CHAPTER IV

CONGRUENCE RELATIONS IN SUPERMODULAR SEMILATTICES

In this chapter some semilattice congruence relations are introduced and the properties are derived. Also, a number of new results have been obtained and main results give connection between congruence relation, principal filter and neutral filter. This result will be the base for the triple representation which is to be discussed in the next chapter. Throughout this chapter $S$ denotes meet semilattice.

Definition 4.1:

Let $S$ be a semilattice and $\Theta$ a relation on $S$. Then $\Theta$ is called a congruence relation if

(i) $\Theta$ is reflexive

\[ x \equiv x (\Theta) \quad \text{for all } x \in S \]

(ii) $\Theta$ is symmetric

\[ x \equiv y (\Theta) \implies y \equiv x (\Theta) \quad \text{for all } x, y \in S \]

(iii) $\Theta$ is transitive

\[ x \equiv y (\Theta), \; y \equiv z (\Theta) \implies x \equiv z (\Theta) \quad \text{for all } x, y, z \in S \]

(iv) Substitution property

\[ x \equiv x_1 (\Theta), \; y \equiv y_1 (\Theta) \]

\[ \Rightarrow \quad x \land y \equiv x_1 \land y_1 (\Theta) \quad \text{for all } x, x_1, y, y_1 \in S \]

Example 4.1:

Let $F$ be a filter of a semilattice $S$. The binary relation $\phi(F)$ on $S$ is defined by $x \equiv y(\phi(F))$ if and only if $x \land t = y \land t$ for some $t \in F$. Then $\phi(F)$ is a congruence relation on $S$. 

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Example 4.2:-
Let $S$ be a semilattice and $b \in S$. The binary relation $\phi_b$ on $S$ is defined by $x = y(\phi_b)$ if and only if $x \land b = y \land b$. Then $\phi_b$ is a congruence relation on $S$.

Proposition 4.1:-
Let $S$ be a semilattice and $b \in S$. The congruence $\phi([b])$ is defined by $x \equiv y(\phi([b]))$ if and only if $x \land t = y \land t$ for some $t \in [b]$, where $[b]$ is a filter in $S$. Then $\phi_b = \phi([b])$.

Proof:-
First to prove $\phi_b \subseteq \phi([b])$

Let $(x, y) \in \phi_b$ be arbitrary

$\Rightarrow x = y(\phi_b)$

$\Rightarrow x \land b = y \land b$.

$\Rightarrow x \equiv y(\phi([b])), \text{since } b \in [b]$

$\Rightarrow (x, y) \in \phi([b])$

Therefore $\phi_b \subseteq \phi([b])$ \hspace{1cm} (1)

Next to prove that $\phi([b]) \subseteq \phi_b$.

Let $(x, y) \in \phi([b])$ be arbitrary

$\Rightarrow x \land b_1 = y \land b_1, \text{ for some } b_1 \in [b]$

$\Rightarrow x \land b_1 \land b = y \land b_1 \land b$

$\Rightarrow x \land b = y \land b, \text{ since } b_1 \geq b$

$\Rightarrow x = y(\phi_b)$

$\Rightarrow (x, y) \in \phi_b$

Therefore $\phi([b]) \subseteq \phi_b$ \hspace{1cm} (2)

From (1) and (2) we have $\phi_b = \phi([b])$.  

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Proposition 4.2:-
Let $S$ be a semilattice and $F$ be a filter in $S$. Then $x = y(\phi(F))$ if and only if $F \lor [x] = F \lor [y]$ holds in $\mathcal{F}(S)$.

Proof:

Assume that $x \equiv y(\phi(F))$ where $x, y \in S$ and $F$ is a filter of $S$.

$\Rightarrow$ $x \land t = y \land t$ for some $t \in F$.

