CHAPTER II
SUPERMODULAR JOIN SEMILATTICE

The concept of distributivity has been extended to join semilattices by T. Katrinak [10] and that of modularity by W.H. Cornish [5]. In this chapter the concept of supermodularity to join semilattices is introduced and the supermodular join semilattice is characterised. Throughout this chapter semilattice stands for join semilattice.

Definition 2.1:-
A semilattice $S$ is said to be supermodular whenever $a, b, c, d, x$ in $S$ and $a \leq x \leq a \lor b, a \lor c, a \lor d$ there exist $s, t, u$ in $S$ such that

$s \leq b, c, a \lor d$

$t \leq c, d, a \lor b$

$u \leq d, b, a \lor c$

and $x = a \lor s \lor t \lor u$.

Theorem 2.1:-
Every supermodular lattice is a supermodular semilattice.

Proof:-
Given $L$ is a supermodular lattice. Then

$$(a \lor b) \land (a \lor c) \land (a \lor d) = a \lor [b \land c \land (a \lor d)] \lor [c \land d \land (a \lor b)]$$

$$\lor [b \land d \land (a \lor c)] \text{ for all } a, b, c, d \text{ in } L \quad (1)$$

To prove that $L$ is a supermodular semilattice

By (1) we get
\[ a \leq x \leq a \vee b, a \vee c, a \vee d \quad \text{where} \quad x = (a \vee b) \wedge (a \vee c) \wedge (a \vee d) \]

\[ \Rightarrow \quad \text{there exist } s, t, u \text{ in } S \text{ such that} \]

\[ s \leq b, c, a \vee d \]

\[ t \leq c, d, a \vee b \]

\[ u \leq d, b, a \vee c \]

and \[ x = a \vee s \vee t \vee u \quad \text{where} \quad s = b \wedge c \wedge (a \vee d) \]

\[ t = c \wedge d \wedge (a \vee b) \]

\[ u = b \wedge d \wedge (a \vee c). \]

Hence \( L \) is a supermodular semilattice.

**Corollary 2.1 :**

Every distributive lattice is a supermodular semilattice.

**Theorem 2.2 :**

Every distributive semilattice is a supermodular semilattice, converse need not be true.

**Proof :**

**First Part :**

Let \( S \) be a distributive semilattice. To prove that \( S \) is supermodular.

Suppose \( a \leq x \leq a \vee b, a \vee c, a \vee d \) where \( a, b, c, d \) and \( x \) in \( S \).

\[ \Rightarrow \quad (a] \subseteq (x] \subseteq (a \vee b], (a \vee c], (a \vee d] \]

\[ \Rightarrow \quad (a] \subseteq (x] \subseteq (a], (a] \vee (b], (a] \vee (c], (a] \vee (d] \]

\[ \Rightarrow \quad A \subseteq X \subseteq A \vee B, A \vee C, A \vee D, \]

where \( A = (a], B = (b], C = (c], D = (d], X = (x] \)

Now \( A \subseteq X \subseteq A \vee B \)

\[ \Rightarrow \quad X = X \wedge (A \vee B) \]

\[ \Rightarrow \quad X = (X \wedge A) \vee (X \wedge B), \text{ since } I(S) \text{ is a distributive lattice.} \]

\[ \Rightarrow \quad X = A \vee (X \wedge B) \]

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Similarly,

\[ X = A \lor (X \land C) \]
\[ X = A \lor (X \land D) \]

Therefore,

\[ X = \left[ A \lor (X \land B) \right] \land \left[ A \lor (X \land C) \right] \land \left[ A \lor (X \land D) \right] \]
\[ X = A \lor [X \land B \land C \lor [A \lor (X \land D)]] \lor [X \land C \land D \lor [A \lor (X \land B)]] \]
\[ \lor [X \land D \land B \lor [A \lor (X \land C)]] \]

since \( I(S) \) is a distributive lattice it is supermodular.

\[ \Rightarrow x \leq a \lor s \lor t \lor u \text{, where } a \text{ in } A, \]
\[ s \text{ in } X \land B \land C \land [A \lor (X \land D)] \]
\[ t \text{ in } X \land C \land D \land [A \lor (X \land B)] \]
\[ u \text{ in } (X \land D \land B \land [A \lor (X \land C)]) \]

\[ \Rightarrow x \leq a \lor s \lor t \lor u \text{, where } s \leq b, c, a \lor d \]
\[ t \leq c, d, a \lor b \]
\[ u \leq d, b, a \lor c \]

But \( s, t, u \) in \( X \); \( a \leq x \)

\[ \Rightarrow s, t, u \leq x; a \leq x \]
\[ \Rightarrow s \lor t \lor u \leq x; a \leq x \]
\[ \Rightarrow a \lor s \lor t \lor u \leq x \]

Therefore \( x = a \lor s \lor t \lor u \)

Hence \( S \) is a supermodular semilattice.

Second part:

To claim that every supermodular semilattice is not necessarily distributive.

It can be proved by an example. Consider the semilattice.
This semilattice is supermodular but not distributive, because even though $x \leq y \lor z$ but there exist no $p \leq y$, $q \leq z$ with $x = p \lor q$.

**Theorem 2.3:**

Every supermodular semilattice is a modular semilattice but the converse need not be.

**Proof:**

Given $S$ is a supermodular semilattice. To prove that $S$ is modular. By definition of supermodular semilattice whenever $a, b, c, d, x$ in $S$ and $a \leq x \leq a \lor b, a \lor c, a \lor d$ there exist $s, t, u$ in $S$ such that

- $s \leq b, c, a \lor d$
- $t \leq c, d, a \lor b$
- $u \leq d, b, a \lor c$

and $x = a \lor s \lor t \lor u$. 

\[ S = \{1, x, y, z, a_0, a_1, ..., a_n \mid a_0 < a_1 < ... < x, y, z < 1 \} \text{ as in fig 2.1} \]
Put \( b = c = d = p \) we get
\[
a \leq x \leq a \lor p
\]
implies that there exist \( s, t, u \) in \( S \) such that \( s \leq p, t \leq p, u \leq p \) and
\[
x = a \lor s \lor t \lor u.
\]
Thus \( a \leq x \leq a \lor p \) implies that there exists \( y \) in \( S \), such that
\[y \leq p \text{ and } x = a \lor y, \text{ where } y = s \lor t \lor u.
\]
Hence, \( S \) is a modular semilattice.

