = \{(x_k) : x_k \in \mathbb{R}(G) \text{ for all } k \in \mathbb{N}\}$$

is a vector space over $\mathbb{R}(G)$ with respect to the algebraic operations $\oplus$ addition and $\odot$ multiplication

$$\oplus : \omega(G) \times \omega(G) \to \omega(G)$$

$$(x, y) \to x \oplus y = (x_k) \oplus (y_k) = (x_k y_k)$$

$$\odot : \mathbb{R}(G) \times \omega(G) \to \omega(G)$$

$$(\alpha, y) \to \alpha \odot y = \alpha \odot (y_k) = (\alpha \ln y_k),$$

where $x = (x_k), y = (y_k) \in \omega(G)$ and $\alpha \in \mathbb{R}(G)$. Then

$$\ell_\infty(G) = \{x = (x_k) \in \omega(G) : \sup_{k \in \mathbb{N}} |x_k|^G < \infty\}$$

$$c(G) = \{x = (x_k) \in \omega(G) : \text{Glim}_{k \to \infty} |x_k \ominus l|^G = 1\}$$
\[ c_0(G) = \{ x = (x_k) \in \omega(G) : G\lim_{k \to \infty} x_k = 1 \} \]

\[ \ell_p(G) = \{ x = (x_k) \in \omega(G) : \sum_{k=0}^{\infty} (|x_k|^G)^p < \infty \}, \text{ where } \sum_G \text{ is the geometric sum.} \]

are classical sequence spaces over the field \( \mathbb{R}(G) \). Also they have shown that \( \ell_\infty(G), c(G) \) and \( c_0(G) \) are Banach spaces with the norm

\[ ||x||_G = \sup_k |x_k|^G, x = (x_1, x_2, x_3, \ldots) \in \lambda(c), \lambda \in \{ \ell_\infty, c, c_0 \}. \]

Here, \( G\lim \) is the \( G \)-limit. For the convenience, we denote \( \ell_\infty(G), c(G), c_0(G) \), respectively as \( \ell_G^{\infty}, c^G, c_0^G \).

In 1981, Kizmaz [40] introduced the notion of difference sequence spaces using forward difference operator \( \Delta \) and studied the classical difference sequence spaces \( \ell_\infty(\Delta), c(\Delta), c_0(\Delta) \). Following Türkmen and Başar [56], Kizmaz [40], we defined geometric sequence space in Chapter 2 as follows:

\[ \ell_G^{\infty}(\Delta_G) = \{ x = (x_k) \in \omega(G) : \Delta_G x \in \ell_G^{\infty} \}, \text{ where } \Delta_G x = x_k \ominus x_{k+1}. \]

where \( \Delta_G x = (\Delta_G x_k) = (x_k \ominus x_{k+1}) \). Then we introduced some theorems, definitions and basic results as follows:

**Theorem 3.1.0.1.** The space \( \ell_G^{\infty}(\Delta_G) \) is a normed linear space w.r.t. the norm

\[ ||x||_{\Delta_G}^G = |x_1|^G \oplus ||\Delta_G x||_{\ell_G^{\infty}}^G. \tag{3.1.1} \]

**Theorem 3.1.0.2.** The space \( \ell_G^{\infty}(\Delta_G) \) is a Banach space w.r.t. the norm \( ||.||_{\Delta_G}^G \).

**Remark 3.1.0.1.** The spaces

(a) \( c^G(\Delta_G) = \{ (x_k) \in \omega(G) : \Delta_G x_k \in c^G \} \)

(b) \( c_0^G(\Delta_G) = \{ (x_k) \in \omega(G) : \Delta_G x_k \in c_0^G \} \)

are Banach spaces with respect to the norm \( ||.||_{\Delta_G}^G \). Also these spaces are BK-spaces.

**Lemma 3.1.0.3.** The following statements are equivalent:

(a) \( \sup_k |x_k \ominus x_{k+1}|^G < \infty \) i.e. \( \sup_k |\Delta_G x_k|^G < \infty \);

(b)(i) \( \sup_k e^{k^{-1}} \ominus |x_k|^G < \infty \) and

\[ (ii) \sup_k |x_k \ominus e^{k(k+1)} \ominus x_{k+1}|^G < \infty. \]
Lemma 3.1.0.4. If $\sup_n \left| \sum_{v=1}^{n} c_v \right|_G^G \leq \infty$ then $\sup_n \left( p_n \odot \sum_{k=1}^{n} \left( \frac{c_{n+k-1}}{p_{n+k}} \right) \right) < \infty$.

Corollary 3.1.0.5. Let $(p_n)$ be monotonically increasing. If $\sup_n \left| \sum_{v=1}^{n} p_v \odot a_v \right|_G^G < \infty$ then

$$\sup_n \left| p_n \odot \sum_{k=n+1}^{\infty} a_k \right|_G^G < \infty.$$

Corollary 3.1.0.6. If $\sum_{k=1}^{\infty} p_k \odot a_k$ is convergent then $\lim_n p_n \odot \sum_{k=n+1}^{\infty} a_k = 1$.

Corollary 3.1.0.7. $\sum_{k=1}^{\infty} e^k \odot a_k$ is convergent if and only if $\sum_{k=1}^{\infty} R_k$ is convergent with $e^n \odot R_n = O(e)$, where $R_n = \sum_{k=n+1}^{\infty} a_k$.

