CHAPTER 6

(\(\tau_1, \tau_2\)) – SEMI-PREGENERALIZED CLOSED SETS
AND SOME FUNCTIONS

In this chapter we define \((\tau_1, \tau_2)\)-semi-generalized closed sets,

\((\tau_1, \tau_2)\)-generalized semi-preclosed sets, \((\tau_1, \tau_2)\)-semi-star-regular sets.

\((\tau_1, \tau_2)\)-semi-pregeneralized closed sets. Also we define some functions on them. And

we define pairwise semi-pre \(T_{1/2}\) spaces, \((\tau_1, \tau_2)\)-semi-pre \(T_{1/2}\) spaces and discuss some
of their properties.

6.1. INTRODUCTION

in the year 1987. J. Dontchev [14] introduced semi-pre \(T_{1/2}\) spaces, Y.Gnanambal
[12] introduced preregular \(T_{1/2}\) spaces. Semi-pre generalized closed sets in unital
topological spaces are introduced by M.K.R.S. Veerakumar [28]. In the present
chapter it is shown that some of their results may be extended to bitopological
spaces. Regarding notations scl\( (A) \), spcl \( (A) \) and pcl \( (A) \) denote the semi-

We recall

**DEFINITION 6.1.1**

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be

(i) $(\tau_1, \tau_2)$-semiopen [26] if $A \subseteq \tau_2 - \text{cl}(\tau_1 - \text{int}(A))$.

(ii) $(\tau_1, \tau_2)$-semiclosed [26] if $\tau_2 - \text{int}(\tau_1 - \text{cl}(A)) \subseteq A$.

(iii) $(\tau_1, \tau_2)$-preclosed [17] if $\tau_1 - \text{cl}(\tau_2 - \text{int}(A)) \subseteq A$.

(iv) $(\tau_1, \tau_2)$-semi-preopen [17] if $A \subseteq \tau_2 - \text{cl}(\tau_1 - \text{int}(\tau_2 - \text{cl}(A)))$.

(v) $(\tau_1, \tau_2)$-semi-preclosed [17] if $\tau_2 - \text{int}(\tau_1 - \text{cl}(\tau_2 - \text{int}(A))) \subseteq A$.

(vi) $(\tau_1, \tau_2)$-$\alpha$ open [5.2.1] if $A \subseteq \tau_1 - \text{int}(\tau_2 - \text{cl}(\tau_1 - \text{int}(A)))$.

**DEFINITION 6.1.2**

A subset $A$ of a topological space $(X, \tau)$ is said to be semi-regular[10] if $A$ is both semiopen and semiclosed.
DEFINITION 6.1.3

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise $\alpha$ open [5.2.1] if it is $(\tau_1, \tau_2)$-$\alpha$ open and $(\tau_2, \tau_1)$-$\alpha$ open.

6.2 $(\tau_1, \tau_2)$—SEMI-PREGENERATLIZED CLOSED SETS

DEFINITION 6.2.1

A subset $A$ of $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$-semi-generalized closed (briefly $(\tau_1, \tau_2)$-sg closed) if $(\tau_1, \tau_2)$-scl($A$) $\subseteq U$ whenever $A \subseteq U$ and $U$ is $(\tau_1, \tau_2)$-semiopen.

DEFINITION 6.2.2

A subset $A$ of $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$-generalized semi-preclosed (briefly $(\tau_1, \tau_2)$-gsp closed) if $(\tau_1, \tau_2)$-spcl($A$) $\subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_1$-open.

DEFINITION 6.2.3

A subset $A$ of $(X, \tau_1, \tau_2)$ is called a $(\tau_1, \tau_2)$-semi-star-regular if $A$ is both
A subset $A$ of $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$-semi-pregeneralized closed (briefly $(\tau_1, \tau_2)$-spg closed) if $(\tau_1, \tau_2)$-spcl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $(\tau_1, \tau_2)$-semiopen.

Here $(\tau_1, \tau_2)$-spcl$(A)$ denotes the closure of $(\tau_1, \tau_2)$-semipreclosed set $A$.

**Theorem 6.2.5**

Every $(\tau_1, \tau_2)$-preclosed set is $(\tau_1, \tau_2)$-spg closed.

**Proof**

Let $A$ is $(\tau_1, \tau_2)$-preclosed set of $(X, \tau_1, \tau_2)$.

$$\Rightarrow \tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A)) \subseteq A.$$  \hspace{1cm} (1)

But $\tau_2 \cdot \text{int} (\tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A))) \subseteq \tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A))$.  \hspace{1cm} (2)

From (1) and (2) we have $\tau_2 \cdot \text{int} (\tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A))) \subseteq A$.

$$\Rightarrow A \text{ is } (\tau_1, \tau_2) \text{-semi-preclosed. And hence } (\tau_1, \tau_2) \cdot \text{spcl} (A) = A.$$
Therefore $A$ is $(\tau_1, \tau_2)$ - spg closed.

The following example shows that the converse of the above theorem is not true.

