CHAPTER 5

JUST TOTAL EXCELLENT GRAPHS

Abstract

In this chapter, we introduce a new class of total excellent graphs, called Just Total Excellent (JTE). We prove that each connected graph is an induced subgraph of a total excellent graph. We obtain a necessary and sufficient condition for a graph to be JTE. An upper bound for \( \gamma_t(G) \), for a JTE graph \( G \) is determined. If \( G \neq mK_2 \) is a JTE graph, we show that \( G \) has no cut vertex.

Fricke, Haynes, Hedetniemi and Laskar defined a graph \( G \) to be total excellent graph if to each \( u \in V \) there is a \( \gamma_t \)-set of \( G \) containing \( u \).

Example: \( K_n, K_{m,n}, C_n \).

Henning et al have characterized trees which are total excellent [30]. For every graph \( G \), the graph \( G \circ K_1 \) is not total excellent. We now prove that every connected graph is an induced subgraph of some total excellent graph.

Theorem 5.1

Given a connected graph \( G \), there exist a total excellent graph \( H \), which contains \( G \) as an induced subgraph.

Proof

Let \( G \) be a given graph with \( n \) vertices. Label the vertices as \( v_0, v_1, ..., v_{n-1} \). Now consider the cycle \( C_{4n} \). Let \( C_{4n} \) be \( u_1, u_2, ..., u_{4n} \). Now let \( E' = \{ u_{4i+1}, u_{4j+1} \mid v_i, v_j \in E(G) \} \). Construct a graph \( H \) with \( V(H) = \{ u_i \mid 1 \leq i \leq 4n \} \), \( E(H) = E(C_{4n}) \cup E' \). Then the resulting graph \( H \) is total excellent containing \( G \) as an induced subgraph. \( \Box \)
We now introduce a new class of total excellent graphs called Just Total Excellent (JTE), as follows.

**Definition**

A graph $G$ is said to be just total excellent (JTE) if for each $u \in V$, there exist a unique $\gamma_t$-set of $G$ containing $u$.

Based on this definition we have the following remarks.

**Remarks**

1. Every JTE is total excellent.

2. If $G$ is JTE, then $\delta(G) \geq n/\gamma_t(G)$.

   **Proof**
   
   Let $V = S_1 \cup S_2 \cup \ldots \cup S_m$ be the partition of $V$ into $\gamma_t$-sets of $G$.
   
   Fix one $u \in V$. Assume that $u \in S_j$. Since each $S_i$ is a $\gamma_t$-set, $u$ is adjacent to at least one vertex of $S_i$. Hence $\delta(u) \geq m = n/\gamma_t(G)$. From this we obtain the following result.

3. If $G \neq mK_2$ is JTE, then $\delta(G) \geq 2$. [In particular any tree $T \neq mK_2$ is not JTE].

4. Every JTE graph $G \neq mK_2$ is connected.

A necessary and sufficient condition for a graph to be JTE is given in the following theorem

**Theorem 5.2**

A graph $G$ is JTE if and only if
1. \( \gamma_t(G) \) divides \( n \).

2. \( d_t(G) = n/\gamma_t(G) \), and

3. \( G \) has exactly \( n/\gamma_t(G) \) distinct \( \gamma_t \) - sets.

Proof is analogous to the proof of theorem 4.1 in Chapter 4.

Definition

If \( D \) is a (total) dominating set of \( G \), for each \( u \in D \), the total private neighbor of \( u \) is defined as \( P\text{N}_t(u, D) = \{ v \in V \mid N(v) \cap D = \{ u \} \} \), where \( u \in D \).

If \( D \) is a minimal total dominating set of \( G \), then \( P\text{N}_t(u, D) \neq \emptyset \), for all \( u \in D \). Now we prove that if \( G \) is JTE, and \( D \) is a \( \gamma_t \) - set of \( G \), then for each \( u \in D \), \( P\text{N}_t(u, D) \) contains atleast two vertices of \( G \).

Theorem 5.3

If \( G \neq mK_2 \) is JTE, then \( |P\text{N}_t(u, D)| \geq 2 \), for all \( u \in D \), where \( D \) is any \( \gamma_t \) - set of \( G \).