Definition 3.1.0.1. If $X$ is a sequence space, it is defined

(i) $X^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X \}$;

(ii) $X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| \text{ is convergent, for each } x \in X \}$;

(iii) $X^\gamma = \{ a = (a_k) : \sup_n \left| \sum_{k=1}^{n} a_k x_k \right| < \infty, \text{ for each } x \in X \}$.

Then $X^\alpha, X^\beta$ and $X^\gamma$ are called $\alpha-$dual (or Köthe-Toeplitz dual), $\beta-$dual (or generalized Köthe-Toeplitz dual) and $\gamma-$dual spaces of $X$, respectively. Then $X^\alpha \subset X^\beta \subset X^\gamma$. If $X \subset Y$, then $Y^\dual \subset X^\dual$, for $\dual = \alpha, \beta$ or $\gamma$. It is clear that $X \subset (X^\alpha)^\alpha = X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$ then $X$ is called $\alpha$-space. $\alpha-$space is also called a Köthe space or a perfect sequence space.

Theorem 3.1.0.8.

(i) If $D_1 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} e^k \odot |a_k|^G < \infty \right\}$ then $(s\ell_{\infty}(\Delta_G))^\alpha = D_1$.

(ii) If $D_2 = \left\{ a = (a_k) : \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent with } \sum_{k=1}^{\infty} |R_k|^G < \infty \right\}$.

Then $(s\ell_{\infty}(\Delta_G))^\beta = D_2$.

(iii) If $D_3 = \left\{ a = (a_k) : \sup_n \left| \sum_{k=1}^{n} e^k \odot a_k \right|^G < \infty, \sum_{k=1}^{\infty} |R_k|^G < \infty \right\}$.

Then $(s\ell_{\infty}(\Delta_G))^\gamma = D_3$.
3.2 Main Results

3.2.1 Generalized geometric difference sequence spaces

Following Çolak and Et [17], Et and Çolak [20] and Chapter 2, now we define the following new sequence spaces

\[ \ell^G_\infty(\Delta^m_G) = \{ x = (x_k) : \Delta^m_G x \in \ell^G_\infty \}, \]

\[ c^G(\Delta^m_G) = \{ x = (x_k) : \Delta^m_G x \in c^G \}, \]

\[ \ell^G_0(\Delta^m_G) = \{ x = (x_k) : \Delta^m_G x \in c^G_0 \}, \]

where \( m \in \mathbb{N} \) and

\[ \Delta^0_G x = (x_k) \]

\[ \Delta_G x = (\Delta_G x_k) = (x_k \ominus x_{k+1}) \]

\[ \Delta^2_G x = (\Delta^2_G x_k) = (\Delta_G x_k \ominus \Delta_G x_{k+1}) \]

\[ = (x_k \ominus x_{k+1} \ominus x_{k+1} \ominus x_{k+2}) \]

\[ = (x_k \ominus e^2 \ominus x_{k+1} \ominus x_{k+1}) \]

\[ \Delta^3_G x = (\Delta^3_G x_k) = (\Delta^2_G x_k \ominus \Delta^2_G x_{k+1}) \]

\[ = (x_k \ominus e^3 \ominus x_{k+1} \ominus e^3 \ominus x_{k+1} \ominus x_{k+3}) \]

\[ \Delta^m_G x = (\Delta^m_G x_k) = (\Delta^{m-1}_G x_k \ominus \Delta^{m-1}_G x_{k+1}) \]

\[ = \left( \sum_{v=0}^{m} (\ominus e)^{v, G} \ominus e^{m, G} \ominus x_{k+v} \right), \text{ with } (\ominus e)^{0, G} = e. \]

Then it can be easily proved that \( \ell^G_\infty(\Delta^m_G), c^G(\Delta^m_G) \) and \( c^G_0(\Delta^m_G) \) are normed linear spaces with norm

\[ \| x \|^G_G = \sum_{i=1}^{m} |x_i|^G \oplus \| \Delta^m_G x \|^G_\infty. \]

Note: Throughout this chapter often we write \( G \sum_k \) instead of \( \sum_{k=1}^{\infty} \) and \( \lim_k \) instead of \( \lim_{k \to \infty} \).

Definition 3.2.1.1 (Geometric associative algebra). An associative algebra is a vector space \( A \subset \mathbb{R}(G) \), equipped with a bilinear map (called multiplication)

\[ \ominus : A \times A \to A \]

\[ (a, b) \to a \ominus b \]
which is associative, i.e.

$$(a \odot b) \odot c = a \odot (b \odot c), \forall a, b, c \in A.$$ 

An algebra is commutative if $a \odot b = b \odot a$ for all $a, b \in A$. An algebra is unital if there exists a unique $e \in A$ such that $e \odot a = a \odot e = a$ for all $a \in A$. A subalgebra of the algebra $A$ is a subspace $B$ that is closed under multiplication, i.e. $a \odot b \in A$ for all $a, b \in B$.

**Definition 3.2.1.2 (Geometric normed algebra).** A normed algebra is a normed space $A \subset \omega(G)$ that is also an associative algebra, such that the norm is sub-multiplicative: $\|a \odot b\|^G \leq \|a\|^G \odot \|b\|^G$ for all $a, b \in A$. A geometric algebra is a complete normed algebra, i.e., a normed algebra which is also a Banach space with respect to its norm.

