CHAPTER IV
FUZZY $\mathcal{I}$-COMPACT SPACES

In this chapter we introduce the concepts of quasi fuzzy $\mathcal{I}$-compact, fuzzy $\mathcal{I}$-compact, fuzzy $\mathcal{I}_\beta$-compact sets and investigate some theorems related to them.

A family $\mathcal{A}$ of fuzzy sets of $X$ is a cover of a fuzzy set $B$ of $X$ if and only if $B \leq \bigvee\{A: A \in \mathcal{A}\}$. If $(X, \tau)$ is a fuzzy topological space and a cover $\mathcal{A}$ of a fuzzy set $B$ is said to be an open cover iff each member of $\mathcal{A}$ is a fuzzy open set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover for $B$.

A fuzzy set $B$ of a fuzzy topological space $X$ is said to be quasi fuzzy compact if each open cover of $B$ has a finite subcover.

A fuzzy topological space $(X, \tau)$ is called fuzzy compact if for every family $\mathcal{U}$ of fuzzy open sets of $X$ and for every $a \in (0, 1]$ such that $\forall u \geq a$ and for every $\epsilon \in (0, a]$ there exists a finite subfamily $\mathcal{U}_1$ of $\mathcal{U}$ that $\forall u \in \mathcal{U}_1 \geq a - \epsilon$.

Definition 4.1

A fuzzy set $f$ of a fuzzy topological space $(X, \tau)$ with a fuzzy ideal $\mathcal{I}$ is said to be quasi fuzzy $\mathcal{I}$-compact if for every family $\mathcal{U}$ of fuzzy open sets of $X$, with $\forall u \in \mathcal{U} \geq f$, there exists a finite subfamily $\mathcal{U}_1$ of $\mathcal{U}$ such that $(f - \bigvee_{u \in \mathcal{U}_1} u) \in \mathcal{I}$.

Definition 4.2

A fuzzy topological space $(X, \tau)$ with fuzzy ideal $\mathcal{I}$ is said to be fuzzy $\mathcal{I}$-compact iff for every family $\mathcal{U}$ of fuzzy open sets of $X$ and for every $a \in (0, 1]$ such that $\forall u \geq a$ and for every $\epsilon \in (0, a]$, there exists a finite subfamily $\mathcal{U}_1$ of $\mathcal{U}$ that $(a - \epsilon - \bigvee_{u \in \mathcal{U}_1} u) \in \mathcal{I}$.

Example 4.3

Let $X$ be a compact set with respect to a crisp topology $\sigma$, consider the fuzzy
topology $\tau = \{ f \in I^X / \text{for every } \lambda \in (0,1], \{ x / f(x) > \lambda \} \text{ is an open set in } (X, \sigma) \}$. Let $\mathcal{U} = \{ f_a \}$ be a collection of fuzzy open sets of $X$ and $\bigwedge_a f_a \geq \bar{a}$ for some $a \in (0,1]$. Let $\varepsilon$ be such that $0 < \varepsilon \leq a$. To each $x \in X$, as $\bigwedge_a f_a(x) \geq a > a - \varepsilon$, there is one $a_x$ such that $f_{a_x}(x) > a - \varepsilon$. As $f_a \in \tau, O_{a_x} = \{ y / f_{a_x}(y) > a - \varepsilon \}$ is an open set in $(X, \sigma)$ and $x \in \{ y / f_a(y) > a - \varepsilon \} = O_{a_x}$.

Then with respect to the topology $\sigma, \{ O_{a_x} / x \in X \} \text{ is an open cover for } X$. As $X$ is compact, there exists a finite subcover $\{ O_{a_{x_i}} \}$ for $X$.

Therefore $X = \bigcup_{i=1}^{n} O_{a_{x_i}}$. As $f_{a_{x_i}}(y) > a - \varepsilon$ for all $y \in O_{a_{x_i}}$, we get $\bigwedge_{i=1}^{n} f_{a_{x_i}} > a - \varepsilon$. Therefore $(X, \tau)$ is fuzzy compact.

