CHAPTER 5

GENERALIZED SETS
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5.0. INTRODUCTION

In 1970, Levine[47] introduced the concept of generalized closed sets as a generalization of closed sets to investigate some topological properties. In fact, a subset $A$ of a space $X$ is generalized closed (Levine[47]) written as $g$-closed in $X$ if $A^- \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$. Following this various mathematicians worked on this, by replacing open sets by nearly open sets and closed sets by nearly closed sets and they obtained several topological properties. In this chapter we discuss various forms of generalized sets. We begin our discussions with preliminaries.
5.1. PRELIMINARIES

Definition 5.1.1. (Murdeswar[58])

A space $X$ is said to be complemented if every open set is closed.

Definition 5.1.2. (Levine[47])

A subset $A$ of a space $X$ is called generalized closed (written as $g$-closed) if $A^c \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 5.1.3. (Maki[52])

A subset $A$ of a space $X$ is called $\alpha$-generalized closed (written as $\alpha g$-closed) if $\alpha cl\ A \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 5.1.4. (Maki[51])

A subset $A$ of a space $X$ is called generalized $\alpha$-closed (written as $g\alpha$-closed) if $\alpha cl\ A \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open.

Definition 5.1.5. (Bhattacharya[9])

A subset $A$ of a space $X$ is called semi-generalized closed (written as $sg$-closed) if $A_\cdot \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open.
Definition 5.1.6. (Arya[6])

A subset $A$ of a space $X$ is called generalized semi-closed (written as gs-closed) if $A \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 5.1.7. (Palaniappan[73])

A subset $A$ of a space $X$ is called regular generalized closed (written as rg-closed) if $A \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open.

Definition 5.1.8. (Balachandran[7])

A subset $A$ of a space $X$ is generalized preclosed (written as gp-closed) if $pclA \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 5.1.9. (Dontchev[21])

A subset $A$ of a space $X$ is generalized semi-preclosed (written as gsp-closed) if $spclA \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 5.1.10. (Gnanambal[32])

A subset $A$ of a space $X$ is called generalized preregular closed (written as gpr-closed) if $pclA \subseteq U$ whenever $A \subseteq U$ and $A$ is regular open.

Generalized open sets can be defined by taking complements in the above definitions.
Definition 5.1.11. (Levine[47])

A subset $A$ of a space $X$ is called generalized open (written as $g$-open) if $F \subseteq A^o$ whenever $F \subseteq A$ and $F$ is closed.

It is easy to note that $A$ is $g$-closed if and only if its complement $X \setminus A$ is $g$-open.

The generalized sets like $\alpha g$-open, $\varrho o$-open, $sg$-open, $gs$-open, $rg$-open, $gp$-open, $gsp$-open and $gpr$-open sets can be analogously defined and their complements can be found to be the corresponding generalized sets namely $\alpha g$-closed, $\varrho o$-closed, $sg$-closed, $gs$-closed, $rg$-closed, $gp$-closed, $gsp$-closed and $gpr$-closed sets.

The following implication diagrams hold among various types of generalized closed sets.

Diagram 5.1.12.

\[ \begin{array}{ccl}
\text{rg-closed} & \Rightarrow & \text{gpr-closed} \\
\uparrow & & \uparrow \\
\text{closed} & \Rightarrow & \alpha g\text{-closed} \\
\downarrow & & \downarrow \\
\text{gs-closed} & \Rightarrow & \text{gsp-closed}
\end{array} \]
Diagram 5.1.13.

(a) $\alpha$-closed $\Rightarrow$ ga-$\alpha$-closed $\neq$ ag-$\alpha$-closed

(b) $\alpha$-closed $\Rightarrow$ ag-$\alpha$-closed $\neq$ ga-$\alpha$-closed.

Diagram 5.1.14.

preclosed $\Rightarrow$ semi-preclosed $=\leq$ semi-closed

$\downarrow$ $\downarrow$ $\downarrow$

gp-closed $\Rightarrow$ gsp-closed $\leq$ sg-closed

The following implication diagrams hold among various types of generalized open sets.

Diagram 5.1.15.

rg-open $\Rightarrow$ gpr-open

$\uparrow$ $\uparrow$

open $\Rightarrow$ g-open $\Rightarrow$ ag-open $\Rightarrow$ gp-open

$\downarrow$ $\downarrow$

gs-open $\Rightarrow$ gsp-open
Diagram 5.1.16.