It is to be noted that the submultiplicativity of the norm means that multiplication in normed algebras is jointly continuous, i.e. if $a_n \xrightarrow{G} a$ and $b_n \xrightarrow{G} b$ then $(a_n)$ is bounded and

$$\|a_n \odot b_n \odot a \odot b\|^G = \|a_n \odot (b_n \odot b) \odot (a_n \odot a) \odot b\|^G$$

$$\leq \|a_n\|^G \odot \|b_n \odot b\|^G \odot \|a_n \odot a\|^G \odot \|b\|^G$$

$$\leq \sup \{\|a_n\|^G\} \odot \|b_n \odot b\|^G \odot \|b\|^G \odot \|a_n \odot a\|^G \xrightarrow{G} 1 \text{ as } n \to \infty.$$ 

**Definition 3.2.1.3 (Geometric sequence algebra).** A geometric sequence space $E(G)$ is said to be sequence algebra if $x \odot y \in E(G)$ for $x = (x_k), y = (y_k) \in E(G)$. i.e. $E(G)$ is closed under the geometric multiplication $\odot$ defined by

$$\odot : E(G) \times E(G) \rightarrow E(G)$$

$$(x, y) \rightarrow x \odot y = (x_k) \odot (y_k) = (x_k \ln y_k)$$

for any two sequences $x = (x_k), y = (y_k) \in E$.

Since $\omega(G)$ is closed under geometric multiplication $\odot$, hence, $\omega(G)$ is a sequence algebra. Also sequence algebra $\omega(G)$ is unital as $\|e_{\alpha}\|^G = e$, where $e_{\alpha} = (e, e, e, \ldots) \in \omega(G)$.

**Definition 3.2.1.4 (Continuous dual space).** If $X$ is a normed space, a linear map $f : X \rightarrow \mathbb{R}(G)$ is called linear functional. $f$ is called continuous linear functional or bounded linear functional if $\|f\|^G < \infty$, where

$$\|f\|^G = \sup \{|f(x)|^G : \|x\|^G \leq e \text{ for all } x \in X\}$$

Let $X^*$ be the collection of all bounded linear functionals on $X$. If $f, g \in X^*$ and $\alpha \in \mathbb{R}(G)$, we define $(\alpha \odot f \oplus g)(x) = \alpha \odot f(x) \oplus g(x)$; $X^*$ is called the continuous dual space of $X$. 

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Theorem 3.2.1.1. The sequence spaces \( \ell^G_\infty(\Delta^m_G) \), \( c^G(\Delta^m_G) \) and \( c^G_0(\Delta^m_G) \) are Banach spaces with the norm
\[
\|x\|_{\Delta^m_G}^G = G \sum_{i=1}^{m} |x_i|^G \oplus \|\Delta^m_G x\|_\infty^G.
\]

Proof. Let \((x_n)\) be a Cauchy sequence in \( \ell^G_\infty(\Delta^m_G) \), where \( x_n = (x_i^{(n)}) = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots) \) for \( n \in \mathbb{N} \) and \( x_k^{(n)} \) is the \( k \)th coordinate of \( x_n \). Then
\[
\|x_n \oplus x_l\|_{\Delta^m_G}^G = \sum_{G} \left| \sum_{i=1}^{m} (x_i^{(n)} \oplus x_i^{(l)}) \right|^G \oplus \|\Delta^m_G (x_n \oplus x_l)\|_\infty^G
\]
\[
= \sum_{G} \sum_{i=1}^{m} |x_i^{(n)} \oplus x_i^{(l)}|^G \oplus \sup_k |\Delta^m_G (x_n \oplus x_l)|^G \rightarrow 1 \text{ as } l, n \rightarrow \infty.
\]

Hence we obtain
\[
|x_k^{(n)} \oplus x_k^{(l)}|^G \rightarrow 1
\]
as \( n, l \rightarrow \infty \) and for each \( k \in \mathbb{N} \). Therefore \((x_k^{(n)}) = (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, \ldots)\) is a Cauchy sequence in \( \mathbb{R}(G) \). Since \( \mathbb{R}(G) \) is complete, \((x_k^{(n)})\) is convergent.

Suppose \( \lim_n x_k^{(n)} = x_k \), for each \( k \in \mathbb{N} \). Since \((x_n)\) is a Cauchy sequence, for each \( \varepsilon > 1 \), there exists \( N = N(\varepsilon) \) such that \( \|x_n \oplus x_l\|_{\Delta^m_G}^G < \varepsilon \) for all \( n, l \geq N \). Hence from (3.2.1)
\[
\sum_{G} \sum_{i=1}^{m} |x_i^{(n)} \oplus x_i^{(l)}|^G \leq \varepsilon \quad \text{and} \quad \sum_{G} \sum_{v=0}^{m} (\varepsilon \oplus e)^G \oplus e^{(m)} \oplus (x_k^{(n)} \oplus x_k^{(l)}) \leq \varepsilon
\]
for all \( k \in \mathbb{N} \) and \( n, l \geq N \). So we have
\[
\lim_l \sum_{G} \sum_{i=1}^{m} |x_i^{(n)} \oplus x_i^{(l)}|^G = \sum_{G} \sum_{i=1}^{m} |x_i^{(n)} \oplus x_i|^G < \varepsilon
\]
and
\[
\lim_l \left| \Delta^m_G (x_k^{(n)} \oplus x_k^{(l)}) \right|^G = \left| \Delta^m_G (x_k^{(n)} \oplus x_k)^G \right| < \varepsilon, \quad \forall n \geq N.
\]