$\Rightarrow$ $[x \land t] = [y \land t]$ in $\mathcal{F}(S)$

$\Rightarrow$ $[x] \lor [t] = [y] \lor [t]$ in $\mathcal{F}(S)$

$\Rightarrow$ $[x] \lor [t] \lor F = [y] \lor [t] \lor F$ in $\mathcal{F}(S)$

$\Rightarrow$ $[x] \lor F = [y] \lor F$ in $\mathcal{F}(S)$

Conversely, assume that $F \lor [x] = F \lor [y]$

$\Rightarrow$ $[x] = [y](\phi(\mathcal{F}(S)))$

$\Rightarrow$ $[x] \lor [t] = [y] \lor [t]$ for some $t \in \mathcal{F}(S)$

$\Rightarrow$ $[x \land t] = [y \land t]$

$\Rightarrow$ $x \land t = y \land t$

$\Rightarrow$ $x \equiv y(\phi(F))$

Proposition 4.3:-

Let $S$ be a semilattice and $F$ be a filter in $S$. If $x, y \in S$ and $t \in F$ are such that $x \land y = x \land t$ then $[x] \phi(F) \leq [y] \phi(F)$

Proof :-

Given $x, y \in S$, $t \in F$ and $x \land y = x \land t$

$\Rightarrow$ $(x \land y) \land t = (x \land t) \land t$

$\Rightarrow$ $(x \land y) \land t = x \land t$

$\Rightarrow$ $x \land y \equiv x(\phi(F))$
The next theorem is the converse case of Proposition 4.3 which characterises supermodularity.

**Theorem 4.1:**

Let \( x \) and \( y \) be arbitrary elements of a semilattice \( S \). Then for any filter \( F \), \( [x] \phi(F) \leq [y] \phi(F) \) in \( S/\phi(F) \) implies

\[
x \land y = x \land t \quad \text{for some} \quad t \in F
\]

if and only if \( S \) is supermodular.

**Proof:**

Assume that \( S \) is a supermodular semilattice and \( F \) is any filter in \( S \).

Let \( x, y \in S \) be arbitrary. To prove that \( [x] \phi(F) \leq [y] \phi(F) \) implies

\[
x \land y = x \land t \quad \text{for some} \quad t \in F
\]

Suppose \( [x] \phi(F) \leq [y] \phi(F) \) in \( S/\phi(F) \)

\[
\Rightarrow ([x] \phi(F)) \land ([y] \phi(F)) = [x] \phi(F)
\]

\[
\Rightarrow [x \land y] \phi(F) = [x] \phi(F)
\]

\[
\Rightarrow x \land y \equiv x \phi(F)
\]

\[
\Rightarrow x \land y \land w = x \land w \quad \text{for some} \quad w \in F
\]

\[
\Rightarrow x \geq x \land y \geq x \land w
\]

\[
\Rightarrow x \land y = x \land t \quad \text{for some} \quad t \geq w, \text{ since } S \text{ is supermodular.}
\]

\[
\Rightarrow x \land y = x \land t \quad \text{for some} \quad t \in F \text{ since } F \text{ is a filter and } w \in F.
\]
Thus $[x] \phi(F) \leq [y] \phi(F)$ implies $x \land y = x \land t$ for some $t \in F$.

Conversely, assume that $F$ is any filter of semilattice $S$ and

$[x] \phi(F) \leq [y] \phi(F)$ implies $x \land y = x \land t$ for some $t \in F$.

To prove that $S$ is supermodular.

Suppose that $x, y_1, y_2, y_3, z \in S$ and $x \geq z \geq x \land y_1, x \land y_2, x \land y_3$.

Let $F_1 = [y_1]$

$x \geq z \Rightarrow x \land y_1 \geq z \land y_1$

$z \geq x \land y_1 \Rightarrow z \land y_1 \geq x \land y_1$

$\Rightarrow z \land y_1 = x \land y_1$

$\Rightarrow z = x(\phi(F_1))$ (1)

Now let $F_2 = [y_2]$

$x \geq z \Rightarrow x \land y_2 \geq z \land y_2$

$z \geq x \land y_2 \Rightarrow z \land y_2 \geq x \land y_2$

$\Rightarrow z \land y_2 = x \land y_2$

$\Rightarrow z = x(\phi(F_2))$ (2)

Let $F_3 = [y_3]$

$x \geq z \Rightarrow x \land y_3 \geq z \land y_3$

$z \geq x \land y_3 \Rightarrow z \land y_3 \geq x \land y_3$

$\Rightarrow z \land y_3 = x \land y_3$

$\Rightarrow z = x(\phi(F_3))$ (3)

