The necessary definitions, theorems and results which are used in this thesis are given in this chapter.

The symbols $\subseteq$, \notin, \lor, \land$ will denote inclusion, non-inclusion, cup or join (l.u.b) and cap or meet (g.l.b) in a lattice while symbols $\subseteq$, $\cup$, $\cap$, $\in$, $\notin$ and $\emptyset$ will refer to set inclusion, union, intersection, membership, non-membership and empty set. Small letters $a$, $b$, $c$ ... will denote elements of a lattice. Greek letters $\theta$, $\phi$ will stand for congruence relation on lattice.

Definition 1.1

A partially ordered set $(P, \leq)$ is a non-empty set $P$ with binary relation $\leq$, defined on it and satisfying the following conditions:

(i) "$\leq$" is reflexive;
   
   $x \leq x$ for every $x$ in $P$

(ii) "$\leq$" is Anti symmetric;

   if $x \leq y$ and $y \leq x$ then $x = y$ for all $x$, $y$ in $P$

(iii) "$\leq$" is transitive;

   if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x$, $y$, $z$ in $P$

Definition 1.2

A lattice is a nonempty set $L$ with a binary relation $\leq$ defined on it and satisfying the following:

(i) $(L, \leq)$ is a partially ordered set

(ii) Any two elements in $L$ have a l.u.b and a g.l.b
Definition: 1.3

A lattice is a non empty set L with two binary operations ‘ ∨ ’, ‘ ∧ ’ defined on it and satisfy the following:

(i) Idempotent law;
   \[ a ∨ a = a, a ∧ a = a \] for all a in L

(ii) Commutative law;
   \[ a ∨ b = b ∨ a, a ∧ b = b ∧ a \] for all a, b in L

(iii) Associative law;
   \[ a ∨ (b ∨ c) = (a ∨ b) ∨ c, a ∧ (b ∧ c) = (a ∧ b) ∧ c \] for all a, b, c in L

(iv) Absorption law;
   \[ a ∨ (a ∧ b) = a, a ∧ (a ∨ b) = a \] for all a, b in L

Definition: 1.4

A partially ordered set (P, ≤) is called a linearly ordered set or chain if it satisfies the following:

\[ x, y \in P \Rightarrow \text{either } x \leq y \text{ or } y \leq x \]

Theorem: 1.1

Definition 1.2 and Definition 1.3 for lattice L are equivalent, with respect to the following:

(i) \( a \leq b \iff a ∨ b = b \text{ or } a ∧ b = a \)

(ii) \( a ∨ b = \text{l.u.b of } \{a, b\} \)

(iii) \( a ∧ b = \text{g.l.b of } \{a, b\} \) where a, b are in L
Definition : 1.5

A lattice $L$ is called a modular lattice if it satisfies the following equality:

$$ (a \lor b) \land (a \lor c) = a \lor [b \land (a \lor c)] \text{ for all } a, b, c \text{ in } L $$

Equivalently, a lattice $L$ is called modular if

$$ a \leq c \Rightarrow (a \lor b) \land c = a \lor (b \land c) \text{ for all } a, b, c \text{ in } L $$

Example : 1.1

Consider the lattice $M_3$ of the following figure:

\[ \text{This lattice is modular.} \]

Example : 1.2

Consider the lattice $N_5$ of the following figure:

\[ \text{This lattice is not modular.} \]
Here \((a \lor b) \land (a \lor c) = 1 \land c = c\)

\[a \lor [b \land (a \lor c)] = a \lor (b \land c) = 0\]

Therefore \((a \lor b) \land (a \lor c) \neq a \lor [b \land (a \lor c)]\)

**Definition : 1.6**

A non empty subset \(S\) of a lattice \(L\) is called sublattice if \(a, b\) in \(S\) implies \(a \lor b, a \land b\) in \(S\).

**Theorem : 1.2**

A lattice \(L\) is modular if and only if \(L\) does not contain a sublattice isomorphic to \(N_5\).

**Definition : 1.7**

A lattice \(L\) is called distributive lattice if it satisfies the following equality:

\[a \lor (b \land c) = (a \lor b) \land (a \lor c)\] for all \(a, b, c\) in \(L\).