This implies \( \|x_n \oplus x\|_{\Delta^m_G}^G < \varepsilon^2, \forall n \geq N \), that is \( x_n \xrightarrow{G} x \) as \( n \rightarrow \infty \), where \( x = (x_k) \). Now we have to show that \( x \in \ell^G_\infty(\Delta^m_G) \). We have
\[
|\Delta^m_G x_k|^G = \left| \sum_{G} \sum_{v=0}^{m} (\varepsilon \oplus e)^G \oplus e^{(m)} \oplus x_k^{(n)} \right|^G
\]
\[
= \left| \sum_{G} \sum_{v=0}^{m} (\varepsilon \oplus e)^G \oplus e^{(m)} \oplus (x_k^{(N)} \oplus x_k^{(N)}) \right|^G
\]
\[
\leq \left| \sum_{G} \sum_{v=0}^{m} (\varepsilon \oplus e)^G \oplus e^{(m)} \oplus x_k^{(N)} \right|^G
\]
\[
\leq \|x^N \oplus x\|_{\Delta^m_G}^G \oplus |\Delta^m_G x_k^N|^G = O(\varepsilon).
\]

Therefore we obtain \( x \in \ell^G_\infty(\Delta^m_G) \). Hence \( \ell^G_\infty(\Delta^m_G) \) is a Banach space. \( \square \)
It can be shown that \( c^G(\Delta^m_G) \) and \( c^G_0(\Delta^m_G) \) are closed subspaces of \( \ell^G_\infty(\Delta^m_G) \). Therefore these sequence spaces are Banach spaces with the same norm defined for \( \ell^G_\infty(\Delta^m_G) \), above.

Now we give some inclusion relations between these sequence spaces.

**Lemma 3.2.1.2.**  
(i) \( c^G_0(\Delta^m_G) \subset c^G(\Delta^m_G) \);  
(ii) \( c^G(\Delta^m_G) \subset c^G(\Delta^{m+1}_G) \);  
(iii) \( \ell^G_\infty(\Delta^m_G) \subset \ell^G_\infty(\Delta^{m+1}_G) \).

**Proof.** (i) Let \( x \in c^G_0(\Delta^m_G) \). Since

\[
|\Delta^m_G x_k|^G = |\Delta^m_G x_k \ominus \Delta^m_G x_{k+1}|^G 
\leq |\Delta^m_G x_k|^G + |\Delta^m_G x_{k+1}|^G \rightarrow 1 \text{ as } k \rightarrow \infty.
\]

\( \therefore \) we obtain \( x \in c^G(\Delta^{m+1}_G) \). Thus \( c^G_0(\Delta^m_G) \subset c^G(\Delta^{m+1}_G) \).

This inclusion is strict. For let

\[
x = (e^{km}) = (e, e^{2m}, e^{3m}, e^{4m}, \ldots, e^{km}, \ldots).
\]

Then \( x \in c^G_0(\Delta^{m+1}_G) \) as \((m+1)\)th geometric difference of \( e^{km} \) is 1(geometric zero). But \( x \notin c^G_0(\Delta^m_G) \) as \( m \)th geometric difference of \( e^{km} \) is a constant. Hence the inclusion is strict.

The proofs of (ii) and (iii) are similar to that of (i). \( \square \)

**Lemma 3.2.1.3.**  
(i) \( c^G_0(\Delta^m_G) \subset c^G(\Delta^{m+1}_G) \);  
(ii) \( c^G(\Delta^m_G) \subset \ell^G_\infty(\Delta^m_G) \).

Proofs are similar to that of Lemma (3.2.1.2).

Furthermore, since the sequence spaces \( \ell^G_\infty(\Delta^m_G), c^G(\Delta^m_G) \) and \( c^G_0(\Delta^m_G) \) are Banach spaces with continuous coordinates, that is, \( \|x_n \ominus x\|^G \Delta^m_G \rightarrow 1 \) implies \( \|x^{(n)}_n \ominus x_k\|^G \rightarrow 1, \forall k \in \mathbb{N} \) as \( n \rightarrow \infty \), these are also BK-spaces.

**Remark 3.2.1.1.** It can be easily proved that \( c^G_0 \) is a sequence algebra. But in general, \( \ell^G_\infty(\Delta^m_G), c^G(\Delta^m_G) \) and \( c^G_0(\Delta^m_G) \) are not sequence algebra. For let \( x = (e^k), y = (e^{km-1}) \). Clearly \( x, y \in c^G_0(\Delta^m_G) \). But

\[
x \ominus y = \left( e^k \ominus e^{km-1} \right) = (e^{km}) \notin c^G_0(\Delta^m_G) \text{ for } m \geq 2,
\]