**EXAMPLE 6.2.6**

Let $X = \{a, b, c\}$. Let $\tau_1 = \{\emptyset, X\}$ and

$$\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$
be topologies on $X$.

Take $A = \{a\}$.

Then $A$ is $(\tau_1, \tau_2)$ - spg closed.

But it is not $(\tau_1, \tau_2)$ - preclosed.

For, $\tau_1$ - cl $(\tau_2$ - int $(A)) = \{a, b\} \not\subset A = \{a\}$.

**THEOREM 6.2.7**

Every $(\tau_1, \tau_2)$ - semi-generalized closed set is $(\tau_1, \tau_2)$ - spg closed.

**PROOF**

For any subset $A$ of $(X, \tau_1, \tau_2)$, we have

$$(\tau_1, \tau_2)$ - spcl $(A) \subset (\tau_1, \tau_2)$ - scl $(A).$$
For.

Let $x \in (\tau_1, \tau_2) - \text{spcl}(A)$. Let $G$ be a $(\tau_1, \tau_2) - \text{semiopen}$ neighbourhood of $x$.

Then $G$ is a $(\tau_1, \tau_2) - \text{semi-preopen}$ neighbourhood of $x$.

But $x \in (\tau_1, \tau_2) - \text{spcl}(A)$ and hence $G \cap A \neq \emptyset$.

Then $G \cap A \neq \emptyset$ for all $(\tau_1, \tau_2) - \text{semiopen}$ neighbourhood of $x$.

Hence $x \in (\tau_1, \tau_2) - \text{scl}(A)$. Since $x$ is arbitrary we have,

$$(\tau_1, \tau_2) - \text{spcl}(A) \subseteq (\tau_1, \tau_2) - \text{scl}(A).$$

Let $A$ be $(\tau_1, \tau_2) - \text{semi-generalized closed}$ set. Let $A \subseteq U$ and $U$ is $(\tau_1, \tau_2) - \text{semiopen}$. That implies $(\tau_1, \tau_2) - \text{scl}(A) \subseteq U$.

From (3) and (4) we have

$$(\tau_1, \tau_2) - \text{spcl}(A) \subseteq U, A \subseteq U \text{ and } U \text{ is } (\tau_1, \tau_2) - \text{semiopen}.$$

And hence $A$ is $(\tau_1, \tau_2) - \text{spg closed}$.

The following example shows that the converse of the above theorem is not true.
(τ₁, τ₂) – Semi-pregeneralized closed sets and some functions

**EXAMPLE 6.2.8**

Let \( X = \{ a, b, c \} \). Let \( τ₁ = \{ \emptyset, X, \{ a \} \} \) and \( τ₂ = \{ \emptyset, X, \{ a, b \} \} \) be topologies on \( X \).

Take \( A = \{ a, c \} \). Then \( A \) is \((τ₁, τ₂)\)-spg closed.

But it is not \((τ₁, τ₂)\)-sg closed. For, take \( U = \{ a, c \} \) which is \((τ₁, τ₂)\)-semiopen and \( A \subseteq U \). But \((τ₁, τ₂)\)-scl \( (A) = X \not\subset U \).

**THEOREM 6.2.9**

Every \((τ₁, τ₂)\)-spg closed set is \((τ₁, τ₂)\)-gsp closed but not conversely.

**PROOF**

Let \( A \) is \((τ₁, τ₂)\)-spg closed. Let \( U \) be any \( τ₁ \)-open set containing \( A \).

Then by remark 2.2 of [26] \( U \) is \((τ₁, τ₂)\)-semi open.

Since \( A \) is \((τ₁, τ₂)\)-spg closed, we have \((τ₁, τ₂)\)-spcl \( (A) \subseteq U \).

Thus \((τ₁, τ₂)\)-spcl \( (A) \subseteq U \), \( A \subseteq U \) and \( U \) is \( τ₁ \)-open. That implies \( A \) is \((τ₁, τ₂)\)-gsp closed.
EXAMPLE 6.2.10

Let $X = \{ a, b, c \}$. Let $\tau_1 = \{ \emptyset, X, \{ a \}, \{ a, b \} \}$ and $\tau_2 = \{ \emptyset, X, \{ a \} \}$ be topologies on $X$.

Take $A = \{ a, c \}$.

Then $A$ is $(\tau_1, \tau_2)$-gsp closed, but not $(\tau_1, \tau_2)$-spg closed.

For, take $U = \{ a, c \}$. Then $U$ is $(\tau_1, \tau_2)$-semiopen also

$A \subseteq U$ but $(\tau_1, \tau_2)$-spcl $(A) = \{ a, b, c \} \not\subseteq U$.

THEOREM 6.2.11

If $A$ is $(\tau_1, \tau_2)$-semiopen and $(\tau_1, \tau_2)$-spg closed set of $(X, \tau_1, \tau_2)$, then $A$ is $(\tau_1, \tau_2)$-semi-pre closed set of $(X, \tau_1, \tau_2)$.