Next to claim that every modular semilattice need not be a supermodular semilattice, by an example.

Consider the semilattice
\[
S = \{1, x, y, z, a, a_0, a_1, \ldots, a_n, \ldots \mid a_0 < a_1 < \ldots < a_n < x, y, z, a < 1 \}
\]
as in fig 2.2.

This semilattice is modular but not supermodular because even though
\[a < 1 \leq a \lor x, a \lor y, a \lor z, \text{ there exists no } s, t, u \text{ satisfying the requirements of the definition.}\]
Theorem 2.4:-
Every supermodular semilattice is directed below

Proof:-
Given \( S \) is a supermodular semilattice. Then \( S \) is a modular semilattice by Theorem 2.3. To prove that \( S \) is directed below.
Suppose \( x, y \in S \) be arbitrary
Then \( y \leq y \) and \( y \leq x \lor y \).
\[ \Rightarrow y \leq y \leq x \lor y \]
\[ \Rightarrow y = x_1 \lor y \text{ for some } x_1 \leq x \text{ by supermodularity.} \]
\[ \Rightarrow x_1 \leq y, \ x_1 \leq x. \]
Hence every supermoduler semilattice \( S \) is directed below.

Theorem 2.5:-
A semilattice \( S \) is supermodular if and only if the lattice of ideals \( I(S) \) is supermodular.

Proof:-
Assume that a semilattice \( S \) is supermodular.
To prove that the lattice of ideals \( I(S) \) of \( S \) is supermodular.
That is to show that
\[ (A \lor B) \land (A \lor C) \land (A \lor D) = A \lor [B \land C \land (A \lor D)] \]
\[ \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)], \]
for all \( A, B, C, D \) in \( I(S) \).

Let \( A, B, C, D \) in \( I(S) \) be arbitrary. Then evidently,
\[ A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)] \subseteq (A \lor B) \land (A \lor C) \land (A \lor D) \]
It is enough to verify that 

\[(A \lor B) \land (A \lor C) \land (A \lor D)\]

\[\subseteq A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)]\]

Let \(x\) in \((A \lor B) \land (A \lor C) \land (A \lor D)\) be arbitrary

\[\Rightarrow x\ in\ A \lor B,\ x\ in\ A \lor C\ and\ x\ in\ A \lor D\]

\[\Rightarrow x \leq a_1 \lor b,\ a_2 \lor c,\ a_3 \lor d,\]  

where \(a_1, a_2, a_3\) in \(A,\ b\) in \(B,\ c\) in \(C\) and \(d\) in \(D\).

\[\Rightarrow x \leq a \lor b,\ a \lor c,\ a \lor d,\]  

where \(a = a_1 \lor a_2 \lor a_3\) in \(A\)

\[\Rightarrow a \leq a \lor x \leq a \lor b,\ a \lor c,\ a \lor d\]

\[\Rightarrow\] there exist \(s, t, u\) in \(S\) such that

\[s \leq b,\ c,\ a \lor d\]
\[t \leq c,\ d,\ a \lor b\]
\[u \leq d,\ b,\ a \lor c\]

and \(a \lor x = a \lor s \lor t \lor u\), since \(S\) is supermodular

\[\Rightarrow x \leq a \lor x = a \lor s \lor t \lor u\]  

where

\[a\ in\ A\]

\[s\ in\ [B \land C \land (A \lor D)]\]
\[t\ in\ [C \land D \land (A \lor B)]\]
\[u\ in\ [D \land B \land (A \lor C)]\]

\[\Rightarrow x\ in\ A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)]\]

Therefore \((A \lor B) \land (A \lor C) \land (A \lor D)\)

\[\subseteq A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)]\]

Thus \((A \lor B) \land (A \lor C) \land (A \lor D)\)

\[= A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \lor [D \land B \land (A \lor C)],\]

for all \(A, B, C, D\) in \(I(S)\).

Hence \(I(S)\) is a supermodular lattice.
Second Part:

Assume that \( I(S) \) is a supermodular lattice.

To prove that the semilattice \( S \) is supermodular.

Suppose \( a \leq x \leq a \lor b, a \lor c, a \lor d \) where \( a, b, c, d \) and \( x \) are in \( S \).

\[
\Rightarrow (a \subseteq (x) \subseteq (a \lor b), (a \lor c), (a \lor d))
\]

\[
\Rightarrow (a \subseteq (x) \subseteq (a) \lor (b), (a) \lor (c), (a) \lor (d))
\]

\[
\Rightarrow A \subseteq X \subseteq A \lor B, A \lor C, A \lor D,
\]

where \( A = (a), B = (b), C = (c), D = (d), X = (x) \)

Now \( A \subseteq X \subseteq A \lor B \)

\[
\Rightarrow X = X \land (A \lor B)
\]

\[
X = A \lor (X \land B), \text{since } I(S) \text{ is supermodular, it is modular.}
\]

Similarly \( X = A \lor (X \land C) \)

\[
X = A \lor (X \land D)
\]

Put \( X \land B = U, X \land C = V, X \land D = W \),
We get \( X = (A \lor U) \land (A \lor V) \land (A \lor W) \)

\[
= A \lor [U \land V \land (A \lor W)] \lor [V \land W \land (A \lor U)]
\]

\[
\lor [W \land U \land (A \lor V)],
\]

since \( I(S) \) is a supermodular lattice.

\[
\Rightarrow x \leq a_1 \lor s_1 \lor t_1 \lor u_1, \text{where}
\]

\[
a_1 \text{ in } A
\]

\[
s_1 \text{ in } U \land V \land (A \lor W)
\]

\[
t_1 \text{ in } V \land W \land (A \lor U)
\]

\[
u_1 \text{ in } U \land W \land (A \lor V)
\]

\[
\Rightarrow x \leq a \lor s_1 \lor t_1 \lor u_1, \text{where}
\]

\[
s_1 \leq b, c, a \lor d
\]

\[
t_1 \leq c, d, a \lor b
\]

\[
u_1 \leq d, b, a \lor c
\]
But $a \lor s_1 \lor t_1 \lor u_1 \leq x$, since $a \leq x, s_1 \leq x, t_1 \leq x, u_1 \leq x$.

