Chapter III
Gratzer G and Schmidt E.T. have introduced and characterized the concept of standard ideals in lattices. In this chapter we have introduced and characterized standard filters in lattices.

**Definition 3.1**

A filter $F$ of a lattice $L$ is called a standard filter if and only if

$X \land (F \lor Y) = (X \land F) \lor (X \land Y)$ for all $X, Y$ in $F(L)$

**Example 3.1**

Consider the lattice $N_5$ of the following figure:

![Lattice Diagram]

Take

$F_1 = \{1\} = [1]$  
$F_2 = L = [0]$  
$F_3 = \{a, b, 1\} = [a]$  
$F_4 = \{b, 1\} = [b]$  
$F_5 = \{c, 1\} = [c]$  

Then.
$F_1$, $F_2$ and $F_3$ are standard filters.
$F_5$ is not standard.

**Proposition : 3.1**

Every standard filter is a filter but the converse is not true

**Proof :**

First part follows from the definition

Second part is proved by an example

Consider the lattice $M_1$ of the following figure:

![Diagram of lattice $M_1$]

Take

$$F = \{c\} = \{c, 1\}$$

Then $F$ is a filter but not a standard filter.

Consider $X = \{a\}, Y = \{b\}$

Then

$$X \wedge (F \vee Y) = \{a\} \wedge (\{c\} \vee \{b\}) = \{a\} \wedge (\{c \wedge b\})$$

$$= \{a \vee 0\} = \{a\}$$

$$\text{(X } \wedge F) \vee (X \wedge Y) = (\{a\} \wedge \{c\}) \vee (\{a\} \wedge \{b\})$$

$$= \{a \vee c\} \vee \{a \vee b\}$$

$$= \{(a \vee c) \wedge (a \vee b)\}$$

$$= \{1 \wedge 1\} = \{1\}$$

$$\Rightarrow X \wedge (F \vee Y) \neq (X \wedge F) \vee (X \wedge Y)$$

$$\Rightarrow F \text{ is not standard.}$$
Theorem 3.1: Characterization theorem for standard filters

Let $F$ be a filter of a lattice $L$. Then the following conditions are equivalent:

(i) $F$ is standard

(ii) The binary relation $\theta_F$ on $\mathcal{F}(L)$ is defined by

\[
 X \equiv Y(\theta_F) \iff (X \land Y) \lor F_1 = X \lor Y \text{ for some } F_1 \leq F
\]

is a congruence relation.

(iii) $F$ is a distributive filter and for all $X, Y \in \mathcal{F}(L)$

\[
 F \land X = F \land Y, F \lor X = F \lor Y \text{ imply } X = Y
\]

Proof:

(i) $\Rightarrow$ (ii)
Assume that (i) is true.

To prove (ii), it is sufficient to prove:

1. $\theta_F$ is reflexive
2. $\theta_F$ is symmetric
3. $X \equiv Y(\theta_F) \iff X \land Y \equiv (X \lor Y)(\theta_F)$
4. $X \leq Y \leq Z; X \equiv Y(\theta_F)$ and $Y \equiv Z(\theta_F) \Rightarrow X \equiv Z(\theta_F)$
5. $X \leq Y$ and $X \equiv Y(\theta_F) \Rightarrow X \land Z \equiv (Y \land Z)(\theta_F)$

For $X \lor Z \equiv (Y \lor Z)(\theta_F)$ for all $X, Y, Z \in \mathcal{F}(L)$

(i) $\Rightarrow$ (iii)

Let $X, Y \in \mathcal{F}(L)$ be arbitrary.

Then $(X \land X) \lor F_1 = X \lor X$ for $X = F_1 \leq F$

$\Rightarrow X \equiv X(\theta_F)$

Thus $X \equiv X(\theta_F)$ for all $X \in \mathcal{F}(L)$
For 2:

Let $X, Y \in \mathcal{F}(\mathcal{L})$ be arbitrary

Suppose $X \equiv Y(\theta_f)$

$\Rightarrow (X \land Y) \lor F_1 = X \lor Y$ for some $F_1 \leq F$

$\Rightarrow (Y \land X) \lor F_1 = Y \lor X$ for some $F_1 \leq F$

$\Rightarrow Y \equiv X(\theta_f)$

Thus $X \equiv Y(\theta_f) \Rightarrow Y \equiv X(\theta_f)$ for all $X, Y \in \mathcal{F}(\mathcal{L})$

For 3:

Let $X, Y \in \mathcal{F}(\mathcal{L})$ be arbitrary

Then $X \equiv Y(\theta_f)$

$\Leftrightarrow (X \land Y) \lor F_1 = X \lor Y$ for some $F_1 \leq F$

$\Leftrightarrow [(X \land Y) \land (X \lor Y)] \lor F_1$

$= (X \land Y) \lor (X \lor Y)$ for some $F_1 \leq F$

$\Rightarrow X \land Y \equiv (X \lor Y)(\theta_f)$

Thus $X \equiv Y(\theta_f) \Leftrightarrow (X \land Y) \equiv (X \lor Y)(\theta_f)$ for all $X, Y \in \mathcal{F}(\mathcal{L})$

For 4:

Let $X, Y, Z \in \mathcal{F}(\mathcal{L})$ be arbitrary

Suppose $X \leq Y \leq Z$, $X \equiv Y(\theta_f)$ and $Y \equiv Z(\theta_f)$

$\Rightarrow X \leq Y \leq Z, (X \land Y) \lor F_1 = X \lor Y$ for some $F_1 \leq F$ and

$(Y \land Z) \lor F_2 = Y \lor Z$ for some $F_2 \leq F$

$\Rightarrow X \lor F_1 = Y$ for some $F_1 \leq F$

and $Y \lor F_2 = Z$ for some $F_2 \leq F$

$\Rightarrow X \lor (F_1 \lor F_2) = (X \lor F_1) \lor F_2 = Y \lor F_2 = Z$ with $F_1 \lor F_2 \leq F$

$\Rightarrow (X \land Z) \lor (F_1 \lor F_2) = X \lor Z$ for some $F_1 \lor F_2 \leq F$ since $X \leq Z$

$\Rightarrow X \equiv Z(\theta_f)$

Thus $X \leq Y \leq Z, X \equiv Y(\theta_f)$ and $Y \equiv Z(\theta_f)$

$\Rightarrow X \equiv Z(\theta_f)$ for all $X, Y, Z \in \mathcal{F}(\mathcal{L})$
For 5:

Let $X, Y, Z \in \mathcal{F}(L)$ be arbitrary.

Suppose $X \leq Y$ and $X \equiv Y(\theta_F)$

$\Rightarrow X \leq Y$, $(X \lor Y) \lor F_1 = X \lor Y$ for some $F_1 \leq F$

$\Rightarrow X \lor F_1 = Y$ for some $F_1 \leq F$

$X \leq Y \Rightarrow X \lor Z \leq Y \lor Z$

$\Rightarrow [(X \lor Z) \land (Y \lor Z)] \lor F_1$

$= (X \lor Z) \lor F_1$

$= (X) \lor (Z \lor F_1)$

$= X \lor (F_1 \lor Z)$

$= (X \lor F_1) \lor Z = Y \lor Z$

$= (X \lor Z) \lor (Y \lor Z)$ for some $F_1 \leq F$

$\Rightarrow X \lor Z \equiv (Y \lor Z)(\theta_F)$

$Y \land Z \leq Y = X \lor F_1 \leq X \lor F$ since $F_1 \leq F$

$\Rightarrow Y \land Z = (Y \land Z) \land (X \lor F)$

$= (Y \land Z) \land (F \lor X)$

$= [(Y \land Z) \land F] \lor [(Y \land Z) \land X]$ by (i)

$= [(Y \land Z) \land F] \lor [(Y \land X) \land Z]$

$= [(Y \land Z) \land F] \lor [X \land Z]$ since $Y \land X = X$

$= (X \land Z) \lor [(Y \land Z) \land F]$

$= (X \land Z) \lor F_2$ where $F_2 = (Y \land Z) \land F \leq F$

$\Rightarrow [(X \land Z) \land (Y \land Z)] \lor F_2$

$= (X \land Z) \lor (Y \land Z)$ with $F_2 \leq F$

$\Rightarrow (X \land Z) \equiv (Y \land Z)(\theta_F)$

Thus $X \leq Y$, $X \equiv Y(\theta_F) \Rightarrow X \lor Z \equiv (Y \lor Z) \theta_F$

$X \land Z \equiv (Y \land Z) \theta_F$ for all $X, Y, Z \in \mathcal{F}(L)$

Hence $\theta_F$ is a congruence relation.
(ii) $\Rightarrow$ (iii): Assume (ii) is true

To prove (iii). First we claim that
\[ F \lor (X \land Y) = (F \lor X) \land (F \lor Y) \] for all $X, Y \in \mathcal{F}(L)$

Let $X, Y \in \mathcal{F}(L)$ be arbitrary

Then
\[
\begin{align*}
[X \land (F \lor X)] \lor F &= [(X \land (X \lor F)] \lor F \\
&= X \lor F \\
&= (X \lor F) \lor F \\
&= X \lor (X \lor F) \\
&= X \lor (F \lor X)
\end{align*}
\]
\[
\begin{align*}
[Y \land (F \lor Y)] \lor F &= [(Y \land (Y \lor F)] \lor F \\
&= Y \lor F \\
&= (Y \lor Y) \lor F \\
&= Y \lor (Y \lor F) \\
&= Y \lor (F \lor Y)
\end{align*}
\]

\[\Rightarrow\] $X = (F \lor X) \theta_f$; $Y = (F \lor Y) \theta_f$

\[\Rightarrow\] $X \land Y \equiv [(F \lor X) \land (F \lor Y)] \theta_f$ by (ii)

\[\Rightarrow\] $[(X \land Y) \land [(F \lor X) \land (F \lor Y)]] \lor F$
\[
\begin{align*}
&= (X \land Y) \lor [(F \lor X) \land (F \lor Y)], \text{ by definition of } \theta_f \\
&= (X \land Y) \lor (F \lor Y)
\end{align*}
\]

\[\Rightarrow\] $X \land Y \leq (F \lor X) \land (F \lor Y)$

Thus $F \lor (X \land Y) = (F \lor X) \land (F \lor Y)$ for all $X, Y \in \mathcal{F}(L)$

\[\Rightarrow\] $F$ is a distributive filter.