Remark 4.4

Let $X = [0,1] \cup \{ 2, 3, ..., \}$. Consider the topology $\sigma$ on $X$ induced by the standard topology on $\mathbb{R}$. Let $\mathcal{J}$ be the fuzzy ideal of all fuzzy sets with countable support.

$\mathcal{J} = \{ f : X \rightarrow [0,1] / S(f) \text{ is a countable set in } X \}$

Let $\tau = \{ f : X \rightarrow [0,1] / \text{to each } \lambda < 1, \text{ the set } \{ x / f(x) > \lambda \} \}$

is open set of $X$ w.r.t. $\sigma$.

Then $X$ is not fuzzy compact w.r.t. $\tau$.

For let $f_n(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cup \{ n \} \\ 0 & \text{if } x \notin [0,1] \cup \{ n \} \end{cases}$

Then $\bigwedge_{n=2}^{N} f_n \geq \bar{a}$ for all $a \in (0,1]$. But given $\varepsilon > 0, 0 < \varepsilon \leq a$, there is no $N$ such that $\bigwedge_{n=2}^{N} f_n \geq a - \varepsilon$.

Therefore $X$ is not fuzzy compact w.r.t. $\tau$. But $X$ is fuzzy $\mathcal{J}$-compact.

Now we prove the following lemma.

Lemma 4.5

If $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces and $f : X \rightarrow Y$ then $f(u) - f(v) \leq f(u - v)$ for all $u, v \in I^X$.  

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Proof

\((f(u) - f(v))(y) \leq \max\{0, f(u)(y) - f(v)(y)\}\)
\[\leq \max\left\{0, \left(\bigvee_{f(x)=y} u(x)\right) - \left(\bigvee_{f(x)=y} v(x)\right)\right\}\]
\[\leq \max\{0, \bigvee_{f(x)=y} (u(x) - v(x))\}\]
\[= \bigvee_{f(x)=y} \max\{0, u(x) - v(x)\}\]
\[= \bigvee_{f(x)=y} (u - v)(x)\]
\[= f(u - v)(y)\]

(i.e.,) \(f(u) - f(v) \leq f(u - v)\)

In the following theorem, we show that fuzzy continuous image of quasi fuzzy \(\mathcal{I}\)-compact set is quasi fuzzy \(f(\mathcal{I})\)-compact.

**Theorem 4.6**

Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a fuzzy continuous function. If \(A\) is quasi fuzzy \(\mathcal{I}\)-compact in \(X\), then \(f(A)\) is quasi fuzzy \(f(\mathcal{I})\)-compact.

**Proof**

Suppose \(A\) is quasi fuzzy \(\mathcal{I}\)-compact. Let \((U_\alpha)_{\alpha \in J}\) be a fuzzy open cover for \(f(A)\). Since \(f\) is fuzzy continuous \(f^{-1}(U_\alpha)_{\alpha \in J}\) is a fuzzy open cover for \(A\).

Since \(A\) is quasi fuzzy \(\mathcal{I}\)-compact, there is a finite subcollection \(f^{-1}(U_\alpha)_{i=1}^n\) such that \((A - \bigvee_{i=1}^n f^{-1}(U_\alpha)) \in \mathcal{I}\).

Clearly \(f(A - \bigvee_{i=1}^n f^{-1}(U_\alpha)) \in f(\mathcal{I})\).................(1)

Note that if \(A_i (i = 1, 2, \ldots n)\) are fuzzy sets of \(X\),

\[\left(\bigvee_{i=1}^n f(A_i)\right)(y) = \bigvee_{i=1}^n f(A_i)(y) = \bigvee_{i=1}^n \left(\bigvee_{f(x)=y} A_i(x)\right) = \bigvee_{f(x)=y} \left(\bigvee_{i=1}^n A_i(x)\right)\]
\[= f\left(\bigvee_{i=1}^n A_i\right)(y)\]
Therefore \( \bigvee_{i=1}^{n} f(A_i) = f \left( \bigvee_{i=1}^{n} A_i \right) \)