**Example : 1.3**

Every chain is a distributive lattice.

**Example : 1.4**

Consider the diamond lattice of the following figure:

![Diagram of a diamond lattice](image)

This is a distributive lattice.
Example : 1.5

Consider the $M_3$ lattice of the following figure:

This is a modular lattice and not distributive.

Here

\[ a \lor (b \land c) = a \lor 0 = a \]

\[ (a \lor b) \land (a \lor c) = 1 \land 1 = 1 \]

so

\[ a \lor (b \land c) \neq (a \lor b) \land (a \lor c) \]

Example : 1.6

Consider the lattice $N_5$ of the following figure:
This lattice is not distributive.

Here
\[ a \lor (b \land c) = a \]
\[ (a \lor b) \land (a \lor c) = c \]
so \[ a \lor (b \land c) \neq (a \lor b) \land (a \lor c) \]

**Theorem 1.3**

Every distributive lattice is modular but the converse need not be.

**Theorem 1.4**

A lattice \( L \) is distributive if and only if it does not contain a sublattice isomorphic to \( M_3 \) or \( N_5 \).

**Theorem 1.5**

A modular lattice \( L \) is distributive if and only if it does not contain a sublattice isomorphic to \( M_3 \).

**Definition 1.8**

Let \( L \) be a lattice and \( \text{} \) be an element of \( L \). Then

\( \text{} \) is called distributive element if and only if
\[ a \lor (x \land y) = (a \lor x) \land (a \lor y) \] for all \( x, y \in L \)

\( \text{} \) is called dually distributive element if and only if
\[ a \land (x \lor y) = (a \land x) \lor (a \land y) \] for all \( x, y \in L \)

\( \text{} \) is called standard element if and only if
\[ x \land (a \lor y) = (x \land a) \lor (x \land y) \] for all \( x, y \in L \)

\( \text{} \) is called dually standard element if and only if
\[ x \lor (a \land y) = (x \lor a) \land (x \lor y) \] for all \( x, y \in L \)

\( \text{} \) is called neutral element if and only if
\[ (a \lor x) \land (y \lor y) \land (y \lor a) = (a \land x) \lor (x \land y) \lor (y \land a) \] for all \( x, y \in L \).
Example : 1.7

Consider the lattice $N_S$ of the following figure:

In this lattice:

0. $1$, $b$ and $c$ are distributive elements
0. $1$, $a$ and $c$ are dually distributive elements
0. $1$, $b$ are standard elements
0. $1$, $a$ are dually standard elements
0. $1$ are neutral elements

Example : 1.8

Consider the lattice $M_3$ of the following figure:
In this lattice 0, 1 are distributive, dually distributive, standard, dually standard and neutral elements.

Example : 1.9

In a distributive lattice, every element is distributive, dually distributive, standard, dually standard and neutral.

Definition : 1.9

A binary relation “θ” on a lattice L is said to be a congruence relation if it satisfies the following conditions:

(i) ‘θ’ is reflexive;
\[ x \equiv x(\theta) \text{ for all } x \in L \]

(ii) ‘θ’ is symmetric;
\[ x \equiv y(\theta) \Rightarrow y \equiv x(\theta) \text{ for all } x, y \in L \]

(iii) ‘θ’ is transitive;
\[ x \equiv x(\theta) \text{ and } y \equiv y(\theta) \Rightarrow x \equiv z(\theta) \text{ for all } x, y, z \in L \]

(iv) Substitution property;
\[ x \equiv x_1(\theta) \text{ and } y \equiv y_1(\theta) \]
\[ \Rightarrow x \lor y = (x_1 \lor y_1)(\theta) \]
\[ x \land y = (x_1 \land y_1)(\theta) \text{ for all } x_1, y_1, x, y \in L \]

Theorem : 1.6

A reflexive and symmetric binary relation ‘θ’ on a lattice L is a congruence relation if and only if the following three properties are satisfied for all \(x, y, z\) in L

(i) \(x \equiv y(\theta) \Leftrightarrow x \land y \equiv (x \lor y)(\theta)\)