PROOF

Since $A \subseteq A$. $A$ is $(\tau_1, \tau_2)$-semiopen and $A$ is $(\tau_1, \tau_2)$-spg closed.

we have $(\tau_1, \tau_2)$-spcl $(A) \subseteq A.$ --- --- (5)

But always $A \subseteq (\tau_1, \tau_2)$-spcl $(A)$.

--- --- (6)
From (5) and (6) we have \( A = (\tau_1, \tau_2) - \text{spcl} (A) \). That implies \( A \) is

\((\tau_1, \tau_2) - \text{semipreclosed set of } (X, \tau_1, \tau_2)\).

**THEOREM 6.2.12**

Let \( A \) be a \((\tau_1, \tau_2) - \text{spg closed set of } (X, \tau_1, \tau_2)\). Then

\((\tau_1, \tau_2) - \text{spcl} (A) - A \) does not contain non empty \((\tau_1, \tau_2) - \text{semiclosed set}\).

**PROOF**

Let \( F \) be a \((\tau_1, \tau_2) - \text{semiclosed set contained in } (\tau_1, \tau_2) - \text{spcl}(A) - A\).

Then clearly \( A \subseteq X - F \) where \( A \) is \((\tau_1, \tau_2) - \text{spg closed and } X - F \) is

\((\tau_1, \tau_2) - \text{semiopen set of } (X, \tau_1, \tau_2)\). And hence \((\tau_1, \tau_2) - \text{spcl} (A) \subseteq X - F\).

That implies \( F \subseteq X - (\tau_1, \tau_2) - \text{spcl} (A)\).

Then \( F \subseteq (X - (\tau_1, \tau_2) - \text{spcl} (A)) \cap ( (\tau_1, \tau_2) - \text{spcl} (A) - A ) \).

Hence \( F \subseteq (X - (\tau_1, \tau_2) - \text{spcl} (A)) \cap ( (\tau_1, \tau_2) - \text{spcl} (A) ) = \phi\).

This shows that \( F = \phi\).
THEOREM 6.2.13

If $A$ is pairwise $\alpha$ open and $(\tau_1, \tau_2) - \text{spg}$ closed set of $(X, \tau_1, \tau_2)$, then $A$ is $\tau_2$ - open in $(X, \tau_1, \tau_2)$.

PROOF

By hypothesis $A$ is $(\tau_1, \tau_2) - \alpha$ open.

Then $A \subseteq \tau_1 - \text{int} (\tau_2 - \text{cl} (\tau_1 - \text{int} (A)))$. --- --- (7)

Always $\tau_1 - \text{int} (\tau_2 - \text{cl} (\tau_1 - \text{int} (A))) \subseteq (\tau_2 - \text{cl} (\tau_1 - \text{int} (A)))$. --- (8)

From (7) and (8) we have $A \subseteq \tau_2 - \text{cl} (\tau_1 - \text{int} (A))$.

And hence $A$ is $(\tau_1, \tau_2) -$ semiopen.

Also $A \subseteq A$ and $A$ is $(\tau_1, \tau_2) - \text{spg}$ closed set. That implies $(\tau_1, \tau_2) - \text{spcl} (A) \subseteq A$.

But $(\tau_1, \tau_2) - \text{spcl} (A) \supseteq A \cup \tau_2 - \text{int} (\tau_1 - \text{cl} (\tau_2 - \text{int} (A)))$.

Thus $\tau_2 - \text{int} (\tau_1 - \text{cl} (\tau_2 - \text{int} (A))) \subseteq A$. --- --- (9)

Also $A$ is $(\tau_2, \tau_1) - \alpha$ open. We have $A \subseteq \tau_2 - \text{int} (\tau_1 - \text{cl} (\tau_2 - \text{int} (A)))$. --- --- (10)
From (9) and (10) we have $A = \tau_2 \cdot \text{int} (\tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A)))$.

That implies $A$ is $\tau_2$-open is $(X, \tau_1, \tau_2)$.

**THEOREM 6.2.14**

Let $A$ is $(\tau_1, \tau_2) - \text{spg}$ closed set of $(X, \tau_1, \tau_2)$ and $A \subset B \subset (\tau_1, \tau_2) - \text{spcl} (A)$. Then $B$ is $(\tau_1, \tau_2) - \text{spg}$ closed set of $(X, \tau_1, \tau_2)$.

**PROOF**

Let $U$ be a $(\tau_1, \tau_2) - \text{semiopen}$ set of $(X, \tau_1, \tau_2)$ such that $B \subset U$.

Then $A \subset B \subset U$ and hence $A \subset U$. Since $A$ is $(\tau_1, \tau_2) - \text{spg}$ closed, $A \subset U$, and $U$ is $(\tau_1, \tau_2) - \text{semi open}$, we have, $(\tau_1, \tau_2) - \text{spcl} (A) \subset U$. By hypothesis $B \subset (\tau_1, \tau_2) - \text{spcl} (A)$. Thus $(\tau_1, \tau_2) - \text{spcl} (B) \subset (\tau_1, \tau_2) - \text{spcl} (A) \subset U$.