(a) $\alpha$-open $\Rightarrow$ $g\alpha$-open $\neq$ $\alpha g$-open

(b) $\alpha$-open $\Rightarrow$ $\alpha g$-open $\neq$ $g\alpha$-open.

Diagram 5.1.17.

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preopen  =  semi-preopen  =  semi-open
         \|     \|     \|
  gp-open  =  gsp-open  sg-open
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Examples can be constructed to show that the reverse implications in the above diagrams need not be true. This motivated Dunham[25] to define $T_{1/2}$ spaces.

**Definition 5.1.18.** (Dunham[25])

A space $X$ is said to be $T_{1/2}$ if every $g$-closed set is closed.

**Definition 5.1.19.** (Bhattacharya[9])

A space $X$ is said to be semi $T_{1/2}$ if every $sg$-closed set is semi-closed.
Definition 5.1.20. (Dontchev[21])

A space $X$ is said to be semi-pre $T_{1/2}$ if every gsp-closed set is semi-preclosed.

Definition 5.1.21.(Gnanambal[32])

A space $X$ is said to be preregular $T_{1/2}$ if every gpr-closed set is preclosed.

Definition 5.1.22. (Jankovic[41])

A space $X$ is locally indiscrete if every closed set is regular closed.

The following lemma due to Gnanambal characterizes locally indiscrete spaces.

Lemma 5.1.23. (Gnanambal[32])

For a space $X$, the following are equivalent.

(a) $X$ is preregular $T_{1/2}$

(b) Every nowhere dense singleton of $X$ is regular closed

(c) The only nowhere dense subset of $X$ is the void set

(d) Every subset of $X$ is preopen

(e) $X$ is locally indiscrete.
5.2. APPLICATIONS TO COMPLEMENTED SPACES

In this section we characterize complemented spaces using generalized sets.

Proposition 5.2.1.

Let $X$ be a complemented space. Then

(a) Every semi-open set is open

(b) Every semi-closed set is open

(c) Every $\alpha$-open set is open

(d) Every open set is regular open.

(e) Every $\alpha$-closed set is open.

(f) Every open set is regular closed.

(g) Every closed set is regular clopen.

(h) Every $q$-set is clopen.

Proof:

(a),(b),(c),(d),(e),(f) and (g) follow from the Definition 5.1.1. Now to prove (h), let $A$ be a $q$-set in $X$. Then by Definition 3.3.2 $A^{-O} \subseteq A^{O'}$ which implies $A^{-} \subseteq A^{O}$ so that $A=A^{O} = A^{-}$. This proves (h).

$\square$
Theorem 5.2.2.

Let $X$ be a complemented space with topology $\mathcal{J}$. Then

$SO(\mathcal{J}) = SC(\mathcal{J}) = \mathcal{J} = SR(\mathcal{J}) = RO(\mathcal{J}) = q(\mathcal{J}) = \mathcal{J}^\alpha = RC(\mathcal{J})$

Proof:

Follows from Proposition 5.2.1, Diagram 3.2.3., Diagram 3.2.4., Diagram 4.2.4. and Diagram 4.2.6.

The following theorem characterizes $p$-sets, preopen sets, preclosed sets, semi-preopen sets and semi-preclosed sets in complemented spaces.

Theorem 5.2.3.

Let $X$ be a complemented space. Then

(a) Every subset of $X$ is preopen (resp. preclosed)

(b) Every subset of $X$ is semi-preopen (resp. semi-preclosed).

(c) Every subset of $X$ is a $p$-set.

Proof:

Let $A \subseteq X$. Since $X$ is complemented $A^0 = A^0$ and

$A^{-0} = A^-$. Since $A \subseteq A^- = A^{-0}$, $A$ is preopen.
Since $A^0 = A$ and since $A^0 = A$, $A$ is preclosed. Since $A^0 = A$ and since $A^0 = A^0$, (b) follows easily.

Since $A^0 = A^0 \subseteq A = A^0$, $A$ is a p-set. This completes the proof of Theorem.

The next theorem characterizes various types of generalized sets in complemented spaces.

Theorem 5.2.4.

