CHAPTER - IV

HOMOMORPHISM ON FUZZY MEET SEMI L - IDEAL

4.1: Introduction

In this chapter, the concept of fuzzy meet semi L-ideal homomorphism in fuzzy meet semi L-ideal is introduced and the invariant property of fuzzy meet semi L-ideal is derived. Fuzzy meet semi L-coset in fuzzy meet semi L-ideal, fuzzy meet semi L-quotient ideal of fuzzy meet semi L-ideal and the properties of fuzzy meet semi L-ideal are discussed. Some related theorems are established.

Definition : 4.1.1

A one-one and onto fuzzy meet semi L-ideal homomorphism is called a fuzzy meet semi L-ideal isomorphism.

Theorem : 4.1.2

Let f be a fuzzy meet semi L-ideal homomorphism from a fuzzy meet semi L-ideal of A onto a fuzzy meet semi L-ideal of A'. If S_μ and S_σ are fuzzy meet semi L-ideal of A, then the following are true.

(v) \( f[S_\mu \wedge S_\sigma] = f[S_\mu] \wedge f[S_\sigma] \)

(vi) \( f[S_\mu \cap S_\sigma] \subseteq f[S_\mu] \cap f[S_\sigma] \), with equality if atleast one of \( S_\mu \) or \( S_\sigma \) is f-invariant.

Proof:

Let \( y \in A' \) and \( \epsilon > 0 \) be given.

(i) Let \( S_\alpha = [f(S_\mu \wedge S_\sigma)](y) \) and

\( S_\beta = [f(S_\mu) \wedge f(S_\sigma)](y) \)
Then $S_{\alpha} \in \epsilon < \max_{z \in f^{-1}(y)} [S_{\mu} \wedge S_{\sigma}] (z)$

$\Rightarrow S_{\alpha} \in \epsilon > [S_{\mu} \wedge S_{\sigma}] (x)$, for some $x \in f^{-1}(y(t))$

$\Rightarrow S_{\alpha} \in \epsilon \geq \{S_{\mu} (x_1), S_{\sigma} (x_2)\}$, for some $x_1, x_2 \in \Lambda$ such that $x = x_1 \wedge x_2$.

$\leq \min \{ [(f^{-1}(f(S_{\mu}))(x_1)], [(f^{-1}(f(S_{\sigma}))(x_2)] \}

= \min \{ [(f(S_{\mu}))(x_1)], [(f(S_{\sigma}))(x_2)] \}

\geq [(f(S_{\mu}) \wedge f(S_{\sigma}))(x)$, since $x = x_1 \wedge x_2$.

$= S_{\beta}$

$\Rightarrow S_{\alpha} \geq S_{\beta}$. Since $\epsilon$ is arbitrary

--------- (1)

Claim : $S_{\beta} \geq S_{\alpha}$

$S_{\beta} \in \epsilon < [f(S_{\mu}) \wedge f(S_{\sigma})] (y)$

$= \max_{z \in f^{-1}(y)} \{ \min \{[(f(S_{\mu}))(y_1)], [(f(S_{\sigma}))(y_2)] \}$

Where $y_1, y_2 \in \Lambda'$.

$= \min \{ \max_{z \in f^{-1}(y_1)} \{ \min \{ \max_{z \in f^{-1}(y_1)} S_{\mu} (z), \max_{z \in f^{-1}(y_2)} S_{\sigma} (z) \} \}

< \min \{ \max_{z \in f^{-1}(y_1)} S_{\mu} (z), \max_{z \in f^{-1}(y_2)} S_{\sigma} (z) \}

\Rightarrow S_{\beta} \in \epsilon < \min \{S_{\mu} (x_1), S_{\sigma} (x_2)\}$, for some $x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)$

$\Rightarrow S_{\beta} \in \epsilon \leq [S_{\mu} \wedge S_{\sigma}] (x_1 \wedge x_2)$

$\Rightarrow S_{\beta} \in \epsilon \leq \max_{z \in f^{-1}(y)} \{ [S_{\mu} \wedge S_{\sigma}] (x) \}$ since $y = y_1 \wedge y_2 = f(x_1 \wedge x_2)$

$\Rightarrow [f(S_{\mu} \wedge S_{\sigma})](y) = S_{\sigma}$

78
Therefore (1) and (2) imply $S_\alpha = S_\beta$.

(i.e) $f[S_\mu \wedge S_\sigma] = f[S_\mu] \wedge f[S_\sigma]$

(ii) Now $S_\mu \cap S_\sigma \subseteq S_\mu$ and $S_\mu \cap S_\sigma \subseteq S_\sigma$

$\Rightarrow f[S_\mu \cap S_\sigma] \subseteq f[S_\mu]$ and $f[S_\mu \cap S_\sigma] \subseteq f[S_\sigma]$

$\Rightarrow f[S_\mu \cap S_\sigma] \subseteq f[S_\mu] \cap f[S_\sigma]$  \hspace{1cm} \text{(3)}

Next, assume that $S_\sigma$ is $f$-invariant.