$B \subset U$ and $U$ is $(\tau_1, \tau_2) - \text{semiopen}$.

This proves that $B$ is also $(\tau_1, \tau_2) - \text{spg}$ closed set of $(X, \tau_1, \tau_2)$.

**THEOREM 6.2.15**

For a subset $A$ of $(X, \tau_1, \tau_2)$ the following conditions are equivalent:
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(i) \(A\) is \((\tau_1, \tau_2)\) - semiopen and \((\tau_1, \tau_2)\) - sg closed.

(ii) \(A\) is \((\tau_1, \tau_2)\) - semi-star - regular.

PROOF

(i) \(\Rightarrow\) (ii)

Let \(A\) is \((\tau_1, \tau_2)\) - semiopen and \((\tau_1, \tau_2)\) - sg closed. \hfill (11)

Now \(A\) is \((\tau_1, \tau_2)\) - sg closed, \(A \subseteq A\) and \(A\) is \((\tau_1, \tau_2)\) - semiopen implies that

\((\tau_1, \tau_2)\) - scl\((A)\) \(\subseteq A\). \hfill (12)

Now \((\tau_1, \tau_2)\) - scl\((A) = A \cup \tau_2 \text{ - int}\((\tau_1 \text{ - cl}\((A)\)).\hfill (13)

From (12) and (13) we have \(\tau_2 \text{ - int}\((\tau_1 \text{ - cl}\((A)\)) \subseteq A\).

That implies \(A\) is \((\tau_1, \tau_2)\) - semi closed. \hfill (14)

From (11) and (14) we have \(A\) is \((\tau_1, \tau_2)\) - semi-star - regular.

(ii) \(\Rightarrow\) (i)

Now every \((\tau_1, \tau_2)\) - semi-star - regular set is both \((\tau_1, \tau_2)\) - semiopen and

\((\tau_1, \tau_2)\) - semiclosed. Also every \((\tau_1, \tau_2)\) - semi closed set is \((\tau_1, \tau_2)\) - sg closed.
For, let \( A \) be any \( (\tau_1, \tau_2) \)-semiclosed set such that \( A \subseteq U \) and \( U \) is \( (\tau_1, \tau_2) \)-semiopen in \( (X, \tau_1, \tau_2) \). Since \( A \) is \( (\tau_1, \tau_2) \)-semi closed, we have \( A = (\tau_1, \tau_2) \)-scl \((A)\). That implies \( (\tau_1, \tau_2) \)-scl \((A)\) \(\subseteq A\), \(A \subseteq U\) and \(U\) is \( (\tau_1, \tau_2) \)-semiopen. And hence \( A \) is \( (\tau_1, \tau_2) \)-sg closed in \( (X, \tau_1, \tau_2) \).

**THEOREM 6.2.16**

For a subset \( A \) of \( (X, \tau_1, \tau_2) \) with \( (\tau_1, \tau_2) \)-spcl \((A)\) = \( (\tau_1, \tau_2) \)-scl \((A)\) the following conditions are equivalent:

(i) \( A \) is \( (\tau_1, \tau_2) \)-semi open and \( (\tau_1, \tau_2) \)-spg closed.

(ii) \( A \) is \( (\tau_1, \tau_2) \)-semi-star regular.

**PROOF**

(i) \(\Rightarrow\) (ii)

By hypothesis \( A \) is \( (\tau_1, \tau_2) \)-semi open, \( A \subseteq A \) and \( A \) is \( (\tau_1, \tau_2) \)-spg closed.

That implies \( (\tau_1, \tau_2) \)-spcl \((A)\) \(\subseteq A\).

By Theorem [6.2.11] \( A \) is \( (\tau_1, \tau_2) \)-semi-preclosed.
Thus $A = (\tau_1, \tau_2)$ - spcl$(A)$. Since $(\tau_1, \tau_2)$ - spcl$(A) = (\tau_1, \tau_2)$ - scl$(A)$, we have $A = (\tau_1, \tau_2)$ - scl$(A)$. This implies that $A$ is $(\tau_1, \tau_2)$ - semiclosed.

Therefore we have $A$ is $(\tau_1, \tau_2)$ - semi-star-regular.

(ii) $\Rightarrow$ (i)

Let $A$ is $(\tau_1, \tau_2)$ - semi-star-regular

Then by theorem [6.2.15] $A$ is $(\tau_1, \tau_2)$ - semiopen and $(\tau_1, \tau_2)$ - sg closed.

Also by theorem [6.2.7] every $(\tau_1, \tau_2)$ - sg closed set is $(\tau_1, \tau_2)$ - spg closed.

Therefore $A$ is $(\tau_1, \tau_2)$ - semiopen and $(\tau_1, \tau_2)$ - spg closed. $\square$

6.3. PAIRWISE SEMI-PRE $T_{1/4}$ SPACES

DEFINITION 6.3.1

A space $(X, \tau_1, \tau_2)$ is called a $(\tau_1, \tau_2)$ - semi-pre $T_{1/4}$ space if every $(\tau_1, \tau_2)$ - semi-pre generalized closed set is $(\tau_1, \tau_2)$ - semi-preclosed.