Let $X$ be a complemented space. Then

(a) Every subset of $X$ is g-open (resp. g-closed)

(b) Every subset of $X$ is ag-open (resp. ag-closed)

(c) Every subset of $X$ is ga-open (resp. ga-closed)

(d) Every subset of $X$ is sg-open (resp. sg-closed)

(e) Every subset of $X$ is gs-open (resp. gs-closed)

(f) Every subset of $X$ is rg-open (resp. rg-closed)

(g) Every subset of $X$ is gp-open (resp. gp-closed)

(h) Every subset of $X$ is gsp-open (resp. gsp-closed)

(i) Every subset of $X$ is gpr-open (resp. gpr-closed).
Proof:

To prove (a). Let $A \subseteq X$. Suppose $F \subseteq A$ and $F$ is closed.

Since $X$ is complemented $F^O = F \subseteq A$. Using Definition 5.1.11, $A$ is $g$-open. Since the complement of a $g$-open set is $g$-closed, every subset of $X$ is $g$-closed. This proves (a).

(b), (e), (f), (g), (h) and (i) follow from Diagram 5.1.12, Diagram 5.1.15 and (a). Now it remains to prove (c) and (d) only.

To prove (c) and (d): Let $A \subseteq X$. Suppose $A \subseteq U$ and $U$ is $\alpha$-open (resp. semi-open). Then, by Proposition 5.2.1 $U$ is open. Again, since $X$ is complemented, by Definition 5.1.1, $U$ is closed which implies $\text{acl} A \subseteq \text{acl} U = U$ (resp. $A \subseteq U = U$). This proves that $A$ is $g\alpha$-closed (resp. $sg$-closed). Since the complement of a $g\alpha$-closed set (resp. $sg$-closed) is $g\alpha$-open (resp. $sg$-open) every subset of $X$ is $g\alpha$-open (resp. $sg$-open). This proves (c) (resp. (d)). This completes the proof of the Theorem.

The following lemma will be useful in sequel.

Lemma 5.2.5.

Let $X$ be any space. Let $A \subseteq X$ be closed. Then $A$ is open if and only if it is $g$-open.
Proof:

Follows from Definition 5.1.11., Diagram 5.1.15. and the fact $A \subseteq A$.

\[ \square \]

Corollary 5.2.6.

Let $A \subseteq X$ be open in the space $X$.

Then $A$ is closed $\iff A$ is $g$-closed.

Proof:

Follows from Lemma 5.2.5.

\[ \square \]

The next theorem characterizes complemented spaces using generalized sets.

Theorem 5.2.7.

Let $X$ be a space. Then the following are equivalent.

(a) $X$ is complemented.

(b) Every subset of $X$ is $g$-open

(c) Every subset of $X$ is $g$-closed

(d) Every $g$-closed set is $g$-open.
(e) Every g-open set is g-closed.

(f) Every rg-closed set is g-open

(g) Every rg-open set is g-closed.

Proof:

(a) \Rightarrow (b), (a) \Rightarrow (c), (a) \Rightarrow (d), (a) \Rightarrow (e),

(a) \Rightarrow (f) and (a) \Rightarrow (g) follow from Theorem 5.2.4.

(b) \Rightarrow (c), (d) \Rightarrow (e) and (f) \Rightarrow (g) are obvious. So, it suffices to prove (b) \Rightarrow (a), (d) \Rightarrow (a) and (f) \Rightarrow (a) only.

(b) \Rightarrow (a) : Let \( A \subseteq X \) be closed. Since every subset is g-open, \( A \) is g-open. Since \( A \) is closed and since \( A \) is g-open, by Lemma 5.2.5, \( A \) is open. This proves (a).

(d) \Rightarrow (a) : Let \( A \subseteq X \) be closed. Then by Diagram 5.1.12, \( A \) is g-closed. Since every g-closed set is g-open, \( A \) is g-open. Since \( A \) is closed and since \( A \) is g-open, by Lemma 5.2.5, \( A \) is open. This proves (a).

(f) \Rightarrow (a) : Let \( A \subseteq X \) be closed. Then, by Diagram 5.1.12, \( A \) is rg-closed and hence it is g-open. Then, from Lemma 5.2.5 it follows that \( A \) is open. This completes the proof of Theorem.

\[ \square \]

Complemented spaces can be characterized using q-sets as shown in the following theorem.
Theorem 5.2.8.

A space $X$ is complemented if and only if every $q$-set is clopen.

Proof:

Follows from Proposition 5.2.1 (h) and the fact that every open set is a $q$-set.