Proof

Let \( D \) be a \( \gamma_t \) - set of \( G \). Since \( D \) is a \( \gamma_t \) - set, \( P\text{N}_t(u, D) \neq \emptyset \). Assume that for some \( u \in D \), \( |P\text{N}_t(u, D)| = 1 \). If there exist \( w \in D \) such that \( N(w) \cap D = \{ u \} \), then \( P\text{N}_t(u, D) = \{ w \} \). So \( (D - u) \cup \{ y \} \) is a \( \gamma_t \) - set for any \( y \in N(w) \). As \( deg(w) \geq 2 \), select one \( y \neq u \in N(w) \). Then \( D \) and \( (D - u) \cup \{ y \} \) are two distinct \( \gamma_t \) - sets of \( G \) containing the vertex \( w \), a contradiction to the fact that \( G \) is JTE.

If \( P\text{N}_t(u, D) = \{ x \} \) where \( x \notin D \), then select one \( y \neq u \in N(x) \). Then \( (D - u) \cup \{ y \} \) is a \( \gamma_t \) - set. Then \( (D - u) \cup \{ y \} \) is a \( \gamma_t \) - set. Then \( D \cap ((D - u) \cup \{ y \}) \neq \emptyset \), which is a contradiction. Thus \( |P\text{N}_t(u, D)| \geq 2 \). \( \square \)
An upper bound for $\gamma_t(G)$, for a JTE graph $G$, is obtained in the following theorem.

**Theorem 5.4**

If $G \neq mK_2$ is JTE, then $\gamma_t(G) \leq n/3$.

**Proof**

Assume that $\gamma_t(G) > n/3$. Then $d_t(G) = 2$, and let $V = V_1 \cup V_2$, where $V_1$ and $V_2$ are the distinct $\gamma_t$-sets of $G$. By theorem 5.2, for all $u \in V_1$, $|PN_t(u, V_1)| \geq 2$. Let $A = \{ u \in V_1 | PN_t(u, V_1) \cap V_1 \geq 2 \}$.

$B = \{ v \in V_1 | PN_t(u, V_1) \cap V_1 = \{ v \}$ for some $u \in A \}$ and $C = V_1 - (A \cup B)$.

We assume that $A \neq \phi$. For every $v \in B$, $PN_t(v, V_1) \subseteq V_2$. For every $v \in C$, $PN_t(v, V_1) \cap V_2 \neq \phi$. Also $|B| \geq 2|A|$. Hence $|U_{x \in V_1}(PN_t(x, V_1) \cap V_2)| \geq 2|B| + |C| \geq |A| + (|A| + |B| + |C|) > |V_1| = |V_2| = \gamma_t(G)$, which is a contradiction. So $A = \phi$. Hence $PN_t(x, V_1) \cap V_2 \neq \phi$ for all $x \in V_1$. So $|PN_t(x, V_1) \cap V_2| = 1, \forall x \in V_1$.

We have $PN_t(x, V_1) \cap V_1 \neq \phi, \forall x \in V_1$ and $U(PN_t(x, V_1) \cap V_1) = V_1$. Also $PN_t(x, V_1) \cap V_1 = \{ y \} \Leftrightarrow PN_t(y, V_1) \cap V_1 = \{ x \}$. So $deg(x) = 1$ in $<V_1>$ for every $x \in V_1$.

Hence $deg(x) = 2$ in $G, \forall x \in V_1$. Similarly $deg(x) = 2$ in $G, \forall x \in V_2$. As $G$ is 2 regular, each component of $G$ is a cycle. As $G \neq mK_2$ and $G$ is JTE, $G$ is connected. Hence $G$ itself is a cycle. But cycle $C_n$ is not JTE. Thus our assumption that $\gamma_t(G) > n/3$ is wrong. Hence $\gamma_t(G) \leq n/3$. □
Remark

1. The bound obtained in the above theorem is sharp. For example $H_{12,3}$ attains the bound.

2. If $G \not\cong mK_2$ is JTE, then $\delta(G) \geq 3$.

Theorem 5.5

If $G \not\cong mK_2$ is JTE, then $\Delta(G) \leq n - 2k + 2$, where $k = \gamma_t(G)$.