Now \( f(A) - \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) = f(A) - f \left( \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i}) \right) \)
\[
\leq f(A - \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) \in f(\mathcal{F}) \quad \text{[by lemma 4.5 and eqn (1)]}
\]

Therefore \( f(A) - \bigvee_{i=1}^{n} U_{\alpha_i} \leq f(A) - \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \) as \( f(f^{-1}(U_{\alpha})) \leq U_{\alpha} \)
\[
\leq f \left( A - \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i}) \right) \in f(\mathcal{F})
\]

(i.e.,) \( f(A) - \bigvee_{i=1}^{n} U_{\alpha_i} \in f(\mathcal{F}) \)

Hence \( f(A) \) is quasi fuzzy \( f(\mathcal{F}) \)-compact.

**Theorem 4.7**

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a fuzzy continuous function. If \((X, \tau)\) is fuzzy \( \mathcal{F} \)-compact, then \((Y, \sigma)\) is fuzzy \( f(\mathcal{F}) \)-compact.

**Proof**

The proof is similar to that of theorem 4.6

Balasubramanian [7] and Ganesan [7] introduced the concept of fuzzy \( \beta \)-compact spaces. I.M. Hanafy [16] investigates some results related to fuzzy \( \beta \)-compact spaces. Here we defined fuzzy \( \mathcal{F} \beta \)-compactness and established some results associated with them.

**Definition [12] 4.8**

A fuzzy set \( u \) in a fuzzy topological space \( X \) is said to be \( \beta \)-open fuzzy set if \( u \leq cl(int(cl(u))) \).

Note that every fuzzy open set is a \( \beta \)-open fuzzy set. There are some spaces in which every \( \beta \)-open fuzzy set is fuzzy open.

**Example 4.9**

Let \( X \) be a non-empty set and fix \( a, 0 < a < 1 \) and
\[
\tau = \{ f : X \rightarrow [0, 1] / |f(x) - f(y)| \leq a \text{ for } x, y \in X \}.
\]
A is fuzzy closed with respect to $\tau$ iff $(1 - A)$ is fuzzy open

\[ \text{iff } |(1 - A)(x) - (1 - A)(y)| \leq a \quad \forall \, x, y \in X \]

\[ \text{iff } |A(y) - A(x)| \leq a \quad \forall \, x, y \in X \]

\[ \text{iff } A \text{ is } \tau \text{- open} \]

Let $g \in I^X$ and $h(x) = \min\{g(x), (\inf g) + a\}, \forall \, x \in X$. Then $h \leq g$ and $h$ is $\tau$-open, as $|h(x) - h(y)| \leq a$. If $h'$ is $\tau$-open and $h' \leq g$, then $|h'(x) - h'(y)| \leq a \quad \forall \, x, y \in X$ and $h'(x) \leq g(x) \quad \forall \, x \in X$. So $\inf h' \leq \inf g$ and $h'(x) \leq \inf g + a$.

Thus $h'(x) \leq g(x) \land ((\inf g) + a) = h(x)$. Therefore $h = \text{int}(g)$. Therefore $\text{cl}(\text{int}(g)) = \text{cl}(h) = h \leq g$ as $h$ is fuzzy closed and fuzzy open. Therefore if $g \notin \tau, \text{cl}(\text{int}(g)) \leq g$. Therefore $g$ is not $\beta$-open fuzzy set. If $g \in \tau$, then $h = g$ and $g = \text{cl}(\text{int}(g))$.

**Example 4.10**

This is an example for a $\beta$-open fuzzy set which is not fuzzy open.

Let $X = \mathbb{R}$ with usual topology $\sigma$. Let $\tau = \{f : X \to [0,1]/ \text{to each } \lambda \in [0,1), \{x \in X/f(x) > \lambda\} \text{ is open in } \mathbb{R}\}$

Note that $g \in I^X$ is fuzzy closed iff $(1 - g)$ is fuzzy open.

iff to each $\lambda$, $\{x \in X/(1 - g)(x) > \lambda\}$ is an open set in $\mathbb{R}$.

iff to each $\lambda$, $\{x/(1 - \lambda) > g(x)\}$ is an open set in $\mathbb{R}$.