Therefore $x = a \lor s_1 \lor t_1 \lor u_1$

Hence $S$ is a supermodular semilattice.

*Theorem 2.6:-*

Every conditionally complete supermodular semilattice is a lattice.

*Proof:-*

Let $S$ be a conditionally complete supermodular semilattice.

$\implies S$ is directed below, since $S$ is a supermodular semilattice

$\implies$ Every non empty subset in $S$ has a lower bound.

$\implies$ Every non empty subset in $S$ with a lower bound has an infimum, since $S$ is conditionally complete.

$\implies S$ is a lattice.

*Theorem 2.7:-*

Finite supermodular semilattice is a lattice.

*Proof:-*

Given $S$ is a finite supermodular semilattice. To prove that $S$ is a lattice.

$S$ is a finite supermodular semilattice

$\implies S$ is directed below

$\implies S$ has infimum, since $S$ is finite.

$\implies S$ is a lattice.

*Theorem 2.8:-*

Let $S$ be a supermodular semilattice and $I_1, I_2$ in $K(S)$. If $I_1 \lor I_2$ and $I_1 \land I_2$ are principal ideals of $S$, then $I_1, I_2$ are principal ideals of $S$. 

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Proof:-

Let \( I_1 \vee I_2 = \langle a \rangle \) and \( I_1 \wedge I_2 = \langle b \rangle \) for suitable \( a, b \) in \( S \).

Then \( a \leq a_1 \vee a_2 \) for some \( a_1 \) in \( I_1 \) and \( a_2 \) in \( I_2 \).

\[
I_1, I_2 \subseteq I_1 \vee I_2 \\
\Rightarrow I_1, I_2 \subseteq \langle a \rangle \\
\Rightarrow a_1, a_2 \text{ in } \langle a \rangle \\
\Rightarrow a_1 \leq a, a_2 \leq a \\
\Rightarrow a_1 \vee a_2 \leq a
\]

Therefore \( a = a_1 \vee a_2 \)

Take any \( x \) in \( I_1 \). Then \( a_1 \leq x \vee a_1 \leq a = a_1 \vee a_2 \)

By Theorem 2.3 there exists \( a_2 \leq a_2 \) such that

\[
x \vee a_1 = a_1 \vee a_2 \]

Also \( a_2 \leq x \vee a_1 \) in \( I_1 \), \( a_2 \) in \( I_2 \).

\[
\Rightarrow a_2 \text{ in } I_1 \wedge I_2 \\
\Rightarrow a_2 \leq b
\]

Therefore \( x \leq x \vee a_1 = a_1 \vee a_2 \leq a_2 \vee b \)

This is true for all \( x \) in \( I_1 \), so that \( I_1 = \langle a_1 \vee b \rangle \)

Similarly, take any \( y \) in \( I_2 \).

Then \( a_2 \leq y \vee a_2 \leq a = a_1 \vee a_2 \)

By Theorem 2.3, there exists \( a_1 \leq a_1 \) such that

\[
y \vee a_2 = a_1 \vee a_2 \]

Also \( a_1 \leq y \vee a_2 \) in \( I_2 \), \( a_1 \) in \( I_1 \)

\[
\Rightarrow a_1 \text{ in } I_1 \wedge I_2 \\
\Rightarrow a_1 \leq b
\]

Therefore \( y \leq y \vee a_2 = a_1 \vee a_2 \leq a_2 \vee b \)

This is true for all \( y \) in \( I_2 \), so that \( I_2 = \langle a_2 \vee b \rangle \).
Theorem 2.9:-

Let $S$ be a supermodular semilattice and $a, b, c, d$ in $S$.

1. If $a \land b, a \land c, a \land d$ exist then

$$a \land [b \lor c \lor (a \land d)] \land [c \lor d \lor (a \land b)] \land [d \lor b \lor (a \land c)]$$

exists and is equal to $(a \land b) \lor (a \land c) \lor (a \land d)$.

2. If $b \land c \land (a \lor d), c \land d \land (a \lor b), d \land b \land (a \lor c)$ exist then

$(a \lor b) \land (a \lor c) \land (a \lor d)$ exists and is equal to

$$a \lor [b \land c \land (a \lor d)] \lor [c \land d \land (a \lor b)] \lor [d \land b \land (a \lor c)]$$

Proof:

1. Clearly $(a \land b) \lor (a \land c) \lor (a \land d)$ is the lower bound of $a, b \lor c \lor (a \land d), c \lor d \lor (a \land b), d \lor b \lor (a \land c)$

Let $t$ be any lower bound of

$$a, b \lor c \lor (a \land d), c \lor d \lor (a \land b), d \lor b \lor (a \land c)$$

$$\Rightarrow t \leq a,$$

$$t \leq b \lor c \lor (a \land d)$$

$$t \leq c \lor d \lor (a \land b)$$

$$t \leq d \lor b \lor (a \land c)$$

$$\Rightarrow (t) \leq ([a], (b \lor c \lor (a \land d)) \lor (c \lor d \lor (a \land b)) \lor (d \lor b \lor (a \land c)))$$

$$\Rightarrow T \subseteq A, B \lor C \lor (A \land D), C \lor D \lor (A \land B), D \lor B \lor (A \land C)$$

where $T = (t), A = (a), B = (b), C = (c), D = (d)$

$$\Rightarrow T \subseteq [A \land (B \lor C \lor (A \land D))] \land [C \lor D \lor (A \land B)] \land [D \lor B \lor (A \land C)]$$

$$= (A \land B) \lor (A \land C) \lor (A \land D)$$

$$\Rightarrow t \leq s_1 \lor t_1 \lor u_1 \text{ where}$$

$s_1$ in $A \land B$

$t_1$ in $A \land C$

$u_1$ in $A \land D$
\( \leq (a \land b) \lor (a \land c) \lor (a \land d) \), since \( a \land b, a \land c, a \land d \) exist.