Proof

Let $u \in V(G)$. Let $S$ be a $\gamma_t$-set for $G$ which contains $u$. By theorem 5.3, $|PN_t(w, S)| \geq 2$, $\forall w \in S$. If $w_1 \neq w_2 \in S$, then $PN_t(w_1, S) \cap PN_t(w_2, S) = \phi$. [Clearly $PN_t(w_1, S) \cap PN_t(w_2, S) \cap (V - S) = \phi$. If $w \in PN_t(w_1, S) \cap S$, then $w \notin N(y)$, $\forall y \neq w_1 \in S$ and $w \notin PN_t(w_2, S)$]. The vertex $u \in PN_t(w, \delta)$ for at most one $w \in S$. So $u$ is not adjacent to any of the vertices in $\bigcup_{w \neq u \in S} PN_t(w, S)$ and $\deg(u) \leq (n - 1) - 2(|S| - 1 - 1)$. As this is true for all $u \in V(G)$, $\Delta \leq n - 2k + 2$. $\square$

Examples of graphs which are total excellent but not JTE

1. $C_n$ is not JTE.

2. $K_{m,n}$ is not JTE (unless $m = n = 1$).

3. Any connected graph $G \not\cong K_2$, with $\delta \leq 2$ is not JTE.

4. The Peterson's graph is not JTE.

Theorem 5.6

Every JTE graph contains no cut vertex (and hence if $G \not\cong mK_2$, it contains no bridge).
Proof

If possible assume that a JTE graph $G$ contains a cut vertex $u$. So $G \neq mK_2$, and by theorem 5.3, $\delta_t(G) \geq 3$. Let $S_1$ be the $\gamma_t$-set of $G$ that contains $u$. Select two distinct $\gamma_t$-sets $S_2$ and $S_3$ different from $S_1$. As $G$ is JTE, $u \notin S_2 \cup S_3$. Select vertices $v$ and $w$ such that $v \in S_2 \cap N(u)$ and $w \in S_3 \cap N(u)$. Let $G_1$ be the component of $G - u$ that contains $v$ and let $H_1$ be the subgraph of $G$ induced by $G_1 \cup \{u\}$, and let $H_2 = G - H_1$.

Case 1

Assume that $w \in H_1$. Then

1. Both $S_2 \cap H_1$ and $S_3 \cap H_1$ do not dominate any vertex of $H_2$.

2. $|S_1| = |S_2| = |S_3| = \gamma_t(G)$.

3. No vertex of $S_i \cap H_2$, $i = 1, 2$ is isolated in $< S_i \cap H_2 >$.

4. If $|S_2 \cap H_2| < |S_3 \cap H_2|$, then $|S_2 \cap H_1| > |S_3 \cap H_1|$ and $(S_2 \cap H_2) \cup (S_3 \cap H_1)$ is a total dominating set of $G$, which is a contradiction as $|(S_2 \cap H_2) \cup (S_3 \cap H_1)| < |(S_2 \cap H_2) \cup (S_3 \cap H_1)| = |S_3| = \gamma_t(G)$. Similarly if $|S_3 \cap H_2| < |S_2 \cap H_2|$, we get a contradiction.

So $|S_2 \cap H_2| = |S_3 \cap H_2|$ and $(S_2 \cap H_1) \cup (S_3 \cap H_2)$ and $S_3$ are distinct $\gamma_t$-sets of $G$ containing $S_3 \cap H_2$. Note that $v \in S_2 \cap H_1$ dominates $u$, which is a contradiction as $G$ is JTE.

Case 2

Assume that $w \in H_2$. Then $S_2 \cap H_2$ and $S_3 \cap H_1$ are total dominating sets for $H_2$ and $H_1 - u$ respectively and hence $|S_2 \cap H_2| \geq \gamma_t(H_2)$ and $|S_3 \cap H_1| \geq \gamma_t(H_1 - u)$. If $|S_2 \cap H_2| > \gamma_t(H_2)$, then $(S_2 \cap H_1) \cup D$ is a total dominating
set of $G$ for any $\gamma_t$-set $D$ of $H_2$. As $|(S_2 \cap H_1) \cup D| = |S_2 \cap H_1| + |D| < |S_2 \cap H_1| + |S_2 \cap H_2| = \gamma_t(G)$, we get a contradiction.

Similarly if $|S_3 \cap H_1| > \gamma_t(H_1 - u)$, $(S_3 \cap H_2) \cup D$ is a total dominating set of $G$, for any $\gamma_t$-set $D$ of $H_1 - u$, and we get a contradiction.

Thus $|S_2 \cap H_2| = \gamma_t(H_2)$ and $|S_3 \cap H_1| = \gamma_t(H_1 - u)$.