since \( m \)th geometric difference of \( e^{km} \) is constant.
Let us define the operator
\[ D : \ell_\infty^G(\Delta^m_G) \to \ell_\infty^G(\Delta^m_G) \]
\[ Dx = (1,1,\ldots,1,x_{m+1},x_{m+2},\ldots), \]
where \( x = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}, \ldots) \in \ell_\infty^G(\Delta^m_G) \). It is trivial that \( D \) is a bounded linear operator on \( \ell_\infty^G(\Delta^m_G) \). Furthermore, the set
\[ D[\ell_\infty^G(\Delta^m_G)] = D\ell_\infty^G(\Delta^m_G) = \{ x = (x_k) : x \in \ell_\infty^G(\Delta^m_G), x_1 = x_2 = \ldots = x_m = 1 \} \]
is a subspace of \( \ell_\infty^G(\Delta^m_G) \) and
\[ \| x \|^G_{\Delta_G} = |x_1|^G + |x_2|^G + \ldots + |x_m|^G + \| \Delta^m_G x \|^G_{\infty} \]
\[ = 1 \oplus 1 \oplus \ldots \oplus 1 + \| \Delta^m_G x \|^G_{\infty} \]
\[ = \| \Delta^m_G x \|^G_{\infty} \]
\[ \therefore \| x \|^G_{\Delta_G} = \| \Delta^m_G x \|^G_{\infty} \text{ in } D\ell_\infty^G(\Delta^m_G). \]

Now let us define
\[ \Delta^m : D\ell_\infty^G(\Delta^m_G) \to \ell_\infty^G \]
\[ \Delta^m_G x = y = (\Delta^m_G^{-1}x_k \oplus \Delta^m_G^{-1}x_{k+1}). \]

**\( \Delta^m_G \) is a linear homomorphism:** Let \( x, y \in D\ell_\infty^G(\Delta^m_G) \). Then
\[ \Delta^m_G(x_k \oplus y_k) = \sum_{v=0}^m (\oplus \alpha) e^v_G \circ e^m_G \circ (x_k \oplus y_k) \]
\[ = \sum_{v=0}^m (\oplus \alpha) e^v_G \circ e^m_G \circ x_k \oplus \sum_{v=0}^m (\oplus \alpha) e^v_G \circ e^m_G \circ y_k \]
\[ = \Delta^m_G x_k \oplus \Delta^m_G y_k \]
\[ \therefore \Delta^m_G(x \oplus y) = \Delta^m_G x \oplus \Delta^m_G y. \]

For \( \alpha \in \mathbb{R}(G) \)
\[ \Delta^m_G(\alpha \circ x) = (\Delta^m_G \alpha \circ x_k) \]
\[ = \left( \sum_{v=0}^m (\oplus \alpha) e^v_G \circ e^m_G \circ \alpha \circ x_k \right) \]
\[ = \left( \alpha \circ \sum_{v=0}^m (\oplus \alpha) e^v_G \circ e^m_G \circ x_k \right) \]
\[ = \alpha \circ \Delta^m_G \circ x. \]

This implies that \( \Delta^m_G \) is a linear homomorphism. Hence \( D\ell_\infty^G(\Delta^m_G) \) and \( \ell_\infty^G \) are equivalent as topological spaces [44]. \( \Delta^m_G \) and \( (\Delta^m_G)^{-1} \) are norm preserving and
\[ \| \Delta^m_G \|^G_{\infty} = \| (\Delta^m_G)^{-1} \|^G_{\infty} = e. \]
Let \([\ell^G_\infty]'\) and \([D\ell^G_\infty(\Delta^m_G)]'\) denote the continuous duals of \(\ell^G_\infty\) and \(D\ell^G_\infty(\Delta^m_G)\), respectively.

It can be shown that \(s : [D\ell^G_\infty(\Delta^m_G)]' \rightarrow [\ell^G_\infty]'\)

\[f_\Delta \rightarrow f_\Delta \circ (\Delta^m_G)^{-1} = f\]
is a linear isometry. So \([D\ell^G_\infty(\Delta^m_G)]'\) is equivalent to \([\ell^G_\infty]'\).

In the same way, it can be shown that \(Dc^G(\Delta^m_G)\) and \(Dc^G_0(\Delta^m_G)\) are equivalent as topological space to \(c^G\) and \(c^G_0\), respectively. Also

\([Dc^G(\Delta^m_G)]' \sim [Dc^G_0(\Delta^m_G)]' \sim \ell^G_1\),

where \(\ell^G_1 = \{x = (x_k) : \sum_k |x_k|^G < \infty\}\).

### 3.2.2 Dual spaces of \(\ell^G_\infty(\Delta^m_G)\) and \(c^G(\Delta^m_G)\)

In this section we construct the \(\alpha\)-dual spaces of \(\ell^G_\infty(\Delta^m_G)\) and \(c^G(\Delta^m_G)\). Also we show that these spaces are not perfect spaces.

**Lemma 3.2.2.1.** The following statements are equivalent:

(a) \(\sup_k |x_k \odot x_{k+1}|^G < \infty\) i.e. \(\sup_k |\Delta_G x_k|^G < \infty\);

(b) (i) \(\sup_k e^{k-1} \odot |x_k|^G < \infty\) and

(ii) \(\sup_k |x_k \odot e^{(k+1)-1} \odot x_{k+1}|^G < \infty\).