Hence \( (a \land b) \lor (a \land c) \lor (a \land d) \) is the greatest lower bound of \( a, b \lor c \lor (a \land d), c \lor d \lor (a \land b), d \lor b \lor (a \land c) \).

But the greatest lower bound of

\( a, b \lor c \lor (a \land d), c \lor d \lor (a \land b), d \lor b \lor (a \land c) \)

is \( a \land [b \lor c \lor (a \land d)] \land [c \lor d \lor (a \land b)] \land [d \lor b \lor (a \land c)] \)

Thus \( (a \land b) \lor (a \land c) \lor (a \land d) =

\( a \land [b \lor c \lor (a \land d)] \land [c \lor d \lor (a \land b)] \land [d \lor b \lor (a \land c)] \)

2. Clearly \( a \lor [b \land c \land (a \land d)] \lor [c \land d \land (a \land b)] \lor [d \land b \land (a \land c)] \)
is a lower bound of \( a \lor b, a \lor c, a \lor d \).

Let \( t \) be any lower bound of \( a \lor b, a \lor c, a \lor d \)

\[ t \leq a \lor b, a \lor c, a \lor d \]

\[ (t) \subseteq (a \lor b), (a \lor c), (a \lor d) \]

\[ (t) \subseteq (a \land (b), (a \land (c), (a \land (d)) \]

\[ T \subseteq A \land B, A \land C, A \land D \]

\[ T = (t), A = (a), B = (b), C = (c), D = (d) \]

where \( T \subseteq (A \lor B) \land (A \lor C) \land (A \lor D) \)

\[ T \subseteq A \lor [B \land C \land (A \lor D)] \lor [C \land D \land (A \lor B)] \]

\[ \lor [D \land B \land (A \lor C)] \]

\[ t \leq a \lor s_1 \lor t_1 \lor u_1, \text{ where } a_1 \text{ in } A \]

\[ s_1 \text{ in } B \land C \land (A \land D) \]

\[ t_1 \text{ in } C \land D \land (A \land B) \]

\[ u_1 \text{ in } D \land B \land (A \land C) \]

\[ t \leq a \lor [b \land c \land (a \land d)] \lor [c \land d \land (a \land b)] \]

\[ \lor [d \land b \land (a \land c)], \]

since \( b \land c \land (a \land d), c \land d \land (a \land b), d \land b \land (a \land c) \) exist.
Hence \( a \lor [ b \land c \land (a \lor d)] \lor [c \land d \land (a \lor b)] \lor [d \land b \land (a \lor c)] \) is the greatest lower bound of \( a \lor b, a \lor c, a \lor d \). But the greatest lower bound of \( a \lor b, a \lor c, a \lor d \) is \( (a \lor b) \land (a \lor c) \land (a \lor d) \).

Thus \( (a \lor b) \land (a \lor c) \land (a \lor d) = a \lor [b \land c \land (a \lor d)] \lor [c \land d \land (a \lor b)] \lor [d \land b \land (a \lor c)] \).

Corollary 2.2:–

If \((L, \lor, \land)\) is a lattice then the following are equivalent

(i) \((L, \lor, \land)\) is a supermodular lattice.

(ii) \((L, \lor)\) is a supermodular join semilattice.

(iii) \((L, \land)\) is a supermodular meet semilattice.

Theorem 2.10:–

A subsemilattice of a supermodular semilattice is not necessarily supermodular.

Proof:– Let us prove this by an example.

Let \((S, \lor)\) be the semilattice of all positive integers, where for any \(a\) and \(b\) in \(S\), \(a \lor b\) is the least common multiple of \(a\) and \(b\) and

\[ R = \{ p z / p \text{ varies over all prime numbers and } z \text{ is an integer} \} \]

Then \(S\) is a supermodular semilattice and \(R\) is a subsemilattice of \(S\). Now \(R\) is not supermodular because \(R\) is not modular. For example,

\[ 2 \leq 2 \leq 2 \lor 3 \text{ in } R \]

but there does not exist \( y \in R \) such that

\[ y \leq 3 \text{ and } 2 = 2 \lor y. \]

Therefore \(R\) is not modular.
Theorem 2.11 :-

Let S be a supermodular semilattice. Let J be a filter of S and I an ideal of S such that $J \cap I \neq \emptyset$. Then $J \cap I$ is a supermodular subsemilattice of S.

Proof :-

Given S is a supermodular semilattice. J a filter of S and I an ideal of S such that $J \cap I \neq \emptyset$. To prove that $J \cap I$ is a supermodular semilattice of S. First we claim that $J \cap I$ is a subsemilattice of S.

Let $a, b \in J \cap I$. Then $a, b \in J$ and $a, b \in I$

$\Rightarrow a \land b \in J$ since $a \leq a \land b, a \in J$ and $a \land b \in I$, since I is an ideal.

$\Rightarrow a \land b \in J \cap I$

Hence $J \cap I$ is a subsemilattice of S.

Next we claim that $J \cap I$ is supermodular. Suppose $a, b, c, d$ and $x \in J \cap I$

such that $a \leq x \leq a \lor b, a \lor c, a \lor d$.

$\Rightarrow a, b, c, d, x \in S$ such that $a \leq x \leq a \lor b, a \lor c, a \lor d$, since $J \cap I \subseteq S$.