**Claim**: $F \land X = F \land Y$ and $F \lor X = F \lor Y$ for all $X, Y \in \mathcal{F}(L)$

\[\Rightarrow\] $X = Y$

Suppose $F \land X = F \land Y$ and $F \lor X = F \lor Y$ for all $X, Y \in \mathcal{F}(L)$
We have \( X \equiv (F \lor X) \theta_F \), \( Y \equiv (F \lor Y) \theta_F \).

\[ X \land Y \equiv [(F \lor X) \land (F \lor Y)] \theta_F \text{ by (ii)} \]

\[ = [(F \lor X) \land (F \lor X)] \theta_F \text{ since } F \lor X = F \lor Y \]

\[ = (F \lor X) \theta_F \]

\[ = X(\theta_F) \text{ since } F \lor X \equiv X(\theta_F) \]

\[ \Rightarrow [(X \land Y) \land X] \lor F_1 = [(X \land Y) \lor X] \text{ for some } F_1 \leq F \]

\[ \Rightarrow (X \land Y) \lor F_1 = X \text{ for some } F_1 \leq F \]

Also \( F_1 \leq (X \land Y) \lor F_1 = X, F_1 \leq F \)

\[ \Rightarrow F_1 \leq F \land X \]

\[ \Rightarrow F_1 \leq F \land Y \]

\[ \Rightarrow F_1 \leq Y \]

\[ F_1 \leq X, F_1 \leq Y \Rightarrow F_1 \leq X \land Y \]

\[ \Rightarrow (X \land Y) \lor F_1 = X \land Y \]

\[ \Rightarrow X = X \land Y \]

\[ \Rightarrow X \leq Y \]

\[ \ldots \ldots \ldots (1) \]

Also \( X \equiv (F \lor X) \theta_F \), \( Y \equiv (F \lor Y) \theta_F \).

\[ \Rightarrow X \land Y \equiv [(F \lor X) \land (F \lor Y)](\theta_F) \text{ by substitution property} \]

\[ = (F \lor Y) \land (F \lor Y)(\theta_F) \text{ since } F \lor Y = F \lor X \]

\[ = (F \lor Y) \theta_F \]

\[ = Y(\theta_F) \text{ since } F \lor Y = Y(\theta_F) \]

\[ \Rightarrow Y \equiv (X \land Y)(\theta_F) \]

\[ \Rightarrow [Y \land (X \land Y)] \lor F_2 = Y \lor (X \land Y) \text{ for some } F_2 \leq F \]

\[ \Rightarrow (X \land Y) \lor F_2 = Y \]

Now \( F_2 \leq (X \land Y) \lor F_2 = Y, F_2 \leq F \)

\[ \Rightarrow F_2 \leq F \land X = F \land X \]

\[ \Rightarrow F_2 \leq X \]

\[ F_2 \leq X, F_2 \leq Y \Rightarrow F_2 \leq X \land Y \]
From (1) and (2) we get

\[ X = Y \]

(iii) \( \Rightarrow \) (i); Assume that (iii) is true

To prove (i)

claim: \( X \land (F \lor Y) = (X \land F) \lor (X \land Y) \) for all \( X, Y \in F(L) \)

Let \( X, Y \in F(L) \) be arbitrary

Take \( B = X \land (F \lor Y) \)

\[ C = (X \land F) \lor (X \land Y) \]

To prove \( B = C \)

By (iii) it is enough to prove that

\[ F \land B = F \land C \text{ and } F \lor B = F \lor C \]

We have \( X \land F \leq X, X \land Y \leq Y \)

\[ X \land F \leq X, X \land Y \leq X \]

\[ (X \land F) \lor (X \land Y) \leq X \]

\[ (X \land F) \lor (X \land Y) \leq F \lor Y \]

Therefore

\[ (X \land F) \lor (X \land Y) \leq X \land (F \lor Y) \]

\[ C \leq B \]

\[ F \land C \leq F \land B \]

\[ F \land X \leq F, F \land X \leq (F \land X) \lor (X \land Y) = C \]

\[ F \land X \leq F \land C \leq F \land B = F \land [X \land (F \lor Y)] \]

\[ = [F \land (F \lor Y)] \land X \]

\[ = F \land X \]

\[ F \land B = F \land C \]

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Also \( F \lor B = F \lor [X \land (F \lor Y)] \)
\[ = (F \lor X) \land [(F \lor (F \lor Y)) \land \text{by (iii)}] \]
\[ = (F \lor X) \land [(F \lor F) \lor Y] \]
\[ = (F \lor X) \land (F \lor Y) \cdot \]
\[ = F \lor (X \land Y) \cdot \]
\[ = [F \lor (F \land X)] \lor (X \land Y) \]
\[ = [F \lor (X \land F)] \lor (X \land Y) \]
\[ = F \lor [(X \land F) \lor (X \land Y)] = F \lor C \]

Thus we have
\[ F \land B = F \land C \text{ and } F \lor B = F \lor C \]
\[ \Rightarrow \quad B = C \text{ by (iii)} \]
\[ \Rightarrow \quad X \land (F \lor Y) = (X \land F) \lor (X \land Y) \text{ for all } X, Y \in \mathcal{F}(L) \]

Hence \( F \) is standard.

**Theorem : 3.2**

Every standard filter is a distributive filter but the converse is not true.

**Proof :**

First part follows from the previous theorem.

Next to prove that every distributive filter is not standard.

This is done by an example.

Consider the lattice \( N_5 \) of the following figure:

\[ 1 \]
\[ b \]
\[ a \]
\[ c \]
\[ 0 \]

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Take $F = \{ c \}$

Then $F$ is a distributive filter but not standard.

Consider $X = \{ a, b, 1 \}$ $Y = \{ b, 1 \}$

Then $X \land (F \lor Y) = [a] \land ([c \land b])$

$= [a] \land [0] = [a \lor 0] = [a]$

$(F \land X) \lor (X \land Y) = [a \lor c] \lor [a \lor b]$

$= [1] \lor [b] = [1 \land b] = [b]$

$\Rightarrow X \land (F \lor Y) \neq (X \land F) \lor (X \land Y)$

$\Rightarrow F$ is not standard

Hence every distributive filter need not be standard.

**Theorem : 3.3**

The necessary and sufficient condition for distributive filter $F$ to be a standard filter is that

$F \land X = F \land Y$ and $F \lor X = F \lor Y$ for all $X, Y$ in $F(\mathcal{L})$

implies $X = Y$

**Proof :** Follows from the characterization theorem for standard filters (3.1)

**Theorem : 3.4**

A lattice $L$ is distributive $\Leftrightarrow$ every filter $F$ of $L$ is a standard filter.

**Proof :**

Assuming $L$ is a lattice and every filter of $L$ is a standard filter

To prove that $L$ is distributive.

$F$ is a standard filter in a lattice $L$

$\Rightarrow F$ is distributive filter by characterization theorem

$\Rightarrow F \lor (X \land Y) = (F \lor X) \land (F \lor Y)$ for all $X, Y \in F(\mathcal{L})$

$\Rightarrow F(\mathcal{L})$ is distributive.
Conversely, assume that a lattice \( L \) is a distributive lattice and \( F \) is a filter of \( L \).

To prove that \( F \) is a standard filter, \( L \) is a distributive lattice:

\[ \Rightarrow \quad \mathcal{F}(L) \text{ is distributive lattice} \]

\[ = \quad \text{every element in } \mathcal{F}(L) \text{ is standard, since } \mathcal{F}(L) \text{ does not contain } N_5 \text{ or } M_3 \]

\[ \Rightarrow \quad \text{every filter } F \text{ of } L \text{ is a standard filter.} \]

\textbf{Theorem 3.5}

If \( F_1 \) and \( F_2 \) are standard filters of a lattice \( L \), then \( F_1 \lor F_2 \), \( F_1 \land F_2 \) are standard filters of \( L \).

\textbf{Proof:}

Given \( F_1 \) and \( F_2 \) are standard filters of a lattice \( L \).

To prove \( F_1 \lor F_2 \), \( F_1 \land F_2 \) are standard filters of \( L \).

Let \( X, Y \in \mathcal{F}(L) \) be arbitrary.