**Proof.** Let (a) be true i.e. \(\sup_k |x_k \odot x_{k+1}|^G < \infty\).

Now \(|x_1 \odot x_{k+1}|^G = \left| \sum_{v=1}^k (x_v \odot x_{v+1}) \right|^G\)

\[= \left| \sum_{v=1}^k \Delta_G x_v \right|^G\]

\[\leq \sum_{v=1}^k |\Delta_G x_v|^G = O(e^k)\]

and \(|x_k|^G = |x_1 \odot x_{k+1} \odot x_k \odot x_{k+1}|^G\)

\[\leq |x_1|^G \odot |x_1 \odot x_{k+1}|^G \odot |x_k \odot x_{k+1}|^G = O(e^k).\]

This implies that \(\sup_k e^{k-1} \odot |x_k|^G < \infty\). This completes the proof of b(i).
Again

\[
\sup_k \left| x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G = \left| \left\{ e^{(k+1)} \ominus e^{(k+1)^{-1}} \right\} \ominus x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G
\]

\[
= \left| \left\{ (e^k \oplus e) \ominus e^{(k+1)^{-1}} \right\} \ominus x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G
\]

\[
= \left| \left\{ e^{k(k+1)^{-1}} \ominus x_k \ominus e^{(k+1)^{-1}} \ominus x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right\} \ominus e^{(k+1)^{-1}} \ominus x_k \right|^G
\]

\[
\leq e^{k(k+1)^{-1}} \ominus \left| x_k \ominus x_{k+1} \right|^G \ominus e^{(k+1)^{-1}} \ominus \left| x_k \right|^G
\]

\[
= O(e).
\]

Therefore \( \sup_k \left| x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G < \infty \). This completes the proof of the part \((ii)\) of \((b)\).

Conversely let \((b)\) be true. Then

\[
\left| x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G = \left| e^{(k+1)(k+1)^{-1}} \ominus x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G
\]

\[
\geq e^{k(k+1)^{-1}} \ominus \left| x_k \ominus x_{k+1} \right|^G \ominus e^{(k+1)^{-1}} \ominus \left| x_k \right|^G
\]

i.e. \( e^{k(k+1)^{-1}} \ominus \left| x_k \ominus x_{k+1} \right|^G \leq e^{(k+1)^{-1}} \ominus \left| x_k \right|^G \ominus \left| x_k \ominus e^{k(k+1)^{-1}} \ominus x_{k+1} \right|^G \).

Thus \( \sup_k \left| x_k \ominus x_{k+1} \right|^G < \infty \) as \((i)\) and \((ii)\) hold. \(\square\)

**Corollary 3.2.2.2.** The following statements are equivalent

\((a)\) \( \sup_k \left| \Delta_G^{m-1} x_k \ominus \Delta_G^{m-1} x_{k+1} \right|^G < \infty \);

\((b)\)

\((i)\) \( \sup_k e^{k^{-1}} \ominus \left| \Delta_G^{m-1} x_k \right|^G < \infty \)

\((ii)\) \( \sup_k \left| \Delta_G^{m-1} x_k \ominus e^{k(k+1)^{-1}} \ominus \Delta_G^{m-1} x_{k+1} \right|^G < \infty \).

**Proof.** By putting \( \Delta_G^{m-1} x_k \) instead of \( x_k \) in Lemma (3.2.2.1), results are obvious. \(\square\)

**Lemma 3.2.2.3.** \( \sup_k e^{k^{-1}} \ominus \left| \Delta_G x_k \right|^G < \infty \) implies \( \sup_k e^{-(i+1)} \ominus \left| x_k \right|^G < \infty \), \( \forall i \in \mathbb{N} \).

**Proof.** For \( i = 1 \) it is obvious from the Lemma (3.2.2.1). Let the result be true for \( i = n \). i.e. \( \sup_k e^{k^{-n}} \ominus \left| \Delta_G x_k \right|^G < \infty \). Then

\[
\left| x_k \ominus x_{k+1} \right|^G = \left| \sum_{v=1}^{k} \Delta_G x_k \right|^G
\]

\[
\leq \sum_{v=1}^{k} \left| \Delta_G x_k \right|^G = O \left( (e^k)^k \right) = O \left( e^{k(n+1)} \right), \text{ as } \sup_k e^{k^{-n}} \ominus \left| \Delta_G x_k \right|^G < \infty
\]

and

\[
\left| x_k \right|^G = \left| x_k \ominus x_1 \ominus x_1 \ominus x_{k+1} \ominus x_{k+1} \right|^G
\]

\[
\leq \left| x_1 \right|^G \ominus \left| x_1 \ominus x_{k+1} \right|^G \ominus \left| x_k \ominus x_{k+1} \right|^G = O \left( e^{k(n+1)} \right).
\]

From this we obtain, \( \sup_k e^{k^{-n+1}} \ominus \left| x_k \right|^G < \infty \). Thus \( \sup_k e^{k^{-i+1}} \ominus \left| x_k \right|^G < \infty \), \( \forall i \in \mathbb{N} \). \(\square\)
Lemma 3.2.2.4. \( \sup_k e^{k-1} \circ |\Delta_G^{m-1} x_k|^G < \infty \) implies \( \sup_k e^{-(i+1)} \circ |\Delta_G^{m-i} x_k|^G < \infty \), for all \( i, m \in \mathbb{N} \) and \( 1 \leq i < m \).

Proof. Putting \( \Delta_G^{m-1} x_k \) instead of \( \Delta_G x_k \) in Lemma (3.2.2.3), the result is immediate.

\( \square \)

Corollary 3.2.2.5. \( \sup_k e^{k-1} \circ |\Delta_G^{m-1} x_k| < \infty \) implies \( \sup_k e^{k-m} \circ |x_k| < \infty \).