From (1), (2) and (3) we get

$z = x(\phi(F_1) \cap \phi(F_2) \cap \phi(F_3))$

$\Rightarrow z = x(\phi(F_1 \cap F_2 \cap F_3))$

$\Rightarrow [x](\phi(F)) = [z](\phi(F))$ where $F = F_1 \cap F_2 \cap F_3$.
\[ [x](\phi(F)) \leq [z](\phi(F)) \]
\[ x \land z = x \land t \quad \text{for some } t \in F \text{ by assumption.} \]
\[ t \in F \Rightarrow t \in F_1 \cap F_2 \cap F_3 \]
\[ \Rightarrow t \geq y_1, \ t \geq y_2, \ t \geq y_3 \text{ and } x \geq z \]
\[ \Rightarrow z = x \land z = x \land t \]
\[ \Rightarrow z = x \land t = x \land t \land t \land t \quad \text{where} \]
\[ t \geq y_1, y_2, x \land y_3 \]
\[ t \geq y_1, y_3, x \land y_1 \]
\[ t \geq y_1, y_3, x \land y_2, \]

Hence \( S \) is a supermodular semilattice.

**Theorem 4.2:-**

If \( F \) is a filter on a supermodular semilattice \( S \) then \( S/(\phi(F)) \) is supermodular.

**Proof:**

Given that \( F \) is a filter on a supermodular semilattice \( S \) and \( \phi(F) \) is a congruence relation on \( S \). Then \( S/(\phi(F)) \) is a semilattice.

To prove that \( S/(\phi(F)) \) is supermodular.

For \( x \in S \), let \( \bar{x} \) denote the \( \phi(F) \) class of \( x \).

Suppose that \( a, b, c, d, x \in S \) and \( \bar{a} \geq \bar{x} \geq \bar{a} \land \bar{b} \lor \bar{c} \lor \bar{a} \land \bar{d} \)

\[ \bar{x} \leq \bar{a} \Rightarrow a \land x = x \land w, \quad w \in F \]
\[ \Rightarrow (a \land b) \land x = (x \land w) \land b \quad (1) \]
\[ \overline{a \land b} \leq \overline{x} \Rightarrow (a \land b) \land x = (a \land b) \land u_1, \quad u_1 \in F \quad (2) \]

(1) and (2) \[ \Rightarrow (x \land w) \land b = (a \land b) \land u_1 \]
\[ \Rightarrow x \land w \geq (a \land b) \land u_1 \]
\[ \Rightarrow x \land w \land w \geq (a \land b) \land u_1 \land w \]
\[ \Rightarrow x \land w \geq (a \land w) \land (b \land u_1) \]

Also, \[ x \leq \overline{a} \Rightarrow a \land x \geq x \land w \]
\[ \Rightarrow a \land x \land w \geq x \land w \land w \]
\[ \Rightarrow a \land w \geq x \land w \]

Therefore \[ a \land w \geq x \land w \geq (a \land w) \land (b \land u_1) \]

Let \[ x_1 = x \land w \]
\[ a_1 = a \land w \]
\[ b_1 = b \land u_1 \]

Then we have \[ a_1 \geq x_1 \geq a_1 \land b_1 \]

Similarly from the other inequalities, we have \[ a_1 \geq x_1 \geq a_1 \land c_1 \]

and \[ a_1 \geq x_1 \geq a_1 \land d_1 \]

where \[ c_1 = c \land u_2, \quad u_2 \in F \]
\[ d_1 = d \land u_3, \quad u_3 \in F \]

Therefore \[ a_1 \geq x \geq a_1 \land b_1, a_1 \land c_1, a_1 \land d_1 \]

\[ \Rightarrow \text{there exists } s, t, y \text{ in } S \text{ such that} \]
\[ s \geq b_1, c_1, a_1 \land d_1 \]

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\[ t \geq c_1, d_1, a_1 \land b_1 \]
\[ y \geq d_1, b_1, a_1 \land c_1. \]

and \[ x_1 = a_1 \land s \land t \land y, \text{ since } S \text{ is supermodular} \]

\[ \implies x \land w = a \land w \land s \land t \land y \]