DEFINITION 6.3.2

A space $(X, \tau_1, \tau_2)$ is called pairwise semi-pre $T_{1/4}$ if it is
EXAMPLE 6.3.3

Let \( X = \{ a, b \} \). Let \( \tau_1 = \{ \emptyset, X, \{ a \} \} \) and \( \tau_2 = \{ \emptyset, X, \{ b \} \} \) be topologies on \( X \). Then \((X, \tau_1, \tau_2)\) is pairwise semi-pre \( T_{1/4} \) space.

THEOREM 6.3.4

Let \((X, \tau_1, \tau_2)\) be a bitopological space. If \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\)-semi-pre \( T_{1/4} \) space then every singleton of \( X \) is \((\tau_1, \tau_2)\)-semiclosed or \((\tau_1, \tau_2)\)-semi-preopen.

PROOF

Let \( x \in X \) and assume that \( \{ x \} \) is not \((\tau_1, \tau_2)\)-semiclosed. Then \( X - \{ x \} \) is not \((\tau_1, \tau_2)\)-semiopen. This implies that the only \((\tau_1, \tau_2)\)-semi open set containing \( X - \{ x \} \) is \( X \). And hence \( X - \{ x \} \) is \((\tau_1, \tau_2)\)-semi-pre closed or equivalently \( \{ x \} \) is \((\tau_1, \tau_2)\)-semi-preopen.

THEOREM 6.3.5

If every singleton of \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\)-semiclosed or \((\tau_1, \tau_2)\)-preopen, then \( X \) is \((\tau_1, \tau_2)\)-semipre \( T_{1/4} \).
PROOF

Let \( A \) be a \((\tau_1, \tau_2)\) - spg closed subset of \((X, \tau_1, \tau_2)\).

Clearly \( A \subseteq (\tau_1, \tau_2)\) - spcl \((A)\). ---- ---- (15)

Let \( x \in (\tau_1, \tau_2)\) - spcl \((A)\). By assumption \( \{x\} \) is \((\tau_1, \tau_2)\) - semi closed or

\((\tau_1, \tau_2)\) - preopen.

Case (i) Suppose \( \{x\} \) is \((\tau_1, \tau_2)\) - semiclosed. Then by theorem [6.2.12]

\((\tau_1, \tau_2)\) - spcl \((A)\) - \( A \) does not contain \( \{x\} \).

Since \( x \in (\tau_1, \tau_2)\) - spcl \((A)\), we have \( x \in A \).

Case (ii) Let \( \{x\} \) is \((\tau_1, \tau_2)\) - preopen. Then \( \{x\} \) is \((\tau_1, \tau_2)\) - semi-preopen.

Since \( x \in (\tau_1, \tau_2)\) - spcl \((A)\), we have \( \{x\} \cap A \neq \emptyset \), then \( x \in A \).

So in any case \((\tau_1, \tau_2)\) - spcl \((A)\) \( \subseteq \) \( A \). ---- ---- (16)

From (15) and (16) we have \( A = (\tau_1, \tau_2)\) - spcl \((A)\).

That is \( A \) is \((\tau_1, \tau_2)\) - semi-pre closed.

Therefore \((X, \tau_1, \tau_2)\) is semi-pre \( T_{\frac{1}{4}} \).
DEFINITION 6.3.6

A space \((X, \tau_1, \tau_2)\) is called \((\tau_1, \tau_2)\) - semi-pre \(T_{1/2}\) space if every

\((\tau_1, \tau_2)\) - generalized semi-pre closed set of \((X, \tau_1, \tau_2)\) is

\((\tau_1, \tau_2)\) - semi-preclosed.

THEOREM 6.3.7

Every \((\tau_1, \tau_2)\) - semi-pre \(T_{1/2}\) space is \((\tau_1, \tau_2)\) - semi-pre \(T_{1/4}\) space.

PROOF

Let \((X, \tau_1, \tau_2)\) be a \((\tau_1, \tau_2)\) - semi-pre \(T_{1/2}\) space.

Let \(A\) be any \((\tau_1, \tau_2)\) - semi-pre generalized closed set of \((X, \tau_1, \tau_2)\).

But by theorem [6.2.9] \(A\) is \((\tau_1, \tau_2)\) - generalized semi-pre closed set of

\((X, \tau_1, \tau_2)\). Also \((X, \tau_1, \tau_2)\) is \((\tau_1, \tau_2)\) - semi-pre \(T_{1/2}\) implies \(A\) is

\((\tau_1, \tau_2)\) - semi-preclosed. Thus every \((\tau_1, \tau_2)\) - semi-pre generalized closed set is

\((\tau_1, \tau_2)\) - semi-preclosed. And hence \((X, \tau_1, \tau_2)\) is semi-pre \(T_{1/4}\).