Proposition 5.2.9.

Let $A$ be a closed (resp. open) subset of a space $X$. Then

(a) $A$ is sg-open (sg-closed) if and only if it is semi-open (resp. semi-closed).

(b) $A$ is ag-open (resp. ag-closed) if and only if it is $\alpha$-open (resp. $\alpha$-closed).

(c) $A$ is ga-open (resp. ga-closed) if and only if it is $\alpha$-open (resp. $\alpha$-closed).

(d) $A$ is gp-open (resp. gp-closed) if and only if it is preopen (resp. preclosed).

(e) $A$ is gsp-open (resp. gsp-closed) if and only if it is semi-preopen (resp. semi-preclosed).

(f) $A$ is gs-open (resp. gs-closed) if and only if it is regular closed (resp. regular open).
Proof:

To prove (a): If A is semi-open (resp. semi-closed) then by Diagram 5.1.17 (resp. Diagram 5.1.14) it is sg-open (resp. sg-closed). If A is sg-open (resp. sg-closed) since A is closed (resp. open) and since A \subseteq A we have A \subseteq A^o (resp. A^c \subseteq A) which implies A is semi-open (resp. semi-closed). This proves (a). Proofs for (b), (c), (d) and (e) are analog.

To prove (f): Suppose A is regular closed (resp. regular open). Suppose F \subseteq A (resp. A \subseteq U) and suppose F is closed (resp. U is open). Since every regular closed (resp. regular open) set is semi-open (resp. semi-closed) we have F \subseteq A^o (resp. A^c \subseteq U) which implies that A is gs-open (resp. gs-closed). Conversely suppose A is gs-open (resp. gs-closed). Since A is closed (resp. open) and since A \subseteq A we have A \subseteq A^o (resp. A \subseteq A^c) which implies that A \subseteq A \cap A^o \subseteq A (resp. A \cup A^c \subseteq A) so that A = A \cap A^o \cap (resp. A = A \cup A^c \cap). Since A is closed (resp. open), we have

A \subseteq A^o \subseteq A (resp. A \subseteq A^-^o) so that A = A^o \cap (resp. A = A^-^o).

This proves (f).

This completes the proof of the Proposition.

\square
Corollary 5.2.10.

(a) A closed set is $\alpha g$-open if and only if it is $g\alpha$-open.

(b) An open set is $\alpha g$-closed if and only if it is $g\alpha$-closed.

Proof:

Follows from Proposition 5.2.9(b) and Proposition 5.2.9(c) and from Diagram 5.1.13 and Diagram 5.1.16.

Remark 5.2.11.

Proposition 5.2.9 (a) is also valid if "closed (resp.open)" is replaced by "semi-closed (resp.semi-open)".

Remark 5.2.12.

Proposition 5.2.9 (c) is also valid if "closed (resp.open)" is replaced by "$\alpha$-closed (resp.$\alpha$-open)".

Proposition 5.2.13.

Let A be regular closed (resp.regular open). Then A is rg-open (resp.rg-closed) if and only if it is open (resp.closed).

Proof:

If A is open (resp.closed), then by Diagram 5.1.15 (resp. Diagram 5.1.12), A is rg-open (resp.rg-closed). Conversely if A is rg-open
(resp. rg-closed), since A is regular closed (resp. regular open) and since \( A \subseteq A \) we have \( A \subseteq A^0 \) (resp. \( A^- \subseteq A \)) which implies A is open (resp. closed). This proves the Proposition.

\[ \square \]

**Proposition 5.2.14.**

Let A be regular closed. Then the following are equivalent.

(a) A is open

(b) A is rg-open

(c) A is gpr-open.

**Proof:**

(a) \( \Rightarrow \) (b) \( \Rightarrow \) (c) follows from Diagram 5.1.15.

To prove (c) \( \Rightarrow \) (a): Suppose A is gpr-open. Since A is regular closed and since \( A \subseteq A \) we have \( A \subseteq \operatorname{pint} A \). Then since \( \operatorname{pint} A = A \cap A^- \circ \), \( A \subseteq A \cap A^- \circ \subseteq A \) which implies \( A = A \cap A^- \circ \) so that \( A \subseteq A^- \circ \). Since A is regular closed, \( A \subseteq A^- \circ = A^0 \subseteq A \) which implies A is open. This proves (a) and hence completes the proof of Proposition.

\[ \square \]
Corollary 5.2.15.