Then \( X \land [(F_1 \lor F_2) \lor Y] = X \land [F_1 \lor (F_2 \lor Y)] \)

\[ = (X \land F_1) \lor [X \land (F_2 \lor Y)], \text{ since } F_1 \text{ is standard} \]

\[ = (X \land F_1) \lor [(X \land F_2) \lor (X \land Y)], \text{ since } F_2 \text{ is standard} \]

\[ = [(X \land F_1) \lor (X \land F_2)] \lor (X \land Y) \]

\[ = [X \land (F_1 \lor F_2)] \lor (X \land Y), \text{ since } F_1 \text{ is standard for all } X,Y \in \mathcal{F}(L) \]

\[ \Rightarrow \quad F_1 \lor F_2 \text{ is standard} \]

Given \( F_1 \) and \( F_2 \) are standard filters.

To prove \( F_1 \land F_2 \) is a standard filter.

By characterization theorem it is sufficient to prove that

\[ \theta_{F_1 \land F_2} = \theta_{F_1} \land \theta_{F_2} \]

Let \( (X,Y) \in \theta_{F_1 \land F_2} \) be arbitrary.
\[ \Rightarrow X \equiv Y (\theta_{_{F_1}} \wedge \theta_{_{F_2}}) \]

\[ \Rightarrow (X \land Y) \lor F = X \lor Y \text{ for some } F \subseteq F_1 \land F_2 \]

\[ \Rightarrow (X \land Y) \lor F = X \lor Y \text{ for some } F \subseteq F_1 \]

and 

\[ (X \land Y) \lor F = X \lor Y \text{ for some } F \subseteq F_2 \]

\[ \Rightarrow X \equiv Y(\theta_{_{F_1}}) \text{ and } X \equiv Y(\theta_{_{F_2}}) \]

\[ \Rightarrow X \equiv Y(\theta_{_{F_1}} \land \theta_{_{F_2}}) \]

\[ (X,Y) \in \theta_{_{F_1}} \land \theta_{_{F_2}} \]

Therefore 

\[ \theta_{_{F_1}} \land \theta_{_{F_2}} \subseteq \theta_{_{F_1}} \land \theta_{_{F_2}} \]

\[ \text{.........(1)} \]

Also let 

\[ (X,Y) \in \theta_{_{F_1}} \land \theta_{_{F_2}} \]

be arbitrary

\[ \Rightarrow (X,Y) \in \theta_{_{F_1}} \text{ and } (X,Y) \in \theta_{_{F_2}} \]

\[ \Rightarrow X \equiv Y(\theta_{_{F_1}}) \text{ and } X \equiv Y(\theta_{_{F_2}}) \]

\[ \Rightarrow (X \land Y) \lor F_3 = X \lor Y \text{ for } F_3 \subseteq F_1 \]

\[ (X \land Y) \lor F_4 = X \lor Y \text{ for } F_4 \subseteq F_2 \]

\[ \Rightarrow (X \land Y) \lor (F_3 \lor F_4) = X \lor Y \]

\[ X \equiv Y(\theta_{_{F_1}}) \Rightarrow (X \land Y) \lor F_3 = X \lor Y \text{ for some } F_3 \leq F_1 \]

\[ \text{.........(1)} \]

We have

\[ F_3 = (F_3 \land (F \lor (X \land Y))) \] by absorption law

\[ = F_3 \land [(X \land Y) \lor F] \]

\[ = F_3 \land [X \lor Y], \text{ by (1)} \]

\[ = F_3 \land (X \land Y) \theta_{_{F_2}}, \text{ since } X \equiv Y(\theta_{_{F_2}}) \]

\[ \text{iff } X \land Y \equiv (X \lor Y) \theta_{_{F_2}} \]

\[ \text{iff } X \lor Y \equiv (X \land Y) \theta_{_{F_2}} \]

\[ \Rightarrow [F_3 \land [F_3 \land (X \land Y)]] \lor F_4 \]

\[ = [F_3 \lor [F_3 \land (X \land Y)]] \text{ for some } F_4 \leq F_2 \]

\[ \Rightarrow [F_3 \land (X \land Y)] \land F_4 = F_3 \text{ for some } F_4 \subseteq F_2 \]

\[ \text{.........(2)} \]
Now \((X \land Y) \lor F_4 = [(X \land Y) \lor [(X \land Y) \land F_3] \lor F_4],\) by absorption law
\[= (X \land Y) \lor [(X \land Y) \land F_3] \lor F_4,\] by (2)
\[= (X \land Y) \lor F_3,\] by (1)
\[= X \lor Y,\] by (1)
\[F_4 \leq F_2; F_4 \leq F_3 \leq F_1\]
\[\Rightarrow F_4 \leq F_1 \land F_2\]
Thus \((X \land Y) \lor F_4 = X \lor Y\) with \(F_4 \leq F_1 \land F_2\)
\[\Rightarrow X \equiv Y(\theta_{F_1 \land F_2})\]
\[\Rightarrow (X, Y) \in \theta_{F_1 \land F_2}\]
\[\therefore \theta_{F_1 \land F_2} \subseteq \theta_{F_1 \land F_2},\ldots (11)\]

From (1) and (11)
\[\theta_{F_1 \land F_2} = \theta_{F_1 \land F_2}\] is a congruence relation
\[\Rightarrow F_1 \land F_2\] is a standard filter

**Theorem : 3.6**
Set of all standard filters of a lattice \(L\) for a sublattice of \(\mathcal{F}(L)\)

**Proof :**

Let \(S(L)\) denote set of all standard filters.

To prove that \(S(L)\) is a sublattice of \(\mathcal{F}(L)\)

Clearly \(S(L) \neq \phi, S(L) \subseteq \mathcal{F}(L)\), since \(L\) itself a standard filter.

Let \(F_1, F_2 \in S(L)\)
\[\Rightarrow F_1, F_2\] are standard filters of \(L\)
\[\Rightarrow F_1 \lor F_2, F_1 \land F_2\] are standard filters of \(L\)
\[\Rightarrow F_1 \lor F_2, F_1 \land F_2 \in S(L)\]
\[\Rightarrow S(L)\] is a sublattice of \(\mathcal{F}(L)\)
**Definition : 3.2**

A filter $F$ of lattice $L$ is called dually standard filter if

$$X \lor (F \land Y) = (X \lor F) \land (X \lor Y)$$

for all $X, Y \in \mathcal{F}(L)$

**Example : 3.2**

Consider the lattice

Here

$$F_1 = \{0\} = L$$

$$F_2 = \{a\} = \{a, c, 1\}$$

$$F_3 = \{b\} = \{b, c, 1\}$$

$$F_4 = \{c\} = \{c, 1\}$$

$$F_5 = \{1\} = \{1\}$$

Then $F_1, F_2, F_3, F_4$ and $F_5$ are dually standard filters.

**Proposition : 3.2**

Every dually standard filter is a filter but the converse is not true.

**Proof :**

The first part is trivial and follows from the definition.

Next to prove that a filter need not be a dually standard filter.

This is done with an example.
Consider the lattice given in the following figure:

Take \( F = \{ b \} = \{ b, 1 \} \)
Then \( F \) is a filter but not dually standard.
Consider \( X = \{ d, 1 \} \)
\( Y = \{ c, d, 1 \} \)
Then \( X \lor (F \land Y) = \{ d \} \lor (\{ b \} \land \{ c \}) \]
\[= \{ d \} \lor \{ b \land c \} \]
\[= \{ d \land (b \lor c) \} \]
\[= \{ d \} \land \{ 1 \} = \{ d \} \]
\((X \lor F) \land (X \lor Y) = (\{ d \} \lor \{ b \}) \land (\{ d \} \lor \{ c \}) \]
\[= \{ d \land b \} \land \{ d \land c \} \]
\[= \{ 0 \} \land \{ c \} = \{ 0 \lor c \} = \{ c \} \]
\[\Rightarrow \] \( X \lor (F \land Y) \neq (X \lor F) \land (X \lor Y) \)
\[\Rightarrow \] \( F \) is not dually standard.
Theorem : 3.7

If $F_1$ and $F_2$ are dually standard filters of a lattice $L$, then $F_1 \lor F_2$, $F_1 \land F_2$ are dually standard filters of $L$.

Proof:

First to prove that $F_1 \land F_2$ is a dually standard filter.

Let $X, Y \in \mathcal{F}(L)$ be arbitrary.

Then $X \lor [(F_1 \land F_2) \land Y] = X \lor [F_1 \land (F_2 \land Y)]$

$= (X \lor F_1) \land [X \lor (F_2 \land Y)]$,

since $F_1$ is Dually standard.

$= (X \lor F_1) \land [(X \lor F_2) \land (X \lor Y)]$,

since $F_2$ Dually standard.

$= [(X \lor F_1) \land (X \lor F_2)] \land (X \lor Y)$

$= [X \lor (F_1 \land F_2)] \land (X \lor Y)$,

since $F_1$ is dually Standard.

Thus $X \lor [(F_1 \land F_2) \land Y] = [X \lor (F_1 \land F_2)] \land (X \lor Y)$

for all $X, Y \in \mathcal{F}(L)$.

$\Rightarrow F_1 \land F_2$ is dually standard.