Therefore we can write

\[ x \equiv x \land w (\phi(F)) \]
\[ \equiv a \land w \land s \land t \land y (\phi(F)) \]
\[ \equiv a \land s \land t \land y (\phi(F)) \]

\[ \implies \tilde{x} = a \land s \land t \land y \quad \text{ where} \]
\[ \tilde{s} \geq b, c, \tilde{a} \land \tilde{d} \]
\[ \tilde{t} \geq c, \tilde{d}, \tilde{a} \land \tilde{b} \]
\[ \tilde{u} \geq b, \tilde{d}, \tilde{a} \land \tilde{c} \]

since (i) \[ s \land b \equiv s \land b \land u_1 \phi(F) \]

\[ \equiv b \land u_1 \phi(F) \text{ since } s \geq b \land u_1 \]
\[ \equiv b(\phi(F)) \text{ since } (b \land u_1) \land u_1 = b \land u_1 \]

\[ \implies \tilde{s} \land \tilde{b} = \tilde{b} \]

\[ \implies \tilde{s} \geq \tilde{b} \]

(ii) \[ s \land c \equiv s \land c \land u_2 (\phi(F)) \]

\[ \equiv c \land u_2 \phi(F) \text{ since } s \geq c \land u_2 \]
\[ \equiv c(\phi(F)) \text{ since } c \land u_2 \land u_2 = c \land u_2 \]

\[ \implies \tilde{s} \land \tilde{c} = \tilde{c} \]
\[ \Rightarrow \quad \tilde{s} \geq \tilde{c} \]

(iii) \[ s \land a \land d \land w \land u_3 \equiv s \land (a \land d) \land (w \land u_3) \land (w \land u_3) \]
\[ \Rightarrow \quad s \land (a \land d) \equiv s \land a \land d \land w \land u_3 \quad (\phi(F)) \]
\[ \Rightarrow \quad s \land (a \land d) \equiv a \land d \land w \land u_3 \quad (\phi(F)), \]
since \( s \geq a \land d \land w \land u_3 \)
also \[ \quad a \land d \land w \land u_3 \equiv a \land d \quad (\phi(F)) \]

Combining the above two equations we get
\[ s \land (a \land d) \equiv a \land d \quad (\phi(F)) \]
\[ \Rightarrow \quad \tilde{s} \land (\tilde{a} \land \tilde{d}) = \tilde{a} \land \tilde{d} \]
\[ \Rightarrow \quad \tilde{s} \geq \tilde{a} \land \tilde{d} \]
\[ \Rightarrow \quad \tilde{s} \geq \tilde{b}, \tilde{c}, \tilde{a} \land \tilde{d} \]

In a similar way it can be proved that
\[ \Rightarrow \quad \tilde{t} \geq \tilde{c}, \tilde{d}, \tilde{a} \land \tilde{b} \]
\[ \Rightarrow \quad \tilde{u} \geq \tilde{b}, \tilde{d}, \tilde{a} \land \tilde{c} \]

which implies \( S/(\phi(F)) \) is supermodular.

**Definition 4.3:**

Let \( S \) be a semilattice, \( C \) be a subsemilattice and \( D \) be a filter on \( S \).

For each \( c \in C \), the set \( c\psi \) is defined as \( c\psi = \{ d \in D ; d \geq c \} \).

**Proposition 4.4:**

\[ c\psi = D \cap [c) \] and \( c\psi \) is a filter on \( D \) and \( S \).

**Proof:**

To prove \( c\psi \subseteq D \cap [c) \)
Let \( d_i \in c \psi \) be arbitrary.

\[
\Rightarrow \quad d_i \in D \text{ such that } d_i \geq c \\
\Rightarrow \quad d_i \in D \text{ and } d_i \in [c) \\
\Rightarrow \quad d_i \in D \cap [c)
\]

Therefore \( c \psi \subseteq D \cap [c) \tag{1} \)

To prove \( D \cap [c) \subseteq c \psi \)

Let \( d_2 \in D \cap [c) \) be arbitrary

\[
\Rightarrow \quad d_2 \in D \text{ and } d_2 \in [c) \\
\Rightarrow \quad d_2 \in D \text{ and } d_2 \geq c \\
\Rightarrow \quad d_2 \in c \psi
\]

Therefore \( D \cap [c) \subseteq c \psi \tag{2} \)

From (1) and (2) \( D \cap [c) = c \psi \)

To prove that \( c \psi \) is a filter on \( D \) and on \( S \). It is enough to prove that \( c \psi \) is a filter on \( D \).