Proof. In Lemma (3.2.2.4) putting \( i = 1 \), we get

\[ \sup_k e^{k-1} \circ |\Delta_G^{m-1} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-2} \circ |\Delta_G^{m-2} x_k|^G < \infty \]

Similarly

\[ \sup_k e^{k-2} \circ |\Delta_G^{m-2} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-3} \circ |\Delta_G^{m-3} x_k|^G < \infty . \]

Continuing the process we get

\[ \sup_k e^{k-(m-1)} \circ |\Delta_G^{1} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-m} \circ |\Delta_G^{0} x_k|^G < \infty \]

Thus

\[ \sup_k e^{k-1} \circ |\Delta_G^{m-1} x_k|^G < \infty \quad \Rightarrow \quad \sup_k e^{k-m} \circ |x_k|^G < \infty . \]

\( \square \)

Corollary 3.2.2.6. If \( x \in \ell_G^{\infty}(\Delta_G^{m}) \) then \( \sup_k e^{k-m} \circ |x_k|^G < \infty \).

Proof.

\[ x \in \ell_G^{\infty}(\Delta_G^{m}) \Rightarrow \Delta_G^{m} x \in \ell_G^{\infty} \]

\[ \Rightarrow \sup_k |\Delta_G^{m} x_k|^G < \infty \]

\[ \Rightarrow \sup_k |\Delta_G^{m-1} x_k \odot \Delta_G^{m-1} x_{k+1}|^G < \infty \]

\[ \Rightarrow \sup_k e^{k-1} \circ |\Delta_G^{m} x_k|^G < \infty \quad \text{by Corollary (3.2.2.2)} \]

\[ \Rightarrow \sup_k e^{k-m} \circ |x_k|^G < \infty \quad \text{by Corollary (3.2.5)}. \]

\( \square \)

Lemma 3.2.2.7. If \( U_1 = \{ a = (a_k) : \sum_k e^{k-m} \circ |a_k|^G < \infty \} \). Then \( [D\ell_G^{\infty}(\Delta_G^{m})]^\alpha = U_1 \).
Proof. Let \( a \in U_1 \), then using Corollary (3.2.2.2) for \( x \in D\ell^G_\infty(\Delta^m_G) \), we have
\[
\sum_G \{a_k \odot x_k|G^* = \sum_G \{e^{km} \odot |a_k|G^* \} \odot \{e^{k-m} \odot |x_k|G^* \} < \infty \text{ by Corollary (3.2.2.5).}
\]
This implies that \( a \in [D\ell^G_\infty(\Delta^m_G)]^a \). Therefore
\[
U_1 \subseteq [D\ell^G_\infty(\Delta^m_G)]^a. \tag{3.2.3}
\]
Conversely, let \( a \in [D\ell^G_\infty(\Delta^m_G)]^a \). Then
\[
\sum_G |a_k \odot x_k|G^* < \infty \text{ (by definition of } \alpha\text{-dual) for } x \in D\ell^G_\infty(\Delta^m_G).
\]
So we take
\[
 x_k = \begin{cases} 1, & \text{if } k \leq m \\ e^{km}, & \text{if } k > m \end{cases} \tag{3.2.4}
\]
Then \( x = (1, 1, 1, \ldots, e^{(m+1)m}, e^{(m+2)m}, \ldots) \in D\ell^G_\infty(\Delta^m_G) \). Therefore
\[
\sum_k e^{km} \odot |a_k|G^* = \sum_k e^{km} \odot |a_k|G^* \odot \sum_k e^{k-m} \odot |x_k|G^* = \sum_G |a_k \odot x_k|G^* < \infty
\]
since \( a_k \odot x_k = 1 \) (the geometric zero) for \( k = 1, 2, \ldots, m \).
Therefore \( a \in U_1 \). This implies
\[
[\ell^G_\infty(\Delta^m_G)]^a \subseteq U_1. \tag{3.2.5}
\]
Then from (3.2.3) and (3.2.5), we get \([\ell^G_\infty(\Delta^m_G)]^a = U_1\). \(\square\)

Lemma 3.2.2.8. \([\ell^G_\infty(\Delta^m_G)]^a = [Dc^G(\Delta^m_G)]^a\).

Proof. Since \( Dc^G(\Delta^m_G) \subseteq \ell^G_\infty(\Delta^m_G) \), hence \([\ell^G_\infty(\Delta^m_G)]^a \subseteq [Dc^G(\Delta^m_G)]^a\).

Again let \( a \in [Dc^G(\Delta^m_G)]^a \). Then \( \sum_G |a_k \odot x_k|G^* < \infty \) for each \( x \in Dc^G(\Delta^m_G) \). If we take \( x = (x_k) \) which is defined in (3.2.4), we get
\[
\sum_K e^{km} \odot |a_k|G^* = \sum_{G=1}^m e^{km} \odot |a_k|G^* \odot \sum_k |a_k \odot x_k|G^* < \infty.
\]
This implies that \( a \in [\ell^G_\infty(\Delta^m_G)]^a \). Thus \([\ell^G_\infty(\Delta^m_G)]^a = [Dc^G(\Delta^m_G)]^a\). \(\square\)

Lemma 3.2.2.9. (i) \([\ell^G_\infty(\Delta^m_G)]^a = [D\ell^G_\infty(\Delta^m_G)]^a\).

(ii) \([c^G(\Delta^m_G)]^a = [Dc^G(\Delta^m_G)]^a\).