As $u, v \notin S_3$, $\exists x \in S_3 \cap H_1$, which is adjacent to $v$ and hence $(S_2 \cap H_2) \cup (S_3 \cap H_1) \cup \{v\}$ is a total dominating set of $G$ containing $v$ and $S_3 \cap H_1$. As $G$ is JTE, this total dominating set of $G$ is not a $\gamma_t$-set of $G$. Thus, $\gamma_t(G) \leq |S_3 \cap H_1| + |S_2 \cap H_2| = \gamma_t(H_1 - u) + \gamma_t(H_2)$.

As $S_2 \cap H_1$ and $S_2 \cap H_2$ are total dominating sets of $H_1 - u$ and $H_2$ respectively, 

\[ \gamma_t(G) = |S_2| = |S_2 \cap H_1| + |S_2 \cap H_2| \geq \gamma_t(H_1 - u) + \gamma_t(H_2) \geq \gamma_t(G). \]

Hence $\gamma_t(G) = \gamma_t(H_1 - u) + \gamma_t(H_2)$.

Now as $\gamma_t(G) = |S_2| = |S_2 \cap H_1| + |S_2 \cap H_2| = |S_2 \cap H_1| + \gamma_t(H_2)$, we get $|S_2 \cap H_1| = \gamma_t(H_1 - u)$. Similarly $|S_3 \cap H_2| = \gamma_t(H_2)$.

If $|S_1 \cap H_2| < \gamma_t(H_2)$, then if $D = (S_2 \cap H_1) \cup \{u\} \cup (S_1 \cap H_2)$, then $D$ is a total dominating set of $G$ and as $|D| = |S_2 \cap H_1| + 1 + |S_1 \cap H_2| \leq \gamma_t(H_1 - u) + 1 + \gamma_t(H_2) - 1 = \gamma_t(G)$, $D$ is a $\gamma_t$-set of $G$, again we get a contradiction as both $D$ and $S_1$ contains $u$.

If $|S_1 \cap H_2| > \gamma_t(H_2)$, then $|S_1 \cap H_1| < \gamma_t(H_1 - u)$ and $(S_1 \cap H_1) \cup (S_3 \cap H_2)$ is a total dominating set for $G$. [ Note that $u \in S_1 \cap H_1$ and $w \in S_3 \cap H_2$. So $(S_1 \cap H_1) \cup (S_3 \cap H_2)$ does not contain isolated vertices ]. But 

\[ |(S_1 \cap H_1) \cup (S_3 \cap H_2)| = |S_1 \cap H_1| + |S_3 \cap H_2| < \gamma_t(H_1 - u) + \gamma_t(H_2) = \gamma_t(G), \]

which is a contradiction.

So $|S_1 \cap H_2| = \gamma_t(H_2)$ and $|S_1 \cap H_1| = \gamma_t(H_1 - u)$. Then $(S_1 \cap H_1) \cup (S_3 \cap H_2)$ ia a $\gamma_t$-set of $D$ a contradiction. [ Note that $u \in S_1 \cap H_1$, $w \in S_3 \cap H_2$ and $uw \in E(G)$]. Thus $u$ is not a cut vertex.
Theorem 5.7

In a JTE graph, $G \neq mK_2$ every vertex $u$ is a level vertex, and also $\gamma_l(G - u) = \gamma_l(G)$.

Proof

Let $u$ be a vertex in $G$. Then there exist a $\gamma_l$ - set of $G$ not containing $u$. Hence $\gamma_l(G - u) \leq \gamma_l(G)$. We claim that $\gamma_l(G - u) = \gamma_l(G)$.

If possible assume that $\gamma_l(G - u) < \gamma_l(G)$. Let $S$ be a $\gamma_l$ - set for $G - u$. Then $S \cup \{v\}$ is a $\gamma$ - set for $G$, $\forall v \in N[u]$. As $G$ is connected, $N(u)$ contains at least two vertices $v_1$ and $v_2$. So $S \cup \{v_1\}$ and $S \cup \{v_2\}$ are $\gamma_l$ - sets for $G$, a contradiction as $G$ is JTE. So $\gamma_l(G - u) = \gamma_l(G)$.

If $\gamma_l^u(G, u) < \gamma_l(G)$, let $S$ be a $\gamma_l^u(G, u)$ - set. If $u \in S$, then $S$ is also a dominating set for $G$, which is a contradiction. If $u \notin S$, then $S$ is a $\gamma_l$ - set for $G - u$ and $\gamma_l(G - u) < \gamma_l(G)$ which is also a contradiction. Thus $\gamma_l^u(G, u) = \gamma_l(G)$. $\square$