Let \( x, y \in c \psi \)

\[
\Rightarrow \quad x, y \in D \text{ and } x, y \in [c) \\
\Rightarrow \quad x \wedge y \in D, \quad \text{since } D \text{ is a filter and} \\
\Rightarrow \quad x \wedge y \in [c), \quad \text{since } x \geq c, \; y \geq c \Rightarrow x \wedge y \geq c \\
\Rightarrow \quad x \wedge y \in c \psi
\]

Let \( x \leq t \) where \( x \in c \psi \) and \( t \in D \)

\[
\Rightarrow \quad x \in c \psi \\
\Rightarrow \quad x \in D \\
\Rightarrow \quad t \in D \\
\Rightarrow \quad x \in [c) \\
\Rightarrow \quad x \geq c \\
\Rightarrow \quad t \geq x \geq c
\]
Let $d_i \in c\psi$ be arbitrary.

- $d_i \in D$ such that $d_i \geq c$
- $d_i \in D$ and $d_i \in [c)$
- $d_i \in D \cap [c)$

Therefore $c\psi \subseteq D \cap [c)$  \hspace{1cm} (1)

To prove $D \cap [c) \subseteq c\psi$

Let $d_2 \in D \cap [c)$ be arbitrary

- $d_2 \in D$ and $d_2 \in [c)$
- $d_2 \in D$ and $d_2 \geq c$
- $d_2 \in c\psi$

Therefore $D \cap [c) \subseteq c\psi$  \hspace{1cm} (2)

From (1) and (2) $D \cap [c) = c\psi$

To prove that $c\psi$ is a filter on $D$ and on $S$. It is enough to prove that $c\psi$ is a filter on $D$.

Let $x, y \in c\psi$

- $x, y \in D$ and $x, y \in [c)$
- $x \land y \in D$, since $D$ is a filter and
- $x \land y \in [c)$ since $x \geq c, y \geq c \Rightarrow x \land y \geq c$
- $x \land y \in c\psi$

Let $x \leq t$ where $x \in c\psi$ and $t \in D$

$x \in c\psi$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $x \in D$

$\Rightarrow$ \hspace{1cm} $t \in D$

$x \in c\psi$ \hspace{1cm} $\Rightarrow$ \hspace{1cm} $x \in [c)$

$\Rightarrow$ \hspace{1cm} $x \geq c$

$\Rightarrow$ \hspace{1cm} $t \geq x \geq c$
\[ \Rightarrow \ t \in [c] \]
\[ \Rightarrow \ t \in D \text{ and } t \in [c] \]
\[ \Rightarrow \ t \in c \psi \]

Hence \( c \psi \) is a filter on \( D \)
\[ \Rightarrow \ c \psi \text{ is a filter on } S. \]

Proposition 4.5:-

The map \( \psi : C \to \mathcal{F}(D) \) defined by \( c \psi = \{d \in D : d \geq c\} = D \cap [c] \)

is an order reversing map.

Proof:-

Let \( a, b \in C \) and \( a \leq b \)
\[ a \psi = \{d \in D : d \geq a\} \]
\[ b \psi = \{d \in D : d \geq b\} \]

Let \( x \in b \psi \)
\[ \Rightarrow \ x \in D \text{ and } x \geq b \]
\[ \Rightarrow \ x \in D \text{ and } x \geq b \geq a \]
\[ \Rightarrow \ x \in D \text{ and } x \geq a \]
\[ \Rightarrow \ x \in a \psi \]

Therefore \( b \psi \subseteq a \psi \)

Hence \( \psi \) is an order reversing map.

Result 4.1:-

\( \phi(c(\psi)) \) is a congruence relation on \( S \), defined by \( x = y(\phi(c(\psi))) \) if and only if \( x \wedge t = y \wedge t \) for some \( t \in c \psi \).