$\Rightarrow$ there exist $s, t, u$ in S such that

$s \leq b, c, a \lor d$

$t \leq c, d, a \lor b$

$u \leq d, b, a \lor c$

and $x = a \lor s \lor t \lor u$, since S is supermodular.

Since J is a filter of S there exist $v$ in J such that $v \leq a, b, c, d, x$.

We have $x = x \lor v = (a \lor s \lor t \lor u) \lor v$

$= a \lor (s \lor v) \lor (t \lor v) \lor (u \lor v)$

$\geq a \lor (s \lor v) \lor (t \lor v) \lor (u \lor v)$

Since $b \geq a \lor (s \lor v), b \in I \Rightarrow s \lor v \in I$

$\Rightarrow v \leq s \lor v, v \in J \Rightarrow s \lor v \in J$
\[ c \geq s \vee v \in J \cap I \]
\[ a \vee d \geq s \vee v \in J \cap I \]
\[ c \geq t \vee v \in J \cap I \]
\[ d \geq t \vee v \in J \cap I \]
\[ a \vee b \geq t \vee v \in J \cap I \]
\[ d \geq u \vee v \in J \cap I \]
\[ b \geq u \vee v \in J \cap I \]
\[ a \vee c \geq u \vee v \in J \cap I \]

That is \( a \leq x \leq a \vee b, a \vee c, a \vee d \) implies that there exist
\[ s \vee v, t \vee v, u \vee v \in J \cap I \]

such that
\[ s \vee v \leq b, c, a \vee d \]
\[ t \vee v \leq c, d, a \vee b \]
\[ u \vee v \leq d, b, a \vee c \]

and \( x = a \vee (s \vee v) \vee (t \vee v) \vee (u \vee v) \).

Thus \( J \cap I \) is a supermodular subsemilattice of \( S \).

As a result we obtain

**Theorem 2.12:**

(i) Every non-empty filter of \( S \) is a supermodular subsemilattice of \( S \).

(ii) Every non-empty ideal of \( S \) is a supermodular subsemilattice of \( S \).

**Proof:**

For (i): Assume that \( S \) is a supermodular semilattice and \( J \) is a filter of \( S \).

To prove \( J \) is a supermodular subsemilattice of \( S \). By assumption \( S \) is an ideal of \( S \) and \( J \) is a filter of \( S \).

\[ \Rightarrow J \cap S = J \neq \emptyset \]

\[ \Rightarrow J \text{ is a supermodular subsemilattice of } S \text{ by Theorem 2.11.} \]
For (ii):

Assume that \( S \) is a supermodular semilattice and \( I \) is an ideal of \( S \).

To prove \( I \) is a supermodular subsemilattice of \( S \). By assumption \( S \) is a filter of \( S \) and \( I \) an ideal of \( S \).

\[ \Rightarrow I \cap S = I \neq \emptyset \]

\[ \Rightarrow I \text{ is a supermodular subsemilattice of } S \text{ by Theorem 2.11.} \]

**Theorem 2.13**

For any \( a \) and \( b \) in \( S \) such that \( a \leq b \), \( [a, b] = \{ x \in S / a \leq x \leq b \} \) is a supermodular subsemilattice of \( S \).

**Proof:**

Given \( S \) is a supermodular semilattice, \( a, b \in S \) and \( a \leq b \).

To prove that \( [a, b] \) is a supermodular subsemilattice of \( S \). We have \( [a) \) is a principal filter and \( (b] \) is a principal ideal.

\[ \Rightarrow \{ a) \text{ is a filter, } (b] \text{ is an ideal and } [a) \cap (b] \neq \emptyset, \text{ since } a \in [a) \text{ and } a \in (b] \]

\[ \Rightarrow [a) \cap (b] \text{ is a supermodular subsemilattice by Theorem 2.11.} \]

\[ \Rightarrow [a, b] \text{ is a supermodular subsemilattice.} \]

**Definition 2.2:**

A subsemilattice \( T \) is called a retract of a semilattice \( S \) if and only if there are homomorphisms \( f : S \to T \) and \( g : T \to S \) such that \( f \circ g \) is the identity on \( T \).

We observe that a subsemilattice of a supermodular semilattice is not always supermodular. However we have
Theorem 2.14: -

The retract of a supermodular semilattice is supermodular.

Proof: -

Let $R$ be a retract of supermodular semilattice $S$. By definition there are homomorphisms $f : S \rightarrow R$ and $g : R \rightarrow S$ such that $f \circ g$ is the identity on $R$.

That is $(f \circ g)u = u$ for every $u$ in $R$.

Also $f$ is an epimorphism and $g$ is a monomorphism.

To prove that $R$ is supermodular.

Suppose $a, b, c, d$ and $x$ in $R$ such that $a \leq x \leq a \vee b, a \vee c, a \vee d$

$\Rightarrow g(a), g(b), g(c), g(d)$ and $g(x)$ in $S$ such that $g(a) \leq g(x) \leq g(a) \vee g(b), g(a) \vee g(c), g(a) \vee g(d)$

$\Rightarrow$ there exist $s, t, u$ in $S$ such that $s \leq g(b), g(c), g(a) \vee g(d)$

$t \leq g(c), g(d), g(a) \vee g(b)$

$u \leq g(d), g(b), g(a) \vee g(c)$

and $g(x) = g(a) \vee s \vee t \vee u$, since $S$ is supermodular.

$\Rightarrow f(g(x)) = f(g(a)) \vee f(s) \vee f(t) \vee f(u)$

$\Rightarrow x = a \vee f(s) \vee f(t) \vee f(u)$

$\Rightarrow x = a \vee s_1 \vee t_1 \vee u_1$, where $s_1 = f(s); t_1 = f(t); u_1 = f(u)$ in $R$.

Thus $R$ is supermodular.

We have already shown that a supermodular semilattice is always modular. Hence it is natural to investigate the necessary and sufficient condition for a modular semilattice to be supermodular.