Next to prove $F_1 \lor F_2$ is a dually standard filter.

Let $X, Y \in \mathcal{F}(L)$ be arbitrary.

Then $(F_1 \lor F_2) \land Y \subseteq F_1 \lor F_2$

$(F_1 \lor F_2) \land Y \subseteq Y$

$\Rightarrow X \lor [(F_1 \lor F_2) \land Y] \subseteq X \lor (F_1 \lor F_2)$

$X \lor [(F_1 \lor F_2) \land Y] \subseteq X \lor Y$

$\Rightarrow X \lor [(F_1 \lor F_2) \land Y] \subseteq [X \lor (F_1 \lor F_2)] \land (X \lor Y)$ ...(1)

Let $t \in [X \lor (F_1 \lor F_2)] \land (X \lor Y)$ be arbitrary,
\[ \Rightarrow \quad t \in X \lor (F_1 \lor F_2) \quad \text{and} \quad t \in X \lor Y \]
\[ \Rightarrow \quad t \geq x \land f \quad \text{for some} \quad x \in X, \quad f \in F_1 \lor F_2 \]
\[ \quad \text{and} \quad f \geq x \land y \quad \text{for some} \quad x \in X, \quad y \in Y \]
\[ \Rightarrow \quad t \geq x \land f \quad \text{for some} \quad x \in X, \quad f = y \in (F_1 \lor F_2) \land Y \]
\[ \Rightarrow \quad t \in X \lor [(F_1 \lor F_2) \land Y] \]

Therefore, \[ X \lor [(F_1 \lor F_2) \land Y] = [X \lor (F_1 \lor F_2)] \land [X \lor Y] \] \( \ldots \) (2)

From (1) and (2) we have
\[ X \lor [(F_1 \lor F_2) \land Y] = [X \lor (F_1 \lor F_2)] \land [X \lor Y] \quad \text{for all} \quad X, \quad Y \in \mathcal{F}(L) \]
\[ \Rightarrow \quad F_1 \lor F_2 \text{ is dually standard.} \]

**Theorem 3.8** Characterization theorem for Dually Standard Filters

Let \( F \) be a filter of a lattice \( L \). Then the following conditions are equivalent.

1. \( F \) is dually standard
2. The binary relation \( \theta_F \) on \( \mathcal{F}(L) \) is defined by
   \[ "X \equiv Y (\theta_F) \] if and only if \( (X \lor Y) \land F_1 = X \land Y \) for some \( F_1 \geq F \text{ where } X, \quad Y \in \mathcal{F}(L) " \) is a congruence relation
3. \( F \) is dually distributive and for all \( x, \quad Y \in \mathcal{F}(L) \)
   \[ F \land X = F \land Y, \quad F \lor X = F \lor Y \text{ implies } X = Y \]

**Proof:**

(i) \( \Rightarrow \) (ii)

Assume that (i) is true

To prove (ii)

It is sufficient to prove

1. \( X \equiv X (\theta_F) \)
2. \( X \equiv Y(\theta_F) \Rightarrow Y \equiv X(\theta_F) \)
(3) \( X \equiv Y(\theta_F) \) iff \( X \land Y \equiv X \lor Y(\theta_F) \)

(4) \( X \leq Y \leq Z \), \( X \equiv Y(\theta_F) \) and \( Y \equiv Z(\theta_F) \) \( \Rightarrow \) \( X \equiv Z(\theta_F) \)

(5) \( X \leq Y \) and \( X \equiv Y(\theta_F) \)
\[ \Rightarrow \quad X \land Z \equiv Y \land Z(\theta_F) \]
\[ \Rightarrow \quad X \lor Z \equiv Y \lor Z(\theta_F) \] for all \( X, Y, Z \in \mathcal{F}(L) \)

For (1):

Let \( X \in \mathcal{F}(L) \) be arbitrary

Then \( (X \lor X) \land F_1 = X \land X \) for \( X = F_1 \geq F \)
\[ \Rightarrow \quad X \equiv X(\theta_F) \]

Thus \( X \equiv X(\theta_F) \) for all \( X \in \mathcal{F}(L) \)

For (2):

Let \( X, Y \in \mathcal{F}(L) \) be arbitrary

Suppose \( X \equiv Y(\theta_F) \)
\[ \Rightarrow \quad (X \lor Y) \land F_1 = X \land Y \text{ for some } F_1 \geq F \]
\[ \Rightarrow \quad (Y \lor X) \land F_1 = Y \land X \text{ for some } F_1 \geq F \]
\[ \Rightarrow \quad Y \equiv X(\theta_F) \]

Thus \( X \equiv Y(\theta_F) \) \( \Rightarrow \) \( Y \equiv X(\theta_F) \) for all \( X, Y \in \mathcal{F}(L) \)

For (3):

Let \( X, Y \in \mathcal{F}(L) \) be arbitrary

Then \( X \equiv Y(\theta_F) \)
\[ \Rightarrow \quad (X \lor Y) \land F_1 = X \land Y \text{ for some } F_1 \geq F \]
\[ \Rightarrow \quad [(X \land Y) \lor (X \lor Y)] \land F_1 = (X \land Y) \land (X \lor Y) \]
\[ \text{for some } F_1 \geq F \]
\[ \Rightarrow \quad X \land Y \equiv (X \lor Y)(\theta_F) \]

Thus \( X \equiv Y(\theta_F) \) iff \( X \land Y \equiv (X \lor Y)(\theta_F) \), for all \( X, Y \in \mathcal{F}(L) \)
For (4):

Let \( X, Y, Z \in \mathcal{F}(\mathcal{L}) \) be arbitrary.

Suppose \( X \leq Y \leq Z, \ X \equiv Y(\theta_f) \) and \( Y \equiv Z(\theta_f) \)

\[ \Rightarrow \quad X \leq Y \leq Z \]

\[(X \lor Y) \land F_1 = X \land Y \quad \text{for some } F_1 \geq F \quad \text{and} \]

\[(Y \lor Z) \land F_2 = Y \land Z \quad \text{for some } F_2 \geq F \]

\[ \Rightarrow \quad (Y \land F_1) = X \quad \text{for some } F_1 \geq F \quad \text{and} \]

\[ Z \land F_2 = Y \quad \text{for some } F_2 \geq F \]

\[ \Rightarrow \quad Z \land F_3 = Z \land (F_2 \land F_1) \quad \text{where } F_3 = F_2 \land F_1 \]

\[ = (Z \land F_2) \land F_1 \]

\[ = Y \land F_1 \]

\[ = X \quad \text{with } F_3 = F_2 \land F_1 \geq F \]

\[ \Rightarrow \quad (X \lor Z) \land F_3 = (X \land Z) \quad \text{with } F_3 \geq F \]

\[ \Rightarrow \quad X \equiv Z(\theta_f) \]

Thus \( X \leq Y \leq Z, X \equiv Y(\theta_f) \) and \( Y \equiv Z(\theta_f) \) \Rightarrow \( X \equiv Z(\theta_f) \) for all \( X, Y, Z \in \mathcal{F}(\mathcal{L}) \).

For (5):

Let \( X, Y, Z \in \mathcal{F}(\mathcal{L}) \) be arbitrary.

Suppose \( X \leq Y \) and \( X \equiv Y(\theta_f) \)

\[ \Rightarrow \quad X \leq Y \quad \text{and } (X \lor Y) \land F_1 = X \land Y \quad \text{for some } F_1 \geq F \]

\[ \Rightarrow \quad Y \land F_1 = X \quad \text{for some } F_1 \geq F \]

\[ X \leq Y \quad \Rightarrow \quad X \land Z \leq Y \land Z, \ X \lor Z \leq Y \lor Z \]

\[ \Rightarrow \quad [(X \land Z) \lor (Y \land Z)] \land F_1 = (Y \land Z) \land F_1 \]

\[ = (Z \land Y) \land F_1 = Z \land (Y \land F_1) = Z \land X = X \land Z \]

\[ = (X \land Z) \land (Y \land Z) \]

\[ \Rightarrow \quad X \land Z = (Y \land Z)(\theta_f) \]
We have \( X \lor Z \geq X = Y \land F \), \( X \lor Z \geq Y \land F \)

\[ X \lor Z = (X \lor Z) \lor (Y \land F) \]

\[ = (X \lor Z) \lor (F \land Y) \]

\[ = [(X \lor Z) \lor F] \land [(X \lor Z) \lor Y], \text{ since } F \text{ is dually standard}. \]

\[ = [(X \lor Z) \lor F] \land [(X \lor Y) \lor Z] \]

\[ = (Y \lor Z) \land [(X \lor Z) \lor F], \text{ since } X \leq Y \]

\[ = (Y \lor Z) \land F_2, \text{ where } F_2 = (X \lor Z) \lor F \geq F \]

\[ \Rightarrow [(X \lor Z) \lor (Y \lor Z)] \land F_2 = (X \lor Z) \land (Y \lor Z), \text{ for some } F_2 \geq F \]

\[ \Rightarrow X \lor Z = (Y \lor Z)(\theta_F) \]

Thus \( X \leq Y \) and \( X \equiv Y(\theta_F) \)

\[ \Rightarrow X \lor Z = Y \lor Z(\theta_F) \]

\[ X \land Z = Y \land Z(\theta_F) \text{ for all } X, Y, Z \in F(L) \]

Hence \( \theta_F \) is a congruence relation.