(ii) \(x \leq y \leq z, x \equiv y(\theta) \text{ and } y \equiv z(\theta) \Rightarrow x \equiv z(\theta)\)
(iii) \( x = y(\emptyset) \) and \( x \leq y \)

\[ x \wedge t = (y \wedge t)(\emptyset) \text{ and } x \vee t = (y \vee t)(\emptyset) \text{ for all } t \in L \]

**Definition : 1.10**

A non-empty subset \( J \) of a lattice \( L \) is said to be an ideal if and only if

(i) \( x \in J, y \in J \Rightarrow x \vee y \in J \)

(ii) \( a \in J, t \in L \text{ and } t \leq a \Rightarrow t \in J \)

**Definition : 1.11**

Let \( 'a' \) be an element of a lattice \( L \). Then the set \( \{ x \in L / x \leq a \} \) form an ideal of \( L \). This ideal is called the principal ideal generated by \( 'a' \) and is denoted by \( (a) \).

**Theorem : 1.7**

Let \( I(L) \) denote the set of all ideals of a lattice \( L \). Then \( I(L) \) is a lattice with respect to the following:

(i) \( J_1 \leq J_2 \Leftrightarrow J_1 \subseteq J_2 \)

(ii) \( J_1 \vee J_2 = \{ x \in L / x \leq x_1 \vee x_2 \text{ for some } x_1 \in J_1, x_2 \in J_2 \} \)

(iii) \( J_1 \wedge J_2 = J_1 \cap J_2 = \{ x \in L / x \in J_1 \text{ and } x \in J_2 \} \)

where \( J_1, J_2 \in I(L) \)

This lattice \( I(L) \) is called the ideal lattice.

**Theorem : 1.8**

A lattice \( L \) is distributive if and only if the lattice \( I(L) \) is distributive.

**Theorem : 1.9**

A lattice \( L \) is modular if and only if the lattice \( I(L) \) is modular.
Definition: 1.12

A non empty subset F of a lattice L is called a filter if and only if

(i) \( x \in F, y \in F \Rightarrow x \land y \in F \)

(ii) \( x \in F, t \in L \text{ and } t \geq x \Rightarrow t \in F \)

Definition: 1.13

Let \( a \) be an element of a lattice L. Then the set \( \{ x \in L / x \geq a \} \) form a filter of L. This filter is called the principal filter generated by \( a \) and is denoted by \( \langle a \rangle \).

Theorem: 1.10

If \( \mathcal{F}(L) \) is the set of all filters of a lattice L then \( \mathcal{F}(L) \) is a lattice with respect to the following:

(i) \( F_1 \leq F_2 \Leftrightarrow F_1 \subseteq F_2 \)

(ii) \( F_1 \lor F_2 = \{ f \in L / f \geq f_1 \land f_2 \text{ for some } f_1 \in F_1, f_2 \in F_2 \} \)

(iii) \( F_1 \land F_2 = \{ x \in L / x \in F_1 \text{ and } x \in F_2 \} \)

where \( F_1, F_2 \in \mathcal{F}(L) \)

Definition: 1.14

Let L be a lattice and \( H \subseteq L \times L \). The smallest congruence relation such that \( a \equiv b (\theta_H) \), for all \( (a, b) \in H \) is denoted by \( \theta_H \).

If \( H = I \) where I is an ideal, we write \( \theta_I \) for \( \theta_H \).
Definition : 1.15
For any two elements 'a' and 'b' of a lattice L
(i)  \((a \land b) = (a) \lor (b)\)
(ii)  \((a \lor b) = (a) \land (b)\)
(iii)  \([a \land b] = [a] \lor [b]\)
(iv)  \([a \lor b] = [a] \land [b]\)

Definition : 1.16
A non empty subset S of a lattice L is called convex sublattice if \(a, b \in S \Rightarrow [a \land b, a \lor b] \subseteq S\).

Example : 1.10
Let L be a lattice and a, b in L.
Then \([a, b] = \{ x \in L / a \leq x \leq b \}\) is a convex sublattice of L.