In the following theorems, since it is necessary that the semilattices are to be directed below, we assume that all the semilattices are directed below till the end of this chapter.
Theorem 2.15 :-

A semilattice \( S \) is supermodular if and only if \( S \) is modular and there is no retract of \( S \) isomorphic to the lattice \( M_4 \) or to the lattice \( M_{3,3} \).

Proof :-

Assume that a semilattice \( S \) is supermodular. Then clearly it is modular. Let \( R \) be a retract of \( S \). Then \( R \) is supermodular by the previous theorem. Therefore \( R \) is a supermodular lattice since \( R \) is a lattice. \( R \) cannot be isomorphic to the lattice \( M_4 \) or to the lattice \( M_{3,3} \).

Conversely, assume that \( S \) is a modular semilattice, which is not supermodular.

\[ \Rightarrow \text{There exist principal ideals } A = \{a\}, B = \{b\}, C = \{c\} \text{ and } D = \{d\} \]
in \( I(S) \) which will generate \( M_4 \) or \( M_{3,3} \) in \( I(S) \).

Case I :- Let \( A, B, C, D \) generate \( M_4 \) in \( I(S) \).

\[ A \lor B = B \lor C = A \lor D \]
\[ A \land B = B \land C = A \land D \]

\[ \text{fig 2.3} \]

In \( M_4 \)
\[ A \lor B = A \lor C = A \lor D = B \lor C = B \lor D = C \lor D \]
\[ \Rightarrow a \lor b = a \lor c = a \lor d = b \lor c = b \lor d = c \lor d. \]

Let \( u \) be a lower bound of \( a, b, c, d \) in \( S \) and let \( L = \{a, b, c, d, a \lor b, u\} \)

We shall prove that \( L \) is retract of \( [u, a \lor b] \). For this purpose let
\[ W = \{ x \in [u, a \lor b] / x \leq a, b, c, d \}. \]

Observe that when \( x \leq \) any two of \( a, b, c, d \) then \( x \in W \).
as \( A \land B = A \land C = A \land D = B \land D = B \land C = C \land D \).

Now define for all \( x \in [u, a \lor b] \)

\[
\begin{align*}
  f(x) &= \begin{cases} 
  u & \text{if } x \in W \\
  a & \text{if } x \leq a \text{ and } x \not\leq b, c, d \\
  b & \text{if } x \leq b \text{ and } x \not\leq a, c, d \\
  c & \text{if } x \leq c \text{ and } x \not\leq a, b, d \\
  d & \text{if } x \leq d \text{ and } x \not\leq a, b, c \\
  a \lor b & \text{if } x \not\leq a, b, c, d 
  \end{cases}
\end{align*}
\]

We must show that \( f \) is a homomorphism. Let \( x, y \in [u, a \lor b] \). We fix the value of \( f(x) \) and vary the value of \( f(y) \) and consider different cases which arise.

**Case (i) :-**

Let \( f(x) = u \). Then \( x \leq a, b, c, d \).

Suppose that \( f(y) = u \) then \( y \leq a, b, c, d \) so that \( x \lor y \leq a, b, c, d \).

Thus \( f(x \lor y) = u = u \lor u = f(x) \lor f(y) \).

Suppose that \( f(y) = a \) then \( y \leq a \) and \( y \not\leq b, c, d \) so that
\( x \lor y \leq a, x \lor y \not\leq b, c, d \).

Thus \( f(x \lor y) = a = u \lor a = f(x) \lor f(y) \).

Suppose that \( f(y) = b \) then \( y \leq b \) and \( y \not\leq a, c, d \) so that
\( x \lor y \leq b \) and \( x \lor y \not\leq a, c, d \). Thus \( f(x \lor y) = b = u \lor b = f(x) \lor f(y) \).

Suppose that \( f(y) = c \) then \( y \leq c \) and \( y \not\leq a, b, d \) so that
\( x \lor y \leq c \) and \( x \lor y \not\leq a, b, d \). Thus \( f(x \lor y) = c = u \lor c = f(x) \lor f(y) \).
Suppose that \( f(y) = d \) then \( y \leq d \) and \( y \notin a, b, c \) so that

\[
x \lor y \leq d \quad \text{and} \quad x \lor y \notin a, b, c.
\]

Thus \( f(x \lor y) = d = u \lor d = f(x) \lor f(y) \).

Finally, when \( f(y) = a \lor b \) then \( y \notin a, b, c, d \) so that

\[
x \lor y \notin a, b, c, d.
\]

Thus \( f(x \lor y) = a \lor b = u \lor (a \lor b) = f(x) \lor f(y) \).

Case (ii):- Let \( f(x) = a \).

The case \( f(y) = u \) has been considered.

Suppose \( f(y) = a \), then \( y \leq a \) and \( y \notin b, c, d \) so that \( x \lor y \leq a \) and

\[
x \lor y \notin b, c, d.
\]

Thus \( f(x \lor y) = a = a \lor a = f(x) \lor f(y) \).

Suppose \( f(y) = b \), then \( y \leq b \) and \( y \notin a, c, d \) so that

\[
x \lor y \notin a, b, c, d.
\]

Thus \( f(x \lor y) = a \lor b = f(x) \lor f(y) \).

Suppose \( f(y) = c \), then \( f(x \lor y) = a \lor b = a \lor c = f(x) \lor f(y) \).

Suppose \( f(y) = d \), then \( f(x \lor y) = a \lor b = a \lor d = f(x) \lor f(y) \).

Finally, when \( f(y) = a \lor b \) then \( y \notin a, b, c, d \) so that

\[
x \lor y \notin a, b, c, d.
\]

Thus \( f(x \lor y) = a \lor b = (a \lor b) \lor (a \lor b) = f(x) \lor f(y) \).

The cases when \( f(x) = b, c, d \) and \( f(y) \) take variable values which are similar to case (ii)

Case (iii):- Let \( f(x) = a \lor b \).

The cases \( f(y) = u, a, b, c, d \) have been considered. Suppose that \( f(y) = a \lor b \) then \( y \notin a, b, c, d \) so that \( x \lor y \notin a, b, c, d \).