Let $A$ be regular open. Then the following are equivalent.

(a) $A$ is closed

(b) $A$ is rg-closed

(c) $A$ is gpr-closed

Proof:

Follows from Proposition 5.2.14.

Proposition 5.2.16.

Let $A$ be a subset of a space $X$.

(a) If $A$ is closed, gs-open and rg-open then $A$ is open.

(b) If $A$ is closed and if $A$ is $\alpha g$-open, then $A$ is open.

Proof:

To prove (a): Suppose $A$ is closed, gs-open and rg-open. Since $A$ is closed and since $A$ is gs-open, using Proposition 5.2.9 (f), $A$ is regular closed. Since $A$ is rg-open and since $A$ is regular closed, by Proposition 5.2.13, $A$ is open. This proves (a).
To prove (b):

Suppose \( A \) is closed and \( \alpha g \)-open. Then by Proposition 5.2.9 (b), \( A \) is \( \alpha \)-open. Since \( A \) is \( \alpha \)-open and since \( A \) is closed, we have \( A \subseteq A^{\circ \circ} \subseteq A^{-} = A^{\circ} \) which implies that \( A = A^{\circ} \) is open. This completes the proof of Proposition.

\[ \square \]

**Corollary 5.2.17.**

Let \( A \) be a subset of a space \( X \).

(a) If \( A \) is open, \( gs \)-closed and \( rg \)-closed then \( A \) is closed.

(b) If \( A \) is open and if \( A \) is \( \alpha g \)-closed then \( A \) is closed.

**Proof:**

Follows from Proposition 5.2.16.

\[ \square \]

The following theorem characterizes complemented spaces.

**Theorem 5.2.18.**

Let \( X \) be a space. If at least one of the following is true then the space \( X \) is complemented.

(a) Every open subset of \( X \) is both \( gs \)-closed and \( rg \)-closed.

(b) Every open subset of \( X \) is \( \alpha g \)-closed.
(c) Every closed subset of $X$ is $\alpha g$-open.

(d) Every closed subset of $X$ is both $gs$-open and $rg$-open.

(e) Every closed set is preopen

(f) Every $\alpha$-closed set is preopen

(g) Every preclosed set is preopen.

(h) Every open set is preclosed

(i) Every $\alpha$-open set is preclosed.

(j) Every preopen set is preclosed.

(k) Every semi-preopen set is closed.

(l) Every semi-preclosed set is open.

(m) Every semi-open set is closed.

(n) Every semi-closed set is open.

Proof:

If (a) is true then the result follows from Corollary 5.2.17 and the Definition 5.1.1.

Similarly (b) implies that $X$ is complemented.

When (c) or (d) is true, using Proposition 5.2.16, it follows that $X$ is complemented.
Since every closed preopen set is open, it follows that if at least one of (e), (f) and (g) holds then X is complemented. Since every open preclosed set is preopen, it follows that if at least one of (h), (i) and (j) holds then X is complemented. If at least one of (k), (l), (m) and (n) holds, it follows that X is complemented by using Diagram 4.2.5. and Diagram 4.2.6.

Theorem 5.2.19.

Every function defined on a complemented space is both precontinuous and p-continuous.

Proof:

Follows from Theorem 5.2.3.

Theorem 5.2.20.

Let f:X→Y be a map and let X be a complemented space. Then f is totally continuous if and only if f is q-continuous.

Proof:

Follows from Theorem 5.2.8.
5.3. APPLICATIONS TO $T_{1/2}$ SPACES

In this section we characterize $T_{1/2}$, semi $T_{1/2}$, semi-pre $T_{1/2}$ and preregular $T_{1/2}$ spaces. We shall prove that $T_{1/2}$ and semi $T_{1/2}$ are equivalent concepts in complemented spaces. It is note worthy to see that pre regular $T_{1/2}$ spaces are precisely complemented spaces. We also note that the concept of $T_{1/2}$ and the concept of complemented spaces are independent as shown by the following example.

Example 5.3.1 (Example 5.3., Gnanambal[32])

Let $X = \{a, b, c\}$, $\mathcal{J} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{c\}, \{a, b\}\}$. Then $(X, \mathcal{J})$ is $T_{1/2}$ but not complemented. $(X, \sigma)$ is complemented but not $T_{1/2}$.