Let $g \in I^X$, defined by $g(x) = \begin{cases} 1 & \text{for } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

Let $g'$ be given by $g' = \Psi_{[0,1]}$. So to each $\lambda \in [0,1), \{x \in X/(1 - \lambda) > g'(x)\} = (-\infty, 0) \cup (1, \infty)$ and $g'$ is fuzzy closed. Let $h'$ be a fuzzy closed set and $g \leq h'$.

Then to each $\lambda \in [0,1), A = \{x/(1 - \lambda) > h'(x)\} \cap (0,1] = \Phi$, and as $A = \{x \in X/(1 - \lambda) > h'(x)\}$ is an open set in $\mathbb{R}$, it follows that $0 \notin A$.
Therefore $h'(0) = 1$ and $\Psi_{[0,1]} \leq h'$. Now it follows that $cl(g) = g' = \Psi_{[0,1]}$. 

$int(cl(g)) = \Psi_{[0,1]}$ and $cl(int(cl(g))) = \Psi_{[0,1]} = g'$. Therefore $g \leq cl(int(cl(g)))$.

Thus $g$ is a $\beta$-open fuzzy set, but which is not fuzzy open. 

A fuzzy topological space $X$ is said to be fuzzy $\beta$-compact iff for every family $\mu$ of $\beta$-open fuzzy sets such that $\bigvee A = 1_X$, there is a finite subfamily $\eta \subseteq \mu$ such that $\bigvee_{A \in \eta} A = 1_X$.

A fuzzy set $u$ in a fuzzy topological space $X$ is said to be fuzzy $\beta$-compact relative to $X$ iff for every family $\mu$ of $\beta$-open fuzzy sets such that $\bigvee_{A \in \mu} A \geq u$ there is a finite subfamily $\eta \subseteq \mu$ such that $\bigvee_{A \in \eta} A \geq u$.

We extend these concepts to similar concepts with respect to fuzzy ideal.

**Definition 4.11**

A fuzzy set $u$ in a fuzzy topological space $X$ is said to be fuzzy $J$-compact relative to $X$, iff for every family $\mu$ of $\beta$-open fuzzy sets with $\bigvee_{A \in \mu} A \geq u$ there is a finite subfamily $\eta \subseteq \mu$ such that $(u - \bigvee_{A \in \eta} A) \in J$.

**Theorem 4.12**

A fuzzy topological space $X$ is fuzzy $J$-compact iff for every collection $\{A_j : j \in J\}$ of $\beta$-closed fuzzy sets of $X$ with $\bigwedge_{j \in F} A_j \notin J$ for every finite subfamily $F \subseteq J$, then $\bigwedge_{j \in J} A_j \neq \emptyset$.

**Proof**

Let $\{A_j : j \in J\}$ be a collection of $\beta$-closed fuzzy sets with $\bigwedge_{j \in F} A_j \notin J$ for every finite subfamily $F \subseteq J$. Suppose $\bigwedge_{j \in J} A_j = \emptyset$. Then $\bigvee_{j \in J} (1 - A_j) = 1$. That is $\bigvee_{j \in J} A'_j = 1$. Since $\{A'_j : j \in J\}$ is a collection of $\beta$-open fuzzy sets of $X$ which covers $1_X$, from the fuzzy $J$-compactness of $X$, it follows that there exists a finite subfamily $M \subseteq J$ such that $\left(1 - \bigvee_{j \in M} A'_j \right) \in J$. That is $\bigwedge_{j \in M} A_j \in J$, which is a contradiction and therefore $\bigwedge_{j \in J} A_j \neq \emptyset$. Conversely, let $\{A'_j : j \in J\}$
be a collection of $\beta$-open fuzzy sets cover of $X$. That is $\bigvee_{j \in J} A'_j = 1$.