Thus \( f(x \lor y) = a \lor b = (a \lor b) \lor (a \lor b) = f(x) \lor f(y) \).
Hence $L$ is an epimorph of $[u, a \lor b]$. Since $L$ is obviously a subsemilattice, it is a retract of $[u, a \lor b]$. Also, $[u, a \lor b]$ is a retract of $S$ gives $L$ is a retract of $S$, since retract of a semilattice is transitive. Thus $M_4$ is a retract of $S$ in this case.

**Case II**: Let $A, B, C, D$ generate $M_{3,3}$ in $I(S)$.

In $M_{3,3}$,

$$A \lor B = A \lor B \lor D = C \lor D \lor A = A \lor B \lor C = C \lor D \lor B = A \lor B \lor C \lor D.$$  

$$\Rightarrow a \lor b = a \lor b \lor d = c \lor d \lor a = a \lor b \lor c = c \lor d \lor b = a \lor b \lor c \lor d.$$  

Let $v$ be a lower bound of $a, b, c \lor d$ in $S$ such that $v \lor c = v \lor d = c \lor d$. The existence of such a lower bound is a consequence of the fact that $M_{3,3}$ of the figure is a sublattice of $I(S)$.

The existence of such a lower bound is a consequence of the fact that $M_{3,3}$ of the figure is a sublattice of $I(S)$.

Next let $u$ be a lower bound of $c, d, v$. Consider the semilattice

$$L = \{ a \lor b, a, b, c \lor d, v, c, d, u \}.$$

We shall prove that $L$ is a retract of $[u, a \lor b]$. For this purpose, let

$$W = \{ x \in [u, a \lor b] / x \leq a, b, c, d, a \lor b, v \}$$  

and we define

for all $x$ in $[u, a \lor b]$
We must show that \( f \) is a homomorphism. Let \( x, y \in [u, a \vee b] \), we fix the value of \( f(x) \) and consider different cases which arise.

**Case (i) :- Let** \( f(x) = u \).

Suppose that \( f(y) = u \) then \( y \leq c, d, v \) so that \( x \vee y \leq c, d, v \).

Thus \( f(x \vee y) = u = u \vee u = f(x) \vee f(y) \).

Suppose that \( f(y) = v \) then \( y \leq v \) and \( y \notin c, d \) so that \( x \vee y \leq v \) and \( x \vee y \notin c, d \) thus \( f(x \vee y) = v = u \vee v = f(x) \vee f(y) \).

Suppose that \( f(y) = c \) then \( y \leq c \) and \( y \notin v, d \) so that \( x \vee y \leq c \) and \( x \vee y \notin v, d \). Thus \( f(x \vee y) = c = u \vee c = f(x) \vee f(y) \).

Suppose that \( f(y) = d \) then \( y \leq d \) and \( y \notin v, c \) so that \( x \vee y \leq d \) and \( x \vee y \notin v, c \). Thus \( f(x \vee y) = d = u \vee d = f(x) \vee f(y) \).

Suppose that \( f(y) = c \vee d \) then \( y \leq c \vee d \) and \( x \vee y \notin a, b, c, d \) so that \( x \vee y \leq c \vee d \) and \( x \vee y \notin a, b, c, d \).

Thus \( f(x \vee y) = c \vee d = u \vee (c \vee d) = f(x) \vee f(y) \).

Suppose that \( f(y) = a \) then \( y \leq a \) and \( y \notin c \vee d, b \) so that \( x \vee y \leq a \) and \( x \vee y \notin c \vee d, b \).

Thus \( f(x \vee y) = a = u \vee a = f(x) \vee f(y) \).
Suppose that $f(y) = b$ then $y \leq b$ and $y \not\leq c \lor d, a$ so that

$x \lor y \leq b$ and $x \lor y \not\leq c \lor d, a$. Thus $f(x \lor y) = b = u \lor b = f(x) \lor f(y)$.

Finally, when $f(y) = a \lor b$ then $y \not\leq c \lor d, a, b$ so that

$x \lor y \not\leq c \lor d, a, b$. Thus $f(x \lor y) = a \lor b = u \lor (a \lor b) = f(x) \lor f(y)$.

Case (ii) :- Let $f(x) = v$.

The case $f(y) = u$ has been considered.

Suppose that $f(y) = v$ then $y \leq v$ and $v \not\leq c, d$ so that

$x \lor y \leq v$ and $x \lor y \not\leq c, d$. Thus $f(x \lor y) = v = v \lor v = f(x) \lor f(y)$.

Suppose that $f(y) = c$ then $y \leq c$ and $y \not\leq v, d$

so that $x \lor y = c \lor d$ and $x \lor y \not\leq a, b, c, d$.

Thus $f(x \lor y) = c \lor d = v \lor c = f(x) \lor f(y)$.

Suppose that $f(y) = d$ then $y \leq d$ and $y \not\leq v, c$ so that $x \lor y \leq c \lor d$ and

$x \lor y \not\leq a, b, c, d$. Thus $f(x \lor y) = c \lor d = v \lor (c \lor d) = f(x) \lor f(y)$.

Suppose $f(y) = c \lor d$ then $y \leq c \lor d$ and $y \not\leq a, b, c, d$ so that

$x \lor y \leq c \lor d$ and $x \lor y \not\leq a, b, c, d$. Thus $f(x \lor y) = c \lor d = v \lor (c \lor d) = f(x) \lor f(y)$.

Suppose $f(y) = a$ then $y \leq a$ and $y \not\leq c \lor d, b$ so that

$x \lor y \leq a$ and $x \lor y \not\leq c \lor d, b$.

Thus $f(x \lor y) = a = v \lor a = f(x) \lor f(y)$.

Suppose $f(y) = b$ we have $f(x \lor y) = b = v \lor b = f(x) \lor f(y)$.

Finally, when $f(y) = a \lor b$ we get

$f(x \lor y) = a \lor b = v \lor (a \lor b) = f(x) \lor f(y)$. 