Result 4.2:-

\( \phi_c \) is a congruence relation on \( S \), defined by \( x = y(\phi_c), \ c \in C \), \( x, y \in S \) if and only if \( x \wedge c = y \wedge c \).
Definition 4.4:-

Let \( \Theta(c \psi) \) and \( \Theta_c \) be the restrictions of \( \phi(c \psi) \) and \( \phi_c \) respectively to \( D \).

Then \( \phi(c \psi) = \Theta(c \psi) \) on \( D \) and \( \phi_c = \Theta_c \) on \( D \).

The next theorem describes when \( \Theta(c \psi) \) and \( \Theta_c \) have the same restrictions on \( D \).

Theorem 4.3:-

For a semilattice \( S \), the following are equivalent:

(i) \( S \) is supermodular

(ii) \( \Theta(\alpha \psi) = \Theta_\alpha \) for any \( \alpha \in S \) and any filter \( D \).

(iii) \( \Theta(\alpha \psi) = \Theta_\alpha \), for any \( \alpha \in S \) and any principal filter \( D \).

Proof :- (i) \( \Rightarrow \) (ii)

Assume that \( S \) is supermodular. To prove that \( \Theta(\alpha \psi) \subseteq \Theta_\alpha \).

Let \( (x, y) \in \Theta(\alpha \psi) \) be arbitrary

\( \Rightarrow \) \( x = y(\Theta(\alpha \psi)) \)

\( \Rightarrow \) \( x \land t = y \land t, \ t \in \alpha \psi \)

\( \Rightarrow \) \( x \land t \land \alpha = y \land t \land \alpha, \ t \geq \alpha \)

\( \Rightarrow \) \( x \land \alpha = y \land \alpha \)

\( \Rightarrow \) \( (x, y) \in \Theta_\alpha \)

Therefore \( \Theta(\alpha \psi) \subseteq \Theta_\alpha \).

Next to prove that \( \Theta_\alpha \subseteq \Theta(\alpha \psi) \).

Let \( (x, y) \in \Theta_\alpha \) be arbitrary

\( \Rightarrow \) \( x = y(\Theta_\alpha) \) where \( x, y \in D \)

\( \Rightarrow \) \( x \land \alpha = y \land \alpha \), \( \alpha \in S \)

\( \Rightarrow \) \( [x \land \alpha] = [y \land \alpha] \) in \( \mathcal{F}(S) \)

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Consider \[ [x] \lor [a] = [y] \lor [a] \] in \( F(S) \) \hspace{1cm} (1)

\[
[x] \lor a \psi = [x] \lor ([a] \cap D) \\
= ([x] \lor [a]) \cap D, \text{ since } S \text{ is supermodular} \\
= ([y] \lor [a]) \cap D, \text{ using (1)} \\
= [y] \lor ([a] \cap D) \\
= [y] \lor a \psi
\]

\[ \Rightarrow \quad x \equiv y (\phi(a \psi)) \]
\[ \Rightarrow \quad x \equiv y (\Theta (a \psi)) \]
\[ \Rightarrow \quad (x, y) \in \Theta (a \psi) \]

Therefore \( \Theta_a \subseteq \Theta(a \psi) \).

Hence \( \Theta(a \psi) = \Theta_a \)

\[ (ii) \quad \Rightarrow \quad (iii) \]

Assume that \( \Theta(a \psi) = \Theta_a \) for any \( a \in S \) and any filter \( D \).

\[ \Rightarrow \Theta(a \psi) = \Theta_a \text{ for any } a \in S \text{ and any principal filter } D \text{ in particular.} \]

Hence (iii) is true.

\[ (iii) \quad \Rightarrow \quad (i) \]

Assume that (iii) is true. To prove that \( S \) is supermodular.