The following example shows that the converse of the above theorem is not true.
EXAMPLE 6.3.8

Let $X = \{a, b, c\}$. Let $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}\}$ be topologies on $X$. Then $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2)$-semi-pre $T_{1/4}$ space. But $X$ is not $(\tau_1, \tau_2)$-semi-pre $T_{1/2}$.

For, take $A = \{a, b\}$. Then $A$ is $(\tau_1, \tau_2)$-gsp closed. But $A$ is not $(\tau_1, \tau_2)$-semi-pre closed.

Since $\tau_2 \cdot \text{int} (\tau_1 \cdot \text{cl} (\tau_2 \cdot \text{int} (A))) = X \not\subset A$.

DEFINITION 6.3.9

A subset $A$ of a bitological space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$-generalized preregular closed [2.3.1] (briefly $(\tau_1, \tau_2)$-gpr closed) if $\tau_2 \cdot \text{pcl} (A) \subset U$ whenever $A \subset U$ and $U$ is $(\tau_1, \tau_2)$-regular open in $(X, \tau_1, \tau_2)$.

DEFINITION 6.3.10

A space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$-preregular $T_{1/2}$ if every
THEOREM 6.3.11

Let $A$ be $(\tau_1, \tau_2)$ - gpr closed set of $(X, \tau_1, \tau_2)$. Then $\tau_2$ - pcl $(A)$ - A does not contain nonempty $(\tau_2, \tau_1)$ - regular closed set.

PROOF

Let $F$ be a $(\tau_2, \tau_1)$ - regular closed set contained in $\tau_2$ - pcl $(A)$ - A.

Then clearly $A \subseteq X - F$ where $A$ is $(\tau_1, \tau_2)$ - gpr closed and $X - F$ is $(\tau_1, \tau_2)$ - regular open. And hence $\tau_2$ - pcl $(A)$ $\subseteq X - F$.

That implies $F \subseteq X - \tau_2$ - pcl $(A)$.

Thus $F \subseteq (X - \tau_2$ - pcl $(A)) \cap (\tau_2$ - pcl $(A)$ - A). That implies $F = \phi$.

THEOREM 6.3.12

For a bitopological space $(X, \tau_1, \tau_2)$ the following conditions are equivalent:

(i) $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2)$ - preregular $T_{1/2}$.

(ii) Every singleton of $X$ is either $(\tau_2, \tau_1)$ - regular closed or $\tau_2$ - preopen.
PROOF

(i) ⇒ (ii)

Let \( x \in X \) and assume that \( \{ x \} \) is not \((\tau_2, \tau_1)\) - regular closed.

Then \( X - \{ x \} \) is not \((\tau_1, \tau_2)\) - regular open.

This implies that the only \((\tau_1, \tau_2)\) - regular open set containing \( X - \{ x \} \) is \( X \) and hence \( X - \{ x \} \) is \((\tau_1, \tau_2)\) - gpr closed.

By (i) \( X - \{ x \} \) is \( \tau_2 \) - preclosed, and hence \( \{ x \} \) is \( \tau_2 \) - preopen.

(ii) ⇒ (i)

Let \( A \subset X \) be \((\tau_1, \tau_2)\) - gpr closed. Let \( x \in \tau_2 - pcl (A) \).

We will show that \( x \in A \).

For, consider the following two cases

Case (i) Let the set \( \{ x \} \) is \((\tau_2, \tau_1)\) - regular closed. Then \( \tau_2 - pcl (A) - A \)
does not contain \( \{ x \} \) [by theorem 6.3.11].

Since \( \{ x \} \in \tau_2 - pcl (A) \), we have \( x \in A \).
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Case (ii)

Let \( \{ x \} \) is \( \tau_2 \)-pre open. Since \( x \in \tau_2 \- p\ cl(A) \), we have

\[ \{ x \} \cap \tau_2 \- p\ cl(A) \neq \emptyset \] then \( x \in A \). Therefore in both cases \( x \in A \). This shows that \( \tau_2 \- p\ cl(A) \subseteq A \). Always \( A \subseteq \tau_2 \- p\ cl(A) \).

Therefore \( A = \tau_2 \- p\ cl(A) \). And hence \( A \) is \( \tau_2 \)-preclosed set.

THEOREM 6.3.13

Let \( A \) be a \((\tau_1, \tau_2)\) - gsp closed set of \((X, \tau_1, \tau_2)\). Then \((\tau_1, \tau_2) \- spcl(A) \)-A does not contain nonempty \( \tau_1 \)-closed set.

PROOF

Let \( F \) be \( \tau_1 \)-closed set contained in \((\tau_1, \tau_2) \- spcl(A) \)-A.

Then clearly \( A \subseteq X - F \) where \( A \) is \((\tau_1, \tau_2)\) - gsp closed and \( X - F \) is \( \tau_1 \)-open.

And hence \((\tau_1, \tau_2) \- spcl(A) \subseteq X - F \). That implies \( F \subseteq X \- (\tau_1, \tau_2) \- spcl(A) \).