Proof. (i) Since \( D\ell^G_\infty(\Delta^m_G) \subseteq \ell^G_\infty(\Delta^m_G) \), so \([\ell^G_\infty(\Delta^m_G)]^a \subseteq [D\ell^G_\infty(\Delta^m_G)]^a\).

Let \( a \in [D\ell^G_\infty(\Delta^m_G)]^a \) and \( x \in \ell^G_\infty(\Delta^m_G) \). From Corollary (3.2.2.6), we have
\[
\sum_k |a_k \odot x_k|G^* = \sum_k e^{km} \odot |a_k|G^* \odot (e^{k-m} \odot |x_k|G^*) < \infty.
\]
Hence $a \in [\ell^G_\infty(\Delta_m^G)]^\alpha$.

(ii) $Dc^G(\Delta_m^G) \subseteq c^G(\Delta_m^G)$ implies $[c^G(\Delta_m^G)]^\alpha \subseteq [Dc^G(\Delta_m^G)]^\alpha$.

Let $a \in [Dc^G(\Delta_m^G)]^\alpha$ and $x \in c^G(\Delta_m^G)$. From Corollary (3.2.2.6), we have

$$G \sum_k |a_k \odot x_k|^G = G \sum_k e^{k^m} \odot |a_k|^G \odot (e^{k^{-m}} \odot |x_k|^G) < \infty$$

for $x \in c^G(\Delta_m^G) \subseteq l^G_\infty(\Delta_m^G)$. Hence $a \in [c^G(\Delta_m^G)]^\alpha$. This completes the proof.

Theorem 3.2.2.10. Let $X$ stand for $\ell^G_\infty$ or $c^G$. Then

$$[X(\Delta_m^G)]^\alpha = \{a = (a_k) : \sum G_k e^{k^m} \odot |a_k|^G < \infty\}.$$

Proof.

$$[\ell^G_\infty(\Delta_m^G)]^\alpha = [D\ell^G_\infty(\Delta_m^G)]^\alpha \quad \text{by Lemma (3.2.2.9)}$$

$$= \{a = (a_k) : \sum G_k e^{k^m} \odot |a_k|^G < \infty\} \quad \text{by Lemma (3.2.7)}.$$

Again

$$[c^G(\Delta_m^G)]^\alpha = [Dc^G(\Delta_m^G)]^\alpha \quad \text{by Lemma (3.2.2.9)}$$

$$= [D\ell^G_\infty(\Delta_m^G)]^\alpha \quad \text{by Lemma (3.2.2.8)}$$

$$= \{a = (a_k) : \sum G_k e^{k^m} \odot |a_k|^G < \infty\} \quad \text{by Lemma (3.2.7)}.$$

Corollary 3.2.2.11. For $X = \ell^G_\infty$ or $c^G$, we have

$$[X(\Delta_m^G)]^\alpha = \{a = (a_k) : \sum G_k e^k \odot |a_k|^G < \infty\}, \text{ and}$$

$$[X(\Delta_m^G)^2] = \{a = (a_k) : \sum G_k e^{2k} \odot |a_k|^G < \infty\}.$$
Hence \( a \in [X(\Delta^m_G)]^{\alpha\alpha} \).

Conversely, let \( a \in [X(\Delta^m_G)]^{\alpha\alpha} \) and \( a \not\in U_2 \). Then we must have

\[
\sup_k e^{k-m} \odot |a_k|^G = \infty.
\]

Hence there exists a strictly increasing sequence \((e^{k(i)})\) of geometric integers (see [56]), where \( k(i) \) is a strictly increasing sequence of positive integers such that

\[
e^{k(i)-m} \odot |a_{k(i)}|^G > e^m.
\]

Let us define the sequence \( x \) by

\[
x_k = \begin{cases} 
  (|a_{k(i)}|^G)^{-1_G}, & k = k(i) \\
  1, & k \neq k(i).
\end{cases}
\]

where \((|a_{k(i)}|^G)^{-1_G}\) is the geometric inverse of \( |a_{k(i)}|^G \) so that \( |a_{k(i)}|^G \odot (|a_{k(i)}|^G)^{-1_G} = e \).

Then we have

\[
\sum_k e^{k-m} \odot |x_k|^G = \sum_i e^{k(i)-m} \odot [|a_{k(i)}|^G]^{-1_G} \leq e^{i-m} < \infty.
\]

Hence \( x \in [X(\Delta^m_G)]^{\alpha\alpha} \) and \( \sum_k a_k \odot x_k|^G = \sum e = \infty \). This is a contradiction as \( a \in [X(\Delta^m_G)]^{\alpha\alpha} \). Hence \( a \in U_2 \).

**Corollary 3.2.2.13.** For \( X = \ell_\infty^G \) or \( c^G \), we have

\[
[X(\Delta^2_G)]^{\alpha\alpha} = \{ a = (a_k) : \sup_k e^{k-2} \odot |a_k|^G < \infty \}.
\]

**Proof.** In Theorem (3.2.2.12), putting \( m = 2 \) we obtain the result.

**Corollary 3.2.2.14.** The sequence spaces \( \ell_\infty^G(\Delta^m_G) \) and \( c^G(\Delta^m_G) \) are not perfect.

**Proof.** Proof is trivial as \( X^{\alpha\alpha} \neq X \) for \( X = \ell_\infty^G(\Delta^m_G) \) or \( c^G(\Delta^m_G) \).