Suppose \( a, b, c, d, z \in S \) such that \( a \geq z \geq a \land b, a \land c, a \land d. \)

Put \( D = \{ z \}. \)

\[
\begin{align*}
\quad & a \geq z \land z \geq a \land b \\
\Rightarrow & \quad a \land b \geq z \land b \text{ and } z \land b \geq a \land b \\
\Rightarrow & \quad z \land b = a \land b \\
\Rightarrow & \quad z \equiv a (\Theta_b) \\
\Rightarrow & \quad z \equiv a(\Theta(b \psi)). \hspace{1cm} (2)
\end{align*}
\]

Similarly \( a \geq z \) and \( z \geq a \land c \)
\[ z \land c = a \land c \]
\[ z \equiv a (\Theta_c) \]
\[ z \equiv a (\Theta(c\psi)) \]  \hspace{1cm} (3)

(2) and (3)
\[ z \equiv a(\Theta(b\psi) \cap \Theta(c\psi)) \]
\[ z \equiv a(\Theta((b\psi) \cap (c\psi))) \]
\[ z \land y_1 = a \land y_1 \text{ for some } y_1 \in b\psi, y_1 \in c\psi, y_1 \in D = [z] \]
\[ z = z \land y_1 = a \land y_1 \]

Now consider the inequalities  \[ a \geq z \text{ and } z \geq a \land d \]
\[ z \land d = a \land d \]
\[ z \equiv a (\Theta_d) \]
\[ z \equiv a(\Theta(d\psi)) \]  \hspace{1cm} (4)

(3) and (4)
\[ z \equiv a(\Theta(c\psi) \cap \Theta(d\psi)) \]
\[ z \equiv a(\Theta((c\psi) \cap (d\psi))) \]
\[ z \land y_2 = a \land y_2 \text{ for some } y_2 \in c\psi, y_2 \in d\psi, y_2 \in D = [z] \]
\[ z = z \land y_2 = a \land y_2 \]

In a similar way, (2) and (4) implies
\[ z \equiv a(\Theta(b\psi) \cap \Theta(d\psi)) \]
\[ z = z \land y_3 = a \land y_3 \text{ for some } y_3 \in b\psi, y_3 \in d\psi, y_3 \in D = [z] \]
\[ z = z \land y_3 = a \land y_3 \]

Combining the three results we have
\[ z = a \land y_1 \land y_2 \land y_3 \quad \text{where} \quad y_1 \geq b, c, a \land d \]
\[ y_2 \geq c, d, a \land b \]
\[ y_3 \geq b, d, a \land c \]

Hence \( S \) is supermodular.
Theorem 4.4:-

Let $S$ be a supermodular semilattice with the greatest element $1$. $C$ a subsemilattice and $D$ a filter on $S$. Suppose that $S$, $C$ and $D$ are interrelated by the condition:

For all $s \in S$, there exist $c \in C$ and $d \in D$ such that $s = c \land d$.

Then the following are equivalent:

(i) $D$ is a neutral filter

(ii) For any $a, b \in C$, $(a \land b) \psi = a \lor b \psi$

(iii) For any $a, b \in C$, $\Theta_a \land b = \Theta_a \lor \Theta_b$ holds in the lattice of congruences on $D$.

Proof: -

(i) $\Rightarrow$ (ii)

Assume that (i) is true. To prove (ii)

$D$ is a neutral filter

$\Rightarrow$ $D$ is a dually distributive filter, by Theorem 3.5.

$\Rightarrow D \land (X \lor Y) = (D \land X) \lor (D \land Y)$ for all $X, Y$ in $\mathcal{F}(S)$ (1)

Suppose $a, b \in C$

Then $(a \land b) \psi = D \cap [a \land b]$

$\quad = D \land ([a] \lor [b])$

$\quad = (D \land [a]) \lor (D \land [b])$ using (1)

$\quad = a \lor b \psi$

To prove (ii) $\Rightarrow$ (i)

(ii) gives for any $a, b \in C$,

$(a \land b) \psi = a \lor b \psi$

$\Rightarrow ([a \land b]) \cap D = ([a] \cap D) \lor ([b] \cap D)$

$\Rightarrow ([a] \lor [b]) \cap D = ([a] \cap D) \lor ([b] \cap D)$ for any $a, b \in C$. 