(ii) \( \Rightarrow \) (iii) Assume that (ii) is true.

To prove (iii),

First we claim that \( F \) is a dually distributive filter

Let \( X, Y \in F(L) \) be arbitrary.

We have \[ [(X \lor (X \land F)) \land F = X \land F \]

\[ = (X \land X) \land F, \text{ by absorption law} \]

\[ = X \land (X \land F) \]

\[ \Rightarrow X = (X \land F)(\theta_F) \]

Also \[ [(Y \lor (Y \land F)) \land F = Y \land F \]

\[ = (Y \land Y) \land F \]

\[ = Y \land (Y \land F) \]

\[ \Rightarrow Y = (Y \land F)(\theta_F) \]

Therefore \( X \lor Y = [(X \land F) \lor (Y \land F)](\theta_F), \text{ by substitution property} \)
\[ [(X \lor Y) \lor [(X \land F) \lor (Y \land F)] \land F \]
\[ = (X \lor Y) \lor [(X \land F) \lor (Y \land F)], \text{ by definition of } \theta_f \]
\[ \Rightarrow (X \lor Y) \land F = (X \land F) \lor (Y \land F), \]
\[ \text{since } X \lor Y \geq (X \land F) \lor (Y \land F) \]
\[ \Rightarrow F \land (X \lor Y) = (F \land X) \lor (F \land Y), \text{ for all } X, Y \in \mathcal{F}(L) \]
\[ \Rightarrow F \text{ is a dually distributive filter.} \]

**Claim:** \( F \land X = F \land Y \) and \( F \lor X = F \lor Y \) for all \( X, Y \in \mathcal{F}(L) \)

\[ \Rightarrow X = Y \]

We have \( X = (F \land X)(\theta_f) \)
\( Y = (F \land Y)(\theta_f) \)

\[ \Rightarrow X \lor Y = [(F \land X) \lor (F \land Y)](\theta_f), \text{ by substitution property} \]
\[ = [(F \land Y) \lor (F \land Y)](\theta_f), \text{ since } F \land X = F \land Y \]
\[ = (F \land Y)(\theta_f) \]
\[ \Rightarrow X \lor Y = Y(\theta_f), \text{ since } Y = (F \land Y)(\theta_f) \]
\[ \Rightarrow [(X \lor Y) \lor Y] \land F_1 = (X \lor Y) \land Y, \text{ for some } F_1 \geq F \]
\[ \Rightarrow (X \lor Y) \land F_1 = Y, \text{ for some } F_1 \geq F \]
\[ F_1 \geq F, \ F_1 \geq (X \lor Y) \land F_1 = Y \]
\[ \Rightarrow F_1 \geq F \land Y = F \lor X \geq X \]

\( F_1 \geq X, \ F_1 \geq Y \)

\[ \Rightarrow F_1 \geq X \lor Y \]
\[ \Rightarrow (X \lor Y) \land F_1 = X \lor Y \]
\[ \Rightarrow Y = X \lor Y \]
\[ \Rightarrow X \leq Y \quad \text{.................. (1)} \]

Also \( Y = (F \land Y)(\theta_f) \)
\( X = (F \land X)(\theta_f) \)

\[ \Rightarrow Y \lor X = [(F \lor Y) \lor (F \lor X)](\theta_f), \text{ by substitution property} \]
\[ = [(F \land X) \lor (F \land X)](\theta_f), \text{ since } F \land Y = F \land X \]
\[ = (F \land X)(\theta_f) \]

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\[ Y \lor X = X(0) \), since \( X = (F \land X)(0) \)

\[ (Y \lor X) \land F_i = (Y \lor X) \land X, \text{ for some } F_i \geq F \]

\[ (Y \lor X) \land F_i = X, \text{ for some } F_i \geq F \]

If \( F_i \geq F \) and \( F_i \geq X \)

\[ F_i \geq F \lor X = F \lor Y \geq Y \]

If \( F_i \geq Y, F_i \geq X \)

\[ F_i \geq Y \lor X \]

\[ (Y \lor X) \land F_i = Y \lor X \]

\[ X = X \lor Y \]

\[ Y \leq X \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2) \]

From (1) and (2) we have that

\[ X = Y \]

(iii) \( \Rightarrow \) (i) : Assume that (iii) is true

To prove (i)

We claim that \( X \lor (F \land Y) = (X \lor F) \land (X \lor Y) \), for all \( X, Y \in \mathcal{F} \). Let \( X, Y \in \mathcal{F} \) be arbitrary

By (iii) it is sufficient to prove that

\[ F \lor F_1 = F \lor F_2 \quad \text{and} \quad F \land F_1 = F \land F_2 \]

where \( F_1 = X \lor (F \land Y), F_2 = (X \lor F) \land (X \lor Y) \)

\[ F \land F_1 = F \land [X \lor (F \land Y)] \]

\[ = (F \land X) \lor [(F \land (F \land Y)) \land (F \land Y)], \text{ by (iii)} \]

\[ = (F \land X) \lor (F \land Y) \]

\[ F \land F_2 = F \land [(X \lor F) \land (X \lor Y)] \]

\[ = [F \land (X \lor F)] \land (X \lor Y), \text{ by associative law} \]

\[ = F \land (X \lor Y) \]

\[ = (F \land X) \lor (F \land Y), \text{ by (iii)} \]

\[ X \leq X \lor F, \quad X \leq X \lor Y \]
\[ \Rightarrow \quad X \leq (X \lor F) \land (X \lor Y) \]

\[ F \leq X \lor F, \quad Y \leq X \lor Y \]

\[ \Rightarrow \quad F \land Y \leq (X \lor F) \land (X \lor Y) \]

Therefore

\[ X \lor (F \land Y) \leq (X \lor F) \land (X \lor Y) \]

\[ \Rightarrow \quad F_1 \leq F_2 \]

\[ \Rightarrow \quad F \land F_1 \leq F \land F_2 \]

\[ F \lor X \geq (F \lor X) \land (X \lor Y) = F_2, \quad F \lor X \geq F \]

\[ \Rightarrow \quad F \lor X \geq F \lor F_2 \]

\[ \geq F \lor F_1 \]

\[ = F \lor [X \lor (F \land Y)] \]

\[ = F \lor [(F \land Y) \lor X] = [F \lor (F \land Y)] \lor X \]

\[ = F \lor X \]

\[ \Rightarrow \quad F \lor F_1 = F \lor F_2 \]

Thus

\[ F \lor F_1 = F \lor F_2 \]

\[ F \land F_1 = F \land F_2 \]

\[ \Rightarrow \quad F_1 = F_2 \quad \text{by (iii)} \]

\[ \Rightarrow \quad X \lor (F \land Y) = (X \lor F) \land (X \lor Y) \quad \text{for all} \quad X, \quad Y \in F(L) \]

D is dually standard.

Theorem: 3.9

A lattice \( L \) is distributive ⇔ every filter of \( L \) is a dually standard filter.

Proof:

Assume that \( L \) is a lattice and every filter \( F \) of \( L \) is dually standard.

To prove that \( L \) is distributive.

\( F \) is dually standard filter in a lattice \( L \)

\[ \Rightarrow \quad F \text{ is dually distributive and } F(L) \text{ is a lattice for all } F \in F(L) \]
\[ F \land (X \lor Y) = (F \land X) \lor (F \land Y) \text{ for all } F, X, Y \in \mathcal{F}(L) \]

\[ (F \lor X) \land (F \lor Y) = \left[ (F \lor X) \land F \right] \lor \left[ (F \lor X) \land Y \right] \]
\[ = [F \land (F \lor X)] \lor [Y \land (F \lor X)] \]
\[ = F \lor [Y \land (F \lor X)] \]
\[ = F \lor [(Y \land F) \lor (Y \land X)] \]
\[ = [F \lor (Y \land F)] \lor (Y \land X) \]
\[ = [F \lor (F \land Y)] \lor (X \land Y) \]
\[ = F \lor (X \land Y) \text{ for all } F, X, Y \in \mathcal{F}(L) \]

\[ \Rightarrow \quad \mathcal{F}(L) \text{ is a distributive lattice} \]

\[ \Rightarrow \quad L \text{ is a distributive lattice.} \]

conversely, assume that a lattice \( L \) is distributive and \( F \) is filter of \( L \)

\[ \Rightarrow \quad \mathcal{F}(L) \text{ is a distributive lattice and } F \text{ is a filter of } L \]

\[ \Rightarrow \quad F \text{ is dually standard.} \]

**Theorem 3.10**

Every dually standard filter in a lattice is a dually distributive filter but the converse is not true.

**Proof:**

First part follows from the characterization theorem for dually standard filter.
Second part is proved by an example
Consider the lattice $L$.

$\text{Take } F = \{ b \} = \{ b, 1 \}$

Then $F$ is a dually distributive filter but not dually standard.

Consider $X = \{ d, 1 \}$

$Y = \{ c, d, 1 \}$

Then $X \lor (F \land Y) = [d] \lor ([b] \land [c])$

$= [d] \lor ([b] \lor [c])$

$= [d] \lor [1]$

$= [d \land 1]$

$= [d]$.