The following theorem characterizes $T_{1/2}$ spaces in complemented spaces.

Theorem 5.3.2.

Let $X$ be a complemented space. Then the following are equivalent.

(a) $X$ is $T_{1/2}$

(b) Every $g$-closed set is open.

(c) Every $g$-open set is closed.
(d) Every $rg$-closed set is open.

(e) Every $rg$-open set is closed.

Proof:

(b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are obvious.

(a) $\Rightarrow$ (b): Let $A \subseteq X$ be $g$-closed. Since $X$ is $T_{1/2}$, $A$ is closed. Since $X$ is complemented, $A$ is open. This proves (b).

(b) $\Rightarrow$ (a): Let $A \subseteq X$ be $g$-closed. If (b) holds, then $A$ is open which implies that $A$ is closed. This proves (a).

(a)$\Rightarrow$(d): Let $A \subseteq X$ be $rg$-closed. Since $X$ is complemented, by Theorem 5.2.4 (a), $A$ is $g$-closed. If (a) holds then $A$ is closed which implies that $A$ is open. This proves (d).

(d) $\Rightarrow$(a). Let $A \subseteq X$ be $g$-closed. Then from the Diagram 5.1.12, $A$ is $rg$-closed. If (d) holds then $A$ is open which implies $A$ is closed. This proves (a).

This completes the proof of Theorem.

The following theorem characterizes semi $T_{1|2}$ spaces in complemented spaces.
Theorem 5.3.3.

Let $X$ be a complemented space. Then the following are equivalent.

(a) $X$ is semi $T_{1/2}$

(b) Every sg-closed set is open.

(c) Every sg-open set is closed.

Proof:

(b) $\Rightarrow$ (c): Obvious.

(a) $\Rightarrow$ (b): Let $A \subseteq X$ be sg-closed.

If (a) holds, $A$ is semi-closed which, by Proposition 5.2.1, implies that $A$ is open. This proves (b).

(b) $\Rightarrow$ (a): Let $A \subseteq X$ be sg-closed.

If (b) holds, then $A$ is open which, implies that $A$ is closed. This proves (a).

This completes the proof of Theorem.
Theorem 5.3.4.

Let $X$ be a complemented space. Then the following are equivalent.

(a) $X$ is $T_{1/2}$
(b) $X$ is semi $T_{1/2}$
(c) $X$ is Discrete.

Proof:

(c) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a): Obvious.

(a) $\Rightarrow$ (c): Let $A \subseteq X$.

Since $X$ is complemented, by Theorem 5.2.4., $A$ is g-closed. If (a) holds, then $A$ is closed. This proves (c).

(b) $\Rightarrow$ (c): Let $A \subseteq X$.

Since $X$ is complemented by Theorem 5.2.4 (d), $A$ is sg-closed. If (b) holds then $A$ is semi-closed which, by Proposition 5.2.1, implies that $A$ is open. This proves (c).

This completes the proof of Theorem.
Proposition 5.3.5.

(a) If every gsp-closed set is open then the space \( X \) is semi-pre \( T_{1/2} \).

(b) If \( X \) is complemented then the space \( X \) is semi-pre \( T_{1/2} \).

(c) If every gsp-closed set is open then \( X \) is complemented.

Proof:

To prove (a): Suppose (a) holds. Let \( A \subseteq X \) be gsp-closed.

Then, by hypothesis, \( A \) is open. By using Proposition 5.2.9 (e), \( A \) is semi-preclosed. This proves that \( X \) is semi-pre \( T_{1/2} \).

To prove (b): Follows from Theorem 5.2.3 (b).

To prove (c): Follows from Diagram 5.1.12 and Definition 5.1.1.

This completes the proof of the Proposition.

\[ \square \]

Theorem 5.3.6.

For any space \( X \), the following are equivalent

(a) \( X \) is preregular \( T_{1/2} \)

(b) \( X \) is complemented

(c) \( X \) is locally indiscrete
Proof:

Follows from Lemma 5.1.23, Theorem 5.2.3 and Theorem 5.2.18.

5.4. SOME OPEN PROBLEMS

Problem 5.4.1.

Characterize ED spaces using Generalized sets.

Problem 5.4.2.

Theorem 5.2.3 (c) shows that every subset of a complemented space is a p-set. Characterize the conditions under which the converse is true.