Then \( F \subseteq (X \- (\tau_1, \tau_2) \- spcl(A)) \cap ((\tau_1, \tau_2) \- spcl(A) \- A) \).

That implies \( F \subseteq (X \- (\tau_1, \tau_2) \- spcl(A)) \cap ((\tau_1, \tau_2) \- spcl(A) \- A) = \emptyset \).
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This shows that \( F = \phi \).

**THEOREM 6.3.14**

If every singleton of \( X \) is \( \tau_1 \) - closed or \( \tau_1 \) - open, then \( (X, \tau_1, \tau_2) \) is

\[ (\tau_1, \tau_2) - semi-pre \ T_{\nu_2}. \]

**PROOF**

Let \( A \) be a \( (\tau_1, \tau_2) - gsp \) closed subset of \( (X, \tau_1, \tau_2) \).

Clearly \( A \subset (\tau_1, \tau_2) - spcl (A) \). Let \( x \in (\tau_1, \tau_2) - spcl (A) \).

By assumption \( \{x\} \) is \( \tau_1 \) - closed or \( \tau_1 \) - open.

**Case (i)** Suppose \( \{x\} \) is \( \tau_1 \) - closed. Then by theorem [6.3.13]

\[ (\tau_1, \tau_2) - spcl (A) \] does not contain \( \{x\} \).

Since \( x \in (\tau_1, \tau_2) - spcl (A) \), we have \( x \in A \).

**Case (ii)** Let \( \{x\} \) is \( \tau_1 \) - open.

Then \( \{x\} \) is \( (\tau_1, \tau_2) \) - preopen. And hence \( \{x\} \) is \( (\tau_1, \tau_2) \) - semi-preopen

[remark 3.1 of [17]]. Since \( x \in (\tau_1, \tau_2) - spcl (A) \), we have \( \{x\} \cap A \neq \phi \).
And hence $x \in A$. So in any case $(\tau_1, \tau_2) - \text{spcl}(A) \subseteq A$.

Therefore $A = (\tau_1, \tau_2) - \text{spcl}(A)$. That is $A$ is $(\tau_1, \tau_2) - \text{semi-preclosed}$.

That implies $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2) - \text{semi-pre} T_{\frac{1}{2}}$.

\textbf{THEOREM 6.3.15}

A $(\tau_1, \tau_2) - \text{preregular} T_{\frac{1}{2}}$ space is $(\tau_1, \tau_2) - \text{semi-pre} T_{\frac{1}{2}}$ if every $(\tau_2, \tau_1)$ - regular closed set is $\tau_1$ - closed or $\tau_1$ - open.

\textbf{PROOF}

Let $(X, \tau_1, \tau_2)$ be a $(\tau_1, \tau_2) - \text{preregular} T_{\frac{1}{2}}$ space, then by the theorem [6.3.12] every singleton of $(X, \tau_1, \tau_2)$ is $(\tau_2, \tau_1)$ - regular closed or $\tau_2$ - pre open. By hypothesis every singleton of $(X, \tau_1, \tau_2)$ is $\tau_1$ - closed or $\tau_1$ - open. Then by theorem [6.3.14] $(X, \tau_1, \tau_2)$ is $(\tau_1, \tau_2) - \text{semi-pre} T_{\frac{1}{2}}$.

\textbf{THEOREM 6.3.16}

A $(\tau_1, \tau_2) - \text{preregular} T_{\frac{1}{2}}$ space is $(\tau_1, \tau_2) - \text{semi-pre} T_{\frac{1}{4}}$ space if every $(\tau_2, \tau_1)$ - regular closed set is $\tau_1$ - closed or $\tau_1$ - open.
PROOF

Let $X$ be $(\tau_1, \tau_2) -$ preregular $T_{1/2}$ space.

Then by theorem [6.3.15] $X$ is $(\tau_1, \tau_2) -$ semi-pre $T_{1/2}$.

And hence by theorem [6.3.7] $X$ is $(\tau_1, \tau_2) -$ semi-pre $T_{1/4}$. \hfill $\square$

6.4 PAIRWISE SEMI-PREGENERALIZED CONTINUOUS FUNCTIONS

DEFINITION 6.4.1

A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise semi-generalized continuous (briefly pairwise sg continuous) if the induced maps

$f: (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f: (X, \tau_2) \rightarrow (Y, \sigma_2)$ are both semi-generalized (briefly sg continuous) continuous.

DEFINITION 6.4.2

A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise semi-pregeneralized (briefly pairwise spg) continuous if the inverse image of each $\sigma_j -$ closed set of $Y$ is $(\tau_i, \tau_j) -$ spg closed in $X$ where $i \neq j$, $(i, j = 1, 2)$. 
REMARK 6.4.3

Every pairwise sg continuous function is pairwise spg continuous but the converse is not true.

For, let $X = \{a, b, c\} = Y$. Let $\tau_1 = \{\phi, X, \{a, b\}\} = \sigma_1$ and $\tau_2 = \{\phi, X, \{b, c\}\} = \sigma_2$. Then $\tau_1$ and $\tau_2$ are topologies on $X$ and $\sigma_1$ and $\sigma_2$ are topologies on $Y$.