$(X \lor F) \land (X \lor Y) = ([d] \lor [b]) \land ([d] \lor [c])$

$= [d \land b] \land [d \land c]$

$= [0] \land [c]$

$= [c]$

$\Rightarrow X \lor (F \land Y) \neq (X \lor F) \land (X \lor Y)$

$\Rightarrow F$ is not dually standard.
THEOREM : 3.11

The necessary and sufficient condition for the dually distributive filter \( F \) to be dually standard in a lattice is that
\[
F \cap X = F \cap Y \\
F \lor X = F \lor Y \quad \text{for all } X, Y \in \mathcal{F}(\mathcal{L}) \implies X = Y
\]

Proof:

By characterization theorem for dually standard filters. ( theorem 3.8 )

Proposition : 3.3

Standard filter is not a dually standard filter in a lattice \( L \).

Proof:

By an example

Consider the lattice \( L \) in the following figure:

\[
\begin{array}{c}
\emptyset \\
\downarrow & \\
a & b & c \\
\downarrow & \\
1 \\
\end{array}
\]

Take \( F = \{ a \} = \{ a, b, 1 \} \)

This is a standard filter but not dually standard.

Consider \( X = \{ b \} \)

\( Y = \{ c \} \)

Then \( X \lor ( F \cap Y ) = \{ b \} \)
Proposition 3.4

Dually standard filter is not a standard filter in a lattice $L$.

Proof: By an example

Consider the lattice $N_5$ in the following figure:

Take $F = \{c\} = \{c, 1\}$

Then $F$ is dually standard but not standard.

Consider $X = \{a\}$ and $Y = \{b\}$. Then

$X \lor (F \land Y) = \{a\}$

$(X \lor F) \land (X \lor Y) = \{0\}$

$\Rightarrow X \lor (F \land Y) \neq (X \lor F) \land (X \lor Y)$

$\Rightarrow F$ is not standard
Theorem : 3.12

If \( L \) is a distributive lattice and \( F \) a filter of \( L \) then the following are equivalent:

(i) \( F \) is a standard filter

(ii) \( F \) is a dually standard filter

Proof:

(i) \( \Rightarrow \) (ii)

Assume that \( L \) is a distributive lattice and \( F \) is a standard filter.

\[
\Rightarrow \quad \mathcal{F}(L) \text{ is a distributive lattice and } \\
\quad X \land (F \lor Y) = (X \land F) \lor (X \land Y) \quad \text{for all } X, Y \in \mathcal{F}(L) \\
\Rightarrow \quad X \lor (F \land Y) = (X \lor F) \land (X \lor Y) \quad \text{for all } X, Y \in \mathcal{F}(L) \\
\Rightarrow \quad F \text{ is a dually standard filter.}
\]

(ii) \( \Rightarrow \) (i)

Assume that \( L \) is a distributive lattice and \( F \) is a dually standard filter.

\[
\Rightarrow \quad \mathcal{F}(L) \text{ is a distributive lattice and } \\
\quad X \lor (F \land Y) = (X \lor F) \land (X \lor Y) \quad \text{for all } F, X, Y \in \mathcal{F}(L) \\
\Rightarrow \quad X \land (F \lor Y) = (X \land F) \lor (X \land Y) \quad \text{for all } F, X, Y \in \mathcal{F}(L) \\
\Rightarrow \quad F \text{ is a standard filter.}
\]

Theorem : 3.13

If \( L \) and \( L_1 \) are two lattices, \( \phi : L \rightarrow L_1 \) is an onto homomorphism with \( F \) a standard filter of \( L \) and \( \theta_F \) defined by " \( x \equiv y(\theta_F) \) iff \( \phi(x) = \phi(y) \) " then

(i) \( \theta_F \) is a congruence relation on \( L \)

(ii) \( L / \theta_F \) is a lattice

(iii) \( L / \theta_F \cong L_1 \)
Proof:

$\theta_F$ is reflexive;

Let $x \in L$ be arbitrary

Then $\phi(x) = \phi(x)$ for all $x \in L$

$\Rightarrow \quad x = x(\theta_F)$

$\Rightarrow \quad$ Thus $x = x(\theta_F)$ for all $x \in L$

$\theta_F$ is symmetric;

Let $x, y \in L$ be arbitrary

Suppose $x = y(\theta_F)$

$\Rightarrow \quad \phi(x) = \phi(y)$

$\Rightarrow \quad \phi(y) = \phi(x)$

$\Rightarrow \quad y = x(\theta_F)$

Thus $x = y(\theta_F)$ implies $y = x(\theta_F)$ for all $x, y \in L$

$\theta_F$ is transitive;

Let $x, y, z \in L$ be arbitrary

Suppose $x = y(\theta_F)$ and $y = z(\theta_F)$

$\Rightarrow \quad \phi(x) = \phi(y)$ and $\phi(y) = \phi(z)$

$\Rightarrow \quad \phi(x) = \phi(z)$

$\Rightarrow \quad x = z(\theta_F)$

Thus $x = y(\theta_F)$ and $y = z(\theta_F) \Rightarrow x = z(\theta_F)$ for all $x, y, z \in L$

Substitution property

Let $x, y, x_1, y_1 \in L$ be arbitrary

Suppose $x = x_1(\theta_F)$ and $y = y_1(\theta_F)$

$\Rightarrow \quad \phi(x) = \phi(x_1)$ and $\phi(y) = \phi(y_1)$

$\Rightarrow \quad \phi(x \lor y) = \phi(x) \lor \phi(y) = \phi(x_1) \lor \phi(y_1)$
\[ \phi(x_1 \lor y_1) = \phi(x) \land \phi(y) = \phi(x_1) \land \phi(y_1) = \phi(x_1 \land y_1) \]

\[ \Rightarrow x \lor y = (x_1 \lor y_1)(\theta_F) \text{ and } x \land y = (x_1 \land y_1)(\theta_F) \]

Thus \( x \equiv x_1(\theta_F) \) and \( y \equiv y_1(\theta_F) \)

\[ \Rightarrow x \lor y \equiv (x_1 \lor y_1)(\theta_F) \]

\[ x \land y \equiv (x_1 \land y_1)(\theta_F) \text{ for all } x, y, x_1, y_1 \in L \]

Therefore \( \theta_F \) is a congruence relation.

**For (ii)**

Define \( L / \theta_F = \{ [x] \theta_F : x \in L \} \)

To prove \( L / \theta_F \) is a lattice.

Define \( \lor, \land \) on \( L / \theta_F \) by

\[ [x] \theta_F \lor [y] \theta_F = [x \lor y] \theta_F \]

\[ [x] \theta_F \land [y] \theta_F = [x \land y] \theta_F \]

where \([x] \theta_F, [y] \theta_F \) in \( L / \theta_F \)

**Idempotent Law :**

Let \([x] \theta_F \) in \( L / \theta_F \) be arbitrary.

Then \([x] \theta_F \lor [x] \theta_F = [x \lor x] \theta_F = [x] \theta_F \]

\([x] \theta_F \land [x] \theta_F = [x \land x] \theta_F = [x] \theta_F \]

Thus \([x] \theta_F \lor [x] \theta_F = [x] \theta_F \]

\([x] \theta_F \land [x] \theta_F = [x] \theta_F \text{ for all } [x] \theta_F \text{ in } L / \theta_F \)
Absorption Laws:

Let $[x]_{\theta_F}, [y]_{\theta_F} \in L / \theta_F$ be arbitrary.

$\Rightarrow x, y \in L$

$\Rightarrow x \lor (x \land y) = x, x \land (x \lor y) = x$

Then $[x]_{\theta_F} \lor ([x]_{\theta_F} \land [y]_{\theta_F}) = ([x]_{\theta_F}) \lor ([x \land y]_{\theta_F})$

$= [x \lor (x \land y)]_{\theta_F}$

$= [x]_{\theta_F}$

$( [x]_{\theta_F} ) \land ( [x]_{\theta_F} \lor [y]_{\theta_F} ) = ( [x]_{\theta_F} ) \land ( [x \lor y]_{\theta_F} )$

$= [x \land (x \lor y)]_{\theta_F}$

$= [x]_{\theta_F}$

Thus $[x]_{\theta_F} \lor ([x]_{\theta_F} \land [y]_{\theta_F}) = [x]_{\theta_F}$

$[x]_{\theta_F} \land ([x]_{\theta_F} \lor [y]_{\theta_F}) = [x]_{\theta_F}$

for all $[x]_{\theta_F}, [y]_{\theta_F} \in L / \theta_F$


Commutative Law:

Let $[x]_{\theta_F}, [y]_{\theta_F} \in L / \theta_F$ be arbitrary.

Then $[x]_{\theta_F} \lor [y]_{\theta_F} = [x \lor y]_{\theta_F}$

$= [y \lor x]_{\theta_F}$

$= [y]_{\theta_F} \lor [x]_{\theta_F}$

$[x]_{\theta_F} \land [y]_{\theta_F} = [x \land y]_{\theta_F}$

$= [y \land x]_{\theta_F}$

$= [y]_{\theta_F} \land [x]_{\theta_F}$

for all $[x]_{\theta_F}, [y]_{\theta_F} \in L / \theta_F$

Thus commutative law holds.


Associative Law:

Let $[x]_{\theta_F}, [y]_{\theta_F}, [z]_{\theta_F} \in L / \theta_F$ be arbitrary.