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The cases when $f(x) = c, d, c \lor d, a, b$ and $f(y)$ take variable values are similar to case (ii).

**Case (iii) :-** Let $f(x) = a \lor b$.

The cases $f(y) = u$ and $f(y) = v$ have been considered in cases (i) and (ii)

Suppose $f(y) = c$ then $y \leq c$ and $y \notin v, d$ so that $x \lor y \notin c \lor d, a, b$.

Thus $f(x \lor y) = a \lor b = (a \lor b) \lor c = f(x) \lor f(y)$.

Suppose $f(y) = d$ then $y \leq d$ and $y \notin v, c$ so that $x \lor y \notin c \lor d, a, b$.

Thus $f(x \lor y) = a \lor b = (a \lor b) \lor d = f(x) \lor f(y)$.

Suppose $f(y) = c \lor d$ then $y \leq c \lor d$ and $y \notin a, b, c, d$ so that $x \lor y \notin c \lor d, a, b$. Thus $f(x \lor y) = a \lor b = (a \lor b) \lor (c \lor d) = f(x) \lor f(y)$.

Suppose $f(y) = a$ then $y \leq a$ and $y \notin c \lor d, b$ so that $x \lor y \notin c \lor d, a, b$. Thus $f(x \lor y) = a \lor b = (a \lor b) \lor a = f(x) \lor f(y)$.

Suppose $f(y) = b$ then $y \leq b$ and $y \notin c \lor d, a$ so that $x \lor y \notin c \lor d, a, b$. Thus $f(x \lor y) = a \lor b = (a \lor b) \lor b = f(x) \lor f(y)$.

Finally, $f(y) = a \lor b$ then $y \notin c \lor d, a, b$ so that $x \lor y \notin c \lor d, a, b$. Thus $f(x \lor y) = a \lor b = (a \lor b) \lor (a \lor b) = f(x) \lor f(y)$.

Hence $L$ is an epimorph of $[u, a \lor b]$. Since it is obviously a subsemilattice, it is a retract of $[u, a \lor b]$. Also $[u, a \lor b]$ is a retract of $S$ gives $L$ is a retract of $S$, since retract of a semilattice is transitive. Thus $M_{3,3}$ is a retract of $S$ in this case.
Theorem 2.16: -

A semilattice $S$ is modular if and only if there is no retract of $S$ is isomorphic to the non modular five lattice $N_5$.

Proof: -

Assume a semilattice $S$ is modular. Let $R$ be a retract of $S$. Then $R$ is modular. Also $R$ is a modular lattice, since $R$ is a lattice. Hence $R$ cannot be isomorphic to non modular five element lattice $N_5$.

Conversely, assume a semilattice $S$ is non modular. Then there exist principal ideals $A = (a)$, $B = (b)$, $C = (c)$ will generate $N_5$ in $I(S)$.

\[
\begin{align*}
A \vee B &= B \vee C \\
A \wedge B &= B \wedge C \\
A \vee b &= b \vee c.
\end{align*}
\]

Let $u$ be a lower bound for $a$ and $b$ and let $L = \{ u, a, b, c, a \vee b \}$. We shall prove that $L$ is a retract of $[u, a \vee b]$. For this purpose let

\[W = \{ x \in [u, a \vee b] / x \leq b, x \leq c \}\]

Now define for all $x \in [u, a \vee b]$. 44
We must show $f$ is a homomorphism. Let $x, y \in [u, a \lor b]$. We fix the values of $f(x)$ and vary the values of $f(y)$. There are five different cases.

**Case (i):**

Let $f(x) = a$. Then $x \neq b$ and $x \leq a \lor z$ for some $z \in W$.

Suppose $f(y) = a$ then $y \leq a \lor w$ for some $w \in W$ so that $x \lor y \leq a \lor (z \lor w)$ and $z \lor w \in W$. Thus $f(x \lor y) = a = f(x) \lor f(y)$.

Suppose $f(y) = u$ then $x \lor y \leq a \lor (z \lor y)$ and $z \lor y \in W$. Thus $f(x \lor y) = u = f(x) \lor f(y)$.

Suppose $f(y) = c$. Since $y \neq a \lor p$ for every $p \in W$, $x \lor y \neq a \lor p$ for every $p \in W$. Since $z \in W$, $x \leq a \lor z$ and $a \lor c = c$. So $x \lor y \leq c$. Thus $f(x \lor y) = c = f(x) \lor f(y)$.

Suppose $f(y) = b$ or $f(y) = a \lor b$ then $y \neq c$. So $x \lor y \neq c$.

Since $x \lor y \neq b$, $f(x \lor y) = a \lor b = f(x) \lor f(y)$.

**Case (ii):**

Let $f(x) = u$.

The case $f(y) = a$ has been considered.

Suppose $f(y) = b$, we have $x \lor y \leq b$ and $x \lor y \neq c$.

So $f(x \lor y) = b = f(x) \lor f(y)$.

Suppose $f(y) = c$, $x \lor c \leq c$, $x \lor y \neq b$ and $x \lor y \neq a \lor z$.
for every $z \in w$. Thus $f(x \lor y) = c = f(x) \lor f(y)$.

Suppose $f(y) = a \lor b$, $x \lor y \neq b$ and $x \lor y \neq c$. So

$f(x \lor y) = a \lor b = f(x) \lor f(y)$.

The remaining cases $f(x) = a \lor b$, $f(x) = c$, $f(x) = b$ can be easily handled.

Hence $L$ is an epimorph of $[u, a \lor b]$. Since it is obviously a subsemilattice, it is a retract of $[u, a \lor b]$. Also $[u, a \lor b]$ is a retract of $S$ gives $L$ is a retract of $S$, since retract of a semilattice is transitive. Thus $N_5$ is a retract of $S$.

Combining Theorems 2.15 and 2.16 we obtain

**Theorem 2.17** :-

A Semilattice $S$ is supermodular if and only if there is no retract of $S$ isomorphic to $N_5$ or $M_4$ or $M_{3,3}$. 

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