Define $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by

$$f(a) = c, f(b) = b \text{ and } f(c) = a.$$ Then $f$ is pairwise spg continuous.

But it is not pairwise sg continuous.

For, take $V = \{c\}$. Then $V$ is $\sigma_1$-closed. Then $f^1(V) = \{a\}$.

Take $A = \{a\}$ and $U = \{a, b\}$. Then $U$ is $\tau_1$-semiopen and $A \subseteq U$.

Now $\tau_1$-scl $(A) = X \subseteq U$ therefore the function $f: (X, \tau_1) \to (Y, \sigma_1)$ is not sg continuous. And hence $f$ is not pairwise sg continuous.
DEFINITION 6.4.4

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be pairwise generalized semi-pre continuous (briefly gsp continuous) if the inverse image of each $\sigma_j$-closed set of $Y$ is $(\tau_i, \tau_j)$-gsp closed in $X$ where $i \neq j, (i, j = 1, 2)$.

THEOREM 6.4.5

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise spg continuous. Then $f$ is pairwise gsp continuous.

PROOF

Let $V$ be a $\sigma_j$-closed set of $(Y, \sigma_1, \sigma_2), j = 1, 2$

Since $f$ is pairwise spg continuous, we have $f^{-1}(V)$ is $(\tau_i, \tau_j)$-spg closed in $X, i \neq j (i, j = 1, 2)$.

Since every $(\tau_i, \tau_j)$-spg closed is $(\tau_i, \tau_j)$-gsp closed, we have $f^{-1}(V)$ is also $(\tau_i, \tau_j)$-gsp closed $i \neq j (i, j = 1, 2)$. And hence $f$ is pairwise gsp continuous. $\square$

The following example shows that the converse of the above theorem is not true.
EXAMPLE 6.4.6

Let $X = \{a, b, c\} = Y$. Let $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{a, c\}\}$ be topologies on $X$.

Let $\sigma_1 = \{\phi, Y, \{a\}\}$ and $\sigma_2 = \{\phi, Y, \{b\}\}$ be topologies on $Y$.

Let $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be the identity function. Then $f$ is pairwise gsp continuous. But it is not pairwise spg continuous.

For, considering the $\sigma_2$ - closed set $\{a, c\}$, $f^{-1}(\{a, c\}) = \{a, c\}$.

Also $\{a, c\} \subseteq \{a, c\}$ and $\{a, c\}$ is $(\tau_1, \tau_2)$ - semiopen.

Now $(\tau_1, \tau_2)$ - spcl $(\{a, c\}) = X \varsubsetneq \{a, c\}$.

Therefore $f$ is not pairwise spg continuous.

DEFINITION 6.4.7

A function $f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be pairwise precontinuous [17] if the inverse image of each $\sigma_j$ - closed set of $Y$ is $(\tau_i, \tau_j)$ - preclosed in $X$,

$i \neq j \ (i, j = 1, 2)$.
THEOREM 6.4.8

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise precontinuous. Then \( f \) is pairwise spg continuous.

PROOF

Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be pairwise precontinuous.

Let \( V \) be an \( \sigma_j \)-closed set of \( Y \) (\( j = 1, 2 \)). Then \( f^i(V) \) is \((\tau_i, \tau_j)\)-preclosed in \( X \), \( i \neq j \), \( (i, j = 1, 2) \).

By theorem [6.2.5] \( f^i(V) \) is \((\tau_i, \tau_j)\)-spg closed in \( X \).

And hence every closed set \( V \) of \( \sigma_j \), we have \( f^i(V) \) is \((\tau_i, \tau_j)\)-spg closed.

Therefore \( f \) is pairwise spg continuous.

EXAMPLE 6.4.9

The converse of the above theorem is not true, that is pairwise spg continuous function need not be pairwise precontinuous.

Let \( X = \{a, b, c\} = Y \). Let \( \tau_1 = \{\phi, X, \{a\}, \{c\}, \{a, c\}\} \) and
Let $\tau_2 = \{ \emptyset, X, \{ b \} \}$ be topologies on $X$.

Let $\sigma_1 = \{ \emptyset, Y, \{ b \}, \{ b, c \} \}$ and $\sigma_2 = \{ \emptyset, Y, \{ a \}, \{ a, c \} \}$

be topologies on $Y$.

Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity function.

Then $f$ is pairwise spg continuous. But $f$ is not pairwise precontinuous.

For, take $\sigma_1$-closed set $\{ a \}$. Then $f^{-1}(\{ a \}) = \{ a \}$.

Now $\tau_1 - \text{cl} (\tau_2 - \text{int}(\{ a \})) = \{ a, c \} \subsetneq \{ a \}$.

Therefore $\{ a \}$ is not $($ $\tau_2, \tau_1$ $)$-preclosed in $X$. And hence $f$ is not pairwise precontinuous.