Then,

$\{ [x]_{\theta_F} \lor [y]_{\theta_F} \} \lor [z]_{\theta_F} = \{ [x \lor y]_{\theta_F} \} \lor [z]_{\theta_F}$
\[(x \lor y) \lor z \theta_f = (x \lor (y \lor z)) \theta_f = (x) \theta_f \lor (y \lor z) \theta_f = (x) \theta_f \lor (y) \theta_f \lor (z) \theta_f\]

\[
\{ (x) \theta_f \land (y) \theta_f \} \land (z) \theta_f = \{ (x \land y) \theta_f \} \land (z) \theta_f = \{(x \land y) \land z \} \theta_f = (x \land (y \land z)) \theta_f = (x) \theta_f \land (y \land z) \theta_f = (x) \theta_f \land \{ (y) \theta_f \land (z) \theta_f \}
\]

Thus
\[
\{ (x) \theta_f \lor (y) \theta_f \} \lor (z) \theta_f = (x) \theta_f \lor \{ (y) \theta_f \lor (z) \theta_f \}
\]

\[
\{ (x) \theta_f \land (y) \theta_f \} \land (z) \theta_f = (x) \theta_f \land \{ (y) \theta_f \land (z) \theta_f \}
\]

for all \((x) \theta_f, (y) \theta_f, (z) \theta_f\) in \(L/\theta_f\)

Hence \(L/\theta_f\) is a lattice.

**For (iii):**

Define a map \(\psi: L/\theta_f \rightarrow L_1\) by

\[\psi((x) \theta_f) = \phi(x) \text{ where } (x) \theta_f \in L/\theta_f\]

Then we claim that \(\psi\) is a well defined isomorphism.

\(\psi\) is well defined:

suppose \((x) \theta_f = (y) \theta_f\) where \((x) \theta_f, (y) \theta_f \in L/\theta_f\)

\[\Rightarrow x = y(\theta_f)\]

\[\Rightarrow \phi(x) = \phi(y)\]

\[\Rightarrow \psi((x) \theta_f) = \psi((y) \theta_f)\]
ψ is one-one:

suppose \( \psi([x]_{\Theta_F}) = \psi([y]_{\Theta_F}) \) where \( [x]_{\Theta_F}, [y]_{\Theta_F} \in L/\Theta_F \)

\[ \Rightarrow \phi(x) = \phi(y) \]
\[ \Rightarrow x = y(\Theta_F) \]
\[ \Rightarrow [x]_{\Theta_F} = [y]_{\Theta_F} \]

ψ is onto:

Take any \( z_1 \in L_i \Rightarrow \) there exist \( z \in L \) such that \( \phi(z) = z_1 \), since \( \phi \) is onto

\[ \Rightarrow [z]_{\Theta_F} \in L/\Theta_F \]
\[ \Rightarrow \psi([z]_{\Theta_F}) = \phi(z) = z_1 \]

ψ preserves "\( \lor \)" and "\( \land \)"

Let \( [x]_{\Theta_F}, [y]_{\Theta_F} \in L/\Theta_F \) be arbitrary

Then \( \psi([x]_{\Theta_F} \lor [y]_{\Theta_F}) = \psi([x \lor y]_{\Theta_F}) \)

\[ = \phi(x \lor y) \]
\[ = \phi(x) \lor \phi(y) \]
\[ = \psi([x]_{\Theta_F}) \lor \psi([y]_{\Theta_F}) \]

for all \( [x]_{\Theta_F}, [y]_{\Theta_F} \in L/\Theta_F \)

\( \psi([x]_{\Theta_F} \land [y]_{\Theta_F}) = \psi([x \land y]_{\Theta_F}) \)

\[ = \phi(x \land y) \]
\[ = \phi(x) \land \phi(y) \]
\[ = \psi([x]_{\Theta_F}) \land \psi([y]_{\Theta_F}) \]

for all \( [x]_{\Theta_F}, [y]_{\Theta_F} \in L/\Theta_F \)

Therefore \( \psi \) is an onto isomorphism.

Hence \( L/\Theta_i \cong L_i \).
Theorem 3.14

Let $L$ be a lattice, $D$ a standard filter and $F$ an arbitrary filter of $L$. Then $F \cap D$ is a standard filter of $F$ and $F \lor D / D \cong F / F \cap D$.

Proof:

Given $D$ is a standard filter and $F$ an arbitrary filter of a lattice $L$.

Then $F \cap D$ is a filter of $F$.

To prove

(i) $F \cap D$ is a standard filter of $F$

(ii) $F \lor D / D \cong F / F \cap D$

For (i) clearly $F \cap D \subseteq F$, $F \cap D$ is a filter of $F$.

By characterization theorem, it is sufficient to prove

$$\theta_{F \cap D} = \theta_F \land \theta_D$$

Let $(X, Y) \in \theta_{F \cap D}$ be arbitrary

$$\Rightarrow X \equiv Y(\theta_{F \cap D})$$

$$\Rightarrow (X \land Y) \lor D_1 = X \lor Y \text{ for some } D_1 \leq F \land D$$

$$\Rightarrow (X \land Y) \lor D_1 = X \lor Y \text{ for some } D_1 \leq F \land D \leq F$$

and $(X \land Y) \lor D_1 = X \lor Y$ for some $D_1 \leq F \land D \leq D$

$$\Rightarrow X \equiv Y(\theta_F) \text{ and } X \equiv Y(\theta_D)$$

$$\Rightarrow (X, Y) \in \theta_F \text{ and } (X, Y) \in \theta_D$$

$$\Rightarrow (X, Y) \in \theta_F \land \theta_D$$

Therefore $\theta_{F \land D} \subseteq \theta_F \land \theta_D$ ........................ (1)

Let $(X, Y) \in \theta_F \land \theta_D$ be arbitrary

$$\Rightarrow (X, Y) \in \theta_F \text{ and } (X, Y) \in \theta_D$$

$$\Rightarrow X \equiv Y(\theta_F) \text{ and } X \equiv Y(\theta_D)$$

$$X \equiv Y(\theta_D) \Rightarrow (X \land Y) \lor D_1 = X \lor Y \text{ for some } D_1 \leq D$$
\[ D_1 \land [ (X \land Y) \lor D_1 ] = D_1 \land (X \lor Y) \]
\[ D_1 \land [ D_1 \lor (X \land Y)] = D_1 \land (X \lor Y) \]
\[ D_1 = D_1 \land (X \lor Y), \text{ by absorption law} \]
\[ \equiv [ D_1 \land (X \land Y)] \theta_f, \text{ since } X \equiv Y(\theta_f) \]
\[ \Rightarrow X \land Y \equiv X \lor Y(\theta_f) \]
\[ \Rightarrow X \lor Y \equiv X \land Y(\theta_f) \]
\[ [D_1 \land (D_1 \land (X \land Y))] \lor D_2 = D_1 \lor [D_1 \land (X \land Y)], \text{ for some } D_2 \leq F \]
\[ \Rightarrow [D_1 \land (X \land Y)] \lor D_2 = D_1, \text{ for some } D_2 \leq F \]

Now \((X \land Y) \lor D_2 = [(X \land Y) \lor [(X \land Y) \land D_1]] \lor D_2, \text{ by absorption Law}\]
\[ = (X \land Y) \lor [(D_1 \land (X \land Y)) \lor D_2] \]
\[ = (X \land Y) \lor D_1 \]
\[ = X \lor Y \]

Therefore
\[ (X \land Y) \lor D_2 = X \lor Y \text{ for some } D_2 \leq F \land D \]
\[ \text{since } D_2 \leq F \text{ and } D_2 \leq D_1 \leq D \]
\[ \Rightarrow X \equiv Y(\theta_{F \land D}) \]
\[ \Rightarrow (X, Y) \in \theta_{F \land D} \]

Therefore \(\theta_F \land \theta_D \leq \theta_{F \land D} \quad \ldots \ldots \quad (2)\)

From (1) and (2) we have
\[ \theta_{F \land D} = \theta_F \land \theta_D \]
\[ \Rightarrow \theta_{F \land D} \text{ is a congruence relation.} \]

Hence \(F \land D\) is a standard filter of \(F\), by the characterization theorem for dually standard filter (theorem 3.8).

For (ii):

Define \(\phi : F \to F \lor D / \theta_D\) by
\[ \phi(x) = [x]_{\theta_D}, \text{ where } x \in F \]

Then we claim \(\phi\) is a well defined onto homomorphism.
\[ \phi \text{ is well defined :} \]

Suppose \( x = y \) where \( x, y \in F \)

\[ \Rightarrow \quad \theta(x) = \theta(y) \]

\[ \Rightarrow \quad \phi(x) = \phi(y) \]

\[ \phi \text{ is onto :} \]

Take any \( \theta(x) \in F \lor D / \theta_D \)

\[ \Rightarrow \quad y \in F \lor D \]

\[ \Rightarrow \quad y \in F \]

\[ \Rightarrow \quad \phi(y) = \theta(y) \]

\[ \phi \text{ preserves } \lor \text{ and } \land : \]

Let \( x, y \in F \) be arbitrary

Then \( \phi(x \lor y) = \theta(x \lor y) \)

\[ = \theta(x) \lor \theta(y) \]

\[ = \phi(x) \lor \phi(y) \quad \text{for all } x, y \in F \]

\[ \phi(x \land y) = \theta(x \land y) \]

\[ = \theta(x) \land \theta(y) \]

\[ = \phi(x) \land \phi(y) \quad \text{for all } x, y \in F \]

Hence \( \phi \) is an onto homomorphism

Take \( \ker \phi = F \land D \)

Then we have \( \phi \) is an onto homomorphism and \( F \land D \) is a standard filter.

\[ \Rightarrow \quad F \lor D / \theta_D \cong F / \theta_{\lor \land} \quad \text{by fundamental theorem of homomorphism} \]

\[ \Rightarrow \quad F \lor D / D \cong F / F \land D \]
Theorem: 3.15

Let $X$ be an arbitrary filter and $D$ a standard filter of the lattice $L$. If $X \lor D$ and $X \land D$ are principal filters then $X$ is a principal filter.