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Let \( x, y \in S, x = a \land d \) and \( y = b \land e \) for suitable \( a, b \in C \) and \( d, e \in D \). Then
\[
(x) \cap D = ([a \land d]) \cap D \\
= ([a] \lor [d]) \cap D \\
= ([d] \lor [a]) \cap D \\
= [d] \lor ([a] \cap D), \text{ since } S \text{ is supermodular and } \{d\} \subseteq D
\]
Similarly
\[
(y) \cap D = ([b] \lor [e]) \cap D \\
= ([b] \lor [e]) \cap D \\
\]
Thus
\[
(x) \cap D \lor (y) \cap D = ([a] \lor [b]) \cap D \lor [d] \lor [e] \\
= ([a] \lor [b]) \cap D \lor [d] \lor [e], \text{ using (ii)} \\
= ([a \land b] \lor [d \land e]) \cap D, \\
\text{ since } S \text{ is supermodular and } \{d \land e\} \subseteq D.
\]
\[
= [a \land b \land d \land e] \cap D \\
= [x \lor y] \cap D \\
= ([x] \lor [y]) \cap D
\]

Now let \( I \) and \( J \) be arbitrary filters of \( S \). To prove that \( D \) is neutral.

By Theorem 3.8, it is sufficient to show that \( (I \lor J) \cap D = (I \cap D) \lor (J \cap D) \), since \( S \) is supermodular.

For that we need only to prove
\[
(I \lor J) \cap D \subseteq (I \cap D) \lor (J \cap D) 
\]
since the reverse inequality always holds.

Let \( w \in (I \lor J) \cap D \)
\[
\Rightarrow w \in D \text{ and } w \geq x \land y \text{ for some } x \in I, y \in J.
\]
Then
\[
\Rightarrow w \in [x \land y] \cap D \\
\Rightarrow w \in ([x] \lor [y]) \cap D \\
\Rightarrow (I \lor J) \cap D \subseteq (I \cap D) \lor (J \cap D).
\]
\[
\Rightarrow D \text{ is neutral.}
\]
(ii) \implies (iii)

Given that for any $a, b \in C$, $(a \land b) \psi = a \lor b \psi$

To prove that for any $a, b \in C$, $\Theta_{a \land b} = \Theta_a \lor \Theta_b$.

Since $\Theta(a \psi)$ is the congruence on $D$ induced by the filter $a \psi$ on $D$,

$\Theta(a \psi) \lor \Theta(b \psi) = \Theta(a \psi \lor b \psi)$

$\Theta(a \psi) \lor \Theta(b \psi) = \Theta((a \land b) \psi)$, using (ii)

$S$ is supermodular

\[ \implies \Theta_a \lor \Theta_b = \Theta_{a \land b} \quad \text{by Theorem 4.3.} \]

(iii) \implies (ii)

Given that for any $a, b \in C$,

$\Theta_{a \lor b} = \Theta_a \lor \Theta_b$ holds in the lattices of congruences on $D$.

To prove that for any $a, b \in C$,

$(a \land b) \psi = a \psi \lor b \psi$

$\Theta_{a \land b} = \Theta((a \land b) \psi)$, using Theorem 4.3.

$\Theta_a \lor \Theta_b = \Theta(a \psi) \lor \Theta(b \psi)$, using Theorem 4.3.

Since $\Theta_{a \lor b} = \Theta_a \lor \Theta_b$, we have

$\Theta(a \land b) \psi = \Theta(a \psi) \lor \Theta(b \psi)$ \quad (1)

Now as $\psi$ is order reversing by Proposition 4.5.

$(a \land b) \psi \geq a \psi \lor b \psi$

To prove the reverse inequality, let

$x \in (a \land b) \psi$

$x \in (a \land b) \cap D$

\[ \implies x \in (a \land b) \]

\[ \implies x \geq a \land b \]

\[ \implies x \land (a \land b) = a \land b \]

\[ \implies x \land (a \land b) = 1 \land (a \land b) \]

\[ \implies x \equiv 1(\Theta(a \land b) \psi) \]

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\[\Rightarrow x \equiv 1(\Theta(a \psi \lor b \psi)) \quad \text{using (1)}\]

Therefore \((a \land b)\psi \leq a \psi \lor b \psi\)

Hence \((a \land b)\psi = a \psi \lor b \psi\)