Suppose that for every finite subset $F \subseteq J$, we have $\left(1 - \bigvee_{j \in F} A'_j\right) \notin \mathcal{F}$. That is $\bigwedge_{j \in F} A_j \notin \mathcal{F}$. We have $\bigwedge_{j \in F} A_j \neq \emptyset$. Which implies that $1 - \bigwedge_{j \in F} A_j \neq 1$. That is $\bigvee_{j \in F} A'_j \neq 1$. This contradicts that $\{A'_j : j \in J\}$ is a $\beta$-open cover of $X$. Thus $X$ is a fuzzy $\mathcal{F}\beta$-compact.

**Definition 4.13**

A collection of fuzzy sets $\xi$ of a fuzzy topological space $X$ is said to form a fuzzy $\mathcal{F}$-filter bases iff for every finite collection $\{A_j : j = 1, 2, \ldots, n\}$ of $\xi$, $\bigwedge_{j=1}^n A_j \notin \mathcal{F}$.

**Theorem 4.14**

A fuzzy topological space $X$ is fuzzy $\mathcal{F}\beta$-compact iff every fuzzy $\mathcal{F}$-filter bases $\xi$ in $X$, $\bigwedge_{G \in \xi} \beta cl(G) \neq \emptyset$

**Proof**

Suppose for every fuzzy $\mathcal{F}$-filter bases $\xi$ in $X$, $\bigwedge_{G \in \xi} \beta cl(G) \neq \emptyset$. Let $\mu$ be a $\beta$-open fuzzy set cover of $X$ and for every finite sub collection $\{A_1, A_2, \ldots, A_n\}$ of $\mu$, $\left(1 - \bigvee_{j=1}^n A_j\right) \notin \mathcal{F}$. That is $\bigwedge_{j=1}^n A_j \notin \mathcal{F}$. Thus $\{A'_j : A_j \in \mu\} = \xi$ forms a fuzzy $\mathcal{F}$-filter bases in $X$. Since $\mu$ is a $\beta$-open fuzzy set cover of $X$, $\forall A_j = 1$ and hence $\bigwedge_{A_j \in \mu} A_j \notin \mathcal{F}$. Thus $\bigwedge_{A_j \in \mu} \beta cl(A'_j) = \bigwedge_{A_j \in \mu} \beta cl(A_j') = \bigwedge_{A_j \in \mu} A_j' = \emptyset$. That is $\bigwedge_{A_j \in \mu} \beta cl(A'_j) = 0$, which is a contradiction.

Hence if $\mu$ is a $\beta$-open fuzzy set cover of $X$, it has a finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of $\mu$ such that $\left(1 - \bigvee_{j=1}^n A_j\right) \notin \mathcal{F}$. Therefore $X$ is fuzzy $\mathcal{F}\beta$-compact.

Conversely assume that $X$ is fuzzy $\mathcal{F}\beta$-compact space. Suppose there exists a fuzzy $\mathcal{F}$-filter bases $\xi$ such that $\bigwedge_{G \in \xi} \beta cl(G) = \emptyset$, so that $\forall (\beta cl(G))' = 1$.

Then $\mu = \{((\beta cl(G))' : G \in \xi\}$ is a $\beta$-open fuzzy set cover of $X$. Since $X$ is
fuzzy $j\beta$-compact, $\mu$ has a finite subcollection $\{(\beta cl(G_j))' : j = 1, 2, \ldots, n\}$ such that $1 - \bigvee_{j=1}^{n} (\beta cl(G_j))' \in J$. That is $(1 - \bigvee_{j=1}^{n} G_j') \in J$. That is $\bigwedge_{j=1}^{n} G_j \in J$, which is a contradiction, since the $G_j$ are members of fuzzy $J$-filter bases $\xi$.

Therefore $\bigwedge_{G \in \xi} \beta cl(G) \neq \emptyset$ for every fuzzy $J$-filter bases $\xi$.