Proof:

Let $X$ be any arbitrary filter and $D$ a standard filter in $L$. Suppose $D \lor X$, $D \land X$ are principal filters.

To prove that $X$ is a principal filter.

By assumption

$D \lor X = \lfloor a \rfloor$

$D \land X = \lfloor b \rfloor$ where $a, b \in L$

We have $a \leq a \Rightarrow a \in D \lor X$

$\lfloor a \rfloor \subseteq D \lor X$

$\lfloor a \rfloor = \lfloor a \rfloor \land (D \lor X)$

$= \lfloor (a) \land D \rfloor \lor (\lfloor a \rfloor \land X)$ since $D$ is a standard filter

$\Rightarrow a \geq a_1 \land a_2$ where $a_1 \in \lfloor a \rfloor \land D$, $a_2 \in \lfloor a \rfloor \land X$

$a_1 \in \lfloor a \rfloor \land D$, $a_2 \in \lfloor a \rfloor \land X$

$\Rightarrow a_1, a_2 \in \lfloor a \rfloor$

$\Rightarrow a_1 \leq a$, $a_2 \geq a$

$\Rightarrow a_1 \land a_2 \geq a$

Therefore $a_1 \land a_2 = a$

Claim $X = \lfloor a_2 \lor b \rfloor$

Let $y \in X$ be arbitrary. Then

$a_2 \in X$, $y \in X \Rightarrow y \geq a_2, y \geq b \Rightarrow y \geq a_2 \lor b$

$\Rightarrow y \in \lfloor a_2 \lor b \rfloor$

$\Rightarrow X \subseteq \lfloor a_2 \lor b \rfloor$ .......................... (1)
Let \( t \in \{ a_2 \lor b \} \) be arbitrary

\[ \Rightarrow \quad t \geq a_2 \lor b \quad \text{with} \quad a_2 \in X \]

\[ a_2 \lor b \geq a_2, a_2 \in X \Rightarrow a_2 \lor b \in X \]

\textbf{Hence} \( t \geq a_2 \lor b, a_2 \lor b \in X \Rightarrow t \in X \)

Therefore \( \{ a_2 \lor b \} \subseteq X \) \hspace{1cm} \text{(2)}

From (1) and (2) we have

\[ \{ a_2 \lor b \} = X \]

Therefore \( X \) is a principal filter.

**Proposition : 3.5**

Let \( L \) and \( L' \) be two lattices. If \( f : L \rightarrow L' \) is an onto homomorphism

with \( F \) a standard filter of \( L \) then \( f(F) = F' \) is a standard filter of \( L' \)

**Proof :**

Given \( f : L \rightarrow L' \) is an onto homomorphism with \( F \) a standard filter

of \( L \) and \( f(F) = F' \). Then \( F' \) is a filter of \( L' \)

\textbf{To prove} \( F' \) is standard.

Let \( X', Y' \in \mathcal{F}(L') \) be arbitrary

\[ \Rightarrow \quad X' = f(X), Y' = f(Y) \quad \text{where} \quad X, Y \in \mathcal{F}(L) \]

Then \( X' \wedge (F' \lor Y') = f(X) \wedge [f(F) \lor f(Y)] \]

\[ = f(\{ f(X) \wedge \{ f(F) \lor f(Y) \} \}) \]

\[ = f(X \wedge \{ f(F) \lor f(Y) \}) \]

\[ = f((X \wedge F) \lor (X \wedge Y)) \]

\[ = f(X \wedge F) \lor f(X \wedge Y) \]

\[ = \{ f(X) \wedge f(F) \} \lor \{ f(X) \wedge f(Y) \} \]

\[ = (X' \wedge F') \lor (X' \wedge Y') \]

\[ \text{for all} \quad X', Y' \in \mathcal{F}(L') \]

\[ \Rightarrow \quad F' \quad \text{is standard.} \]
Theorem 3.16  Second Isomorphism Theorem

Let L be a lattice, F a filter of L and D a standard filter of L such that

\[ D \subseteq F. \]

Then,

(i) \( F \) is a standard filter in L if and only if \( F / D \) is a standard filter in \( L / D \)

(ii) \( L / F \cong (L / D) / (F / D) \)

Proof:

For (i):

Assume that \( F \) is a standard filter of L.

To prove \( F / D \) is standard filter in \( L / D \).

By proposition 3.5, it is sufficient to prove that homomorphic image of \( F \) is \( F / D \).

Define \( \phi : L \rightarrow L / D \) by

\[ \phi(x) = [x]_D \quad \text{where} \quad x \in L \]

Then we claim that \( \phi \) is an onto homomorphism.

\( \phi \) is well defined:

Suppose \( x = y \) where \( x, y \in L \)

\[ \Rightarrow [x]_D = [y]_D \]

\[ \Rightarrow \phi(x) = \phi(y) \]

\( \phi \) is onto:

Take any \([x]_D \in L / D\).

\[ \Rightarrow x \in L \]

\[ \Rightarrow \phi(x) = [x]_D \]

\( \phi \) preserves "\( \lor \)" , "\( \land \)".

Let \( x, y \in L \) be arbitrary.

Then \( \phi(x \lor y) = [x \lor y]_D \)

\[ = [x]_D \lor [y]_D \]

\[ = \phi(x) \lor \phi(y), \text{ for all } x, y \in L \]
\[
\phi(x \wedge y) = [x \wedge y]_{\theta_D}
\]
\[
= [x]_{\theta_D} \wedge [y]_{\theta_D}
\]
\[
= \phi(x) \wedge \phi(y), \text{ for all } x, y \in L
\]

Hence \( \phi \) is an onto homomorphism

\[\Rightarrow \phi(F) = F / D \text{ is a homomorphic image of } F\]

\[\Rightarrow F / D \text{ is a standard filter, by proposition 3.5}\]

Conversely, assume that \( F / D \) is a standard filter in \( L / D \)

To prove \( F \) is a standard filter in \( L \)

Let \( X, Y \in \mathcal{F}(L) \) be arbitrary

Then \( \bar{X}, \bar{Y} \) be the homomorphic image of \( X \) and \( Y \) under the map

\[\phi : L \rightarrow L / D\]

By assumption \( F / D \) is standard in \( L / D \)

\[\Rightarrow \bar{X} = \bar{Y}(\theta_{F_D})\]

\[\Rightarrow (\bar{X} \wedge \bar{Y}) \vee F_1 = \bar{X} \vee \bar{Y} \text{ for some } \bar{F}_1 \leq \bar{F} = F / D\]

\[\Rightarrow (X \wedge Y) \vee F_1 = X \vee Y \text{ for some } F_1 \leq F\]

\[\Rightarrow X \equiv Y(\theta_1)\]

\[\Rightarrow F \text{ is a standard filter in } L\]

For (ii)

Define \( g : L \rightarrow (L/D) / (F/D) \) by

\[g(x) = [\bar{x}]_{\theta(F_D)} \text{ where } x \in L\]

Then we claim that \( g \) is a well defined onto homomorphism with \( \ker g = F \)

\( g \) is well defined:

Suppose \( x = y \) where \( x, y \in L \)

\[\Rightarrow [\bar{x}]_{\theta(F_D)} = [\bar{y}]_{\theta(F_D)}\]

\[\Rightarrow g(x) = g(y)\]
g is onto:

Take any $[x]_{\theta(F/D)} \in (L/D)/(F/D)$

$\Rightarrow \ x \in L/D$

$\Rightarrow \ x \in L$

$\Rightarrow \ g(x) = [\ x \ ]_{\theta(F/D)}$

Hence $g$ is onto.

'g' preserves '∨', '∧':

Let $x, y \in L$ be arbitrary.

Then $g(x \lor y) = [ \ x \lor \ y \ ]_{\theta(F/D)}$

$\quad = [x]_{\theta(F/D)} \lor [y]_{\theta(F/D)}$

$\quad = g(x) \lor g(y)$ where $x, y \in L$

$g(x \land y) = [x \land \ y \ ]_{\theta(F/D)}$

$\quad = [x]_{\theta(F/D)} \land [y]_{\theta(F/D)}$

$\quad = g(x) \land g(y)$ where $x, y \in L$

Taking $\ker g = F$, then we have $g$ is an onto homomorphism with $\ker g = F$ and $F$ is a standard filter.

$\Rightarrow \ L/F \cong (L/F)/(F/D)$ by fundamental theorem of homomorphism.