**Theorem 4.15**

A fuzzy set $u$ in a fuzzy topological space $X$ is fuzzy $j\beta$-compact relative to $X$ iff for every fuzzy $J$-filter bases $\xi$ such that every finite members $\eta$ of $\xi$ with $(\bigwedge_{A \in \eta} A) \cap u \notin J$ then $(\bigwedge_{A \in \xi} \beta cl(A)) \cap u \neq \emptyset$. $\square$

A function $f : X \to Y$ is said to be fuzzy $\beta$-continuous (resp., $M\beta$-continuous) if the inverse image of every open (resp., $\beta$-open) fuzzy set in $Y$ is $\beta$-open (resp., $\beta$-open) fuzzy set in $X$.

Let $(X, \tau)$ and $(Y, \sigma)$ be two fuzzy topological spaces. A function $f : X \to Y$ is said to be fuzzy open if $u \in \tau$ then $f(u) \in \sigma$.

A function $f : X \to Y$ is said to be fuzzy $M\beta$-open iff the image of every $\beta$-open fuzzy set in $X$ is $\beta$-open in $Y$.

**Theorem 4.16**

If a function $f : X \to Y$ is fuzzy $M\beta$-continuous and $u$ is fuzzy $J\beta$-compact relative to $X$, then so is $f(u)$

**Proof**

Let $\{A_j : j \in J\}$ be a $\beta$-open fuzzy cover for $f(u)$. Since $f$ is fuzzy $M\beta$-continuous, then $\{f^{-1}(A_j) : j \in J\}$ is $\beta$-open fuzzy cover for $u$. Since $u$ is fuzzy $J\beta$-compact relative to $X$, there is a finite subfamily $\{f^{-1}(A_j) : j = 1, 2, \ldots, n\}$ such that $\left(u - \bigvee_{j=1}^{n} f^{-1}(A_j)\right) \in J$.

That is $u - f^{-1}\left(\bigvee_{j=1}^{n} A_j\right) \in J$ (by a result in chapter I)

Now $f\left(u - f^{-1}\left(\bigvee_{j=1}^{n} A_j\right)\right) \in f(J)$
Then \( f(u) - f \left( \left( \bigcup_{j=1}^{n} A_j \right) \right) \in f(\mathcal{I}) \). [by lemma 4.5], That is \( f(u) - \bigcup_{j=1}^{n} A_j \in f(\mathcal{I}) \) [as \( f(f^{-1}(B)) \leq B \) for any fuzzy set \( B \) in \( Y \)]

Therefore \( f(u) \) is fuzzy \( f(\mathcal{I})\beta \)-compact relative to \( Y \).

**Corollary 4.17**

Let \( f : X \to Y \) be fuzzy open and fuzzy continuous function and \( X \) is fuzzy \( \mathcal{I}\beta \)-compact, then \( f(X) \) is fuzzy \( f(\mathcal{I})\beta \)-compact.

**Proof**

It follows directly from theorem 1.34 and theorem 4.16

**Theorem 4.18**

Let \( f : X \to Y \) be a fuzzy \( M\beta \)-open bijective function and \( Y \) is fuzzy \( f(\mathcal{I})\beta \)-compact, then \( X \) is fuzzy \( \mathcal{I}\beta \)-compact.

**Proof**

Let \( \{ A_j : j \in J \} \) be a collection of \( \beta \)-open fuzzy set cover of \( X \), then \( \{ f(A_j) : j \in J \} \) is \( \beta \)-open fuzzy set covering of \( Y \). Since \( Y \) is fuzzy \( f(\mathcal{I})\beta \)-compact, there is a finite subset \( F \subseteq J \) such that \( \left( 1_Y - \bigcup_{j \in F} f(A_j) \right) \in f(\mathcal{I}) \).

As \( f \) is a bijective function, \( f^{-1}\left( 1_Y - \bigcup_{j \in F} f(A_j) \right) \in \mathcal{I} \) and \( f^{-1}(f(A_j)) = A_j \).

So \( 1_X - \bigcup_{j \in F} A_j \in \mathcal{I} \).