Then $f^{-1}[f(S_\sigma)] = f[S_\sigma]$

Now, put $S_\alpha = [f(S_\mu) \cap f(S_\sigma)](y)$ and $S_\beta = [f(S_\mu \cap S_\sigma)](y)$

Then $S_\alpha - \varepsilon < \max \{ [f(S_\mu)](y), [f(S_\sigma)](y) \}$

\[= \max \{ \max_{x \in f^{-1}(y)} \{ S_\mu(x), [f(S_\sigma)](y) \} \} \]

$\Rightarrow S_\alpha - \varepsilon < S_\mu(z)$, for some $z \in f^{-1}(y)$ and $S_\alpha - \varepsilon < [f(S_\sigma)](y)$

$\Rightarrow S_\alpha - \varepsilon < S_\mu(z)$ and $S_\alpha - \varepsilon < [f(S_\sigma)](z) = f^{-1}[f(S_\sigma)](z) = S_\sigma(z)$

$\Rightarrow S_\alpha - \varepsilon < \max \{ S_\mu(z), S_\sigma(z) \}$

$\Rightarrow [S_\mu \cap S_\sigma](z)$

$\Rightarrow S_\alpha - \varepsilon < \max_{z \in f^{-1}(y)} \{ [S_\mu \cap S_\sigma](z) \}$, since $z \in f^{-1}(y)$

\[= f[S_\mu \cap S_\sigma](y) \]

\[= S_\beta \]
Hence \( f(S_\mu) \cap f(S_\sigma) \subseteq f(S_\mu \cap S_\sigma) \) \hspace{1cm} \begin{equation} (4) \end{equation}

Therefore (3) and (4) imply, \( f(S_\mu \cap S_\sigma) = f(S_\mu) \cap f(S_\sigma) \).

**Definition 4.1.3**

Let \((A, \wedge)\) and \((A', \wedge)\) be two fuzzy meet semilattices. Let \( f \) be a fuzzy meet semi \( L \)-ideal homomorphism from a fuzzy meet semi \( L \)-ideal of \( A \) onto a fuzzy meet semi \( L \)-ideal of \( A' \). If \( S_\mu \) and \( S_\sigma \) are fuzzy meet semi \( L \)-ideals of \( A \), then

\[
 f(S_\mu \wedge S_\sigma) = f(S_\mu) \wedge f(S_\sigma), \forall S_\mu, S_\sigma \in A.
\]

**Theorem 4.1.4**

Let \( A \) and \( A' \) be two fuzzy meet semilattices. If \( f \) is a fuzzy meet semi \( L \)-ideal homomorphism from a fuzzy meet semi \( L \)-ideal of \( A \) onto a fuzzy meet semi \( L \)-ideal of \( A' \) then for each fuzzy meet semi \( L \)-ideal \( S_\mu \) of \( A \), \( f(S_\mu) \) is a fuzzy meet semi \( L \)-ideal of \( A' \) and for each fuzzy meet semi \( L \)-ideal \( S_\mu' \) of \( A' \), \( f^{-1}(S_\mu') \) is a fuzzy meet semi \( L \)-ideal of \( A \).

**Proof:**

Let \( f \) be a fuzzy meet semi \( L \)-ideal homomorphism from a fuzzy meet semi \( L \)-filter of \( A \) onto a fuzzy meet semi \( L \)-ideal of \( A' \).

Assume that \( S_\mu \) is a fuzzy meet semi \( L \)-ideal of \( A \) and define \( S_\mu' = [f(x)] = S_\mu(x) \)

and for \( y \in A' \), \( S_\mu'(y) = \sup_{x \in f^{-1}(y)} \{ S_\mu(x) \} \).
To prove that $S_\mu$ is a fuzzy meet semi L-ideal of $A'$ corresponding to the fuzzy meet semi L-ideal $S_\mu$ of $A$.

(ie) to prove $S_\mu' [f(x) \wedge f(y)] \geq \max \{S_\mu'[f(x)], S_\mu'[f(y)]\}$

(ie) $S_\mu' [f(x) \wedge f(y)] = S_\mu' [f(x \wedge y)]$, since $f$ is a fuzzy meet semi L-ideal homomorphism.

$= S_\mu (x \wedge y)$, since $S_\mu'[f(x)] = S_\mu (x)$.

$\leq \max \{S_\mu (x), S_\mu (y)\}$

$= \max \{S_\mu'[f(x)], S_\mu'[f(y)]\}$

Therefore $S_\mu'$ is a fuzzy meet semi L-ideal of $A'$

Let $S_\mu'$ be a fuzzy meet semi L-ideal of $A'$.

To prove that $f^\dagger(S_\mu)$ is a fuzzy meet semi L-ideal of $A'$

$S_\mu' [f(x) \wedge f(y)] = S_\mu' [f(x \wedge y)] = S_\mu (x \wedge y)$

Now, $S_\mu (x_1 \wedge x_2) = S_\mu [f^\dagger(y_1) \wedge f^\dagger(y_2)]$

$= S_\mu [f^\dagger(y_1 \wedge y_2)]$

$= S_\mu' (y_1 \wedge y_2)$

$\geq \max \{S_\mu' (y_1), S_\mu' (y_2)\}$

$= \max \{S_\mu'[f(x_1)], S_\mu'[f(x_2)]\}$

$= \max \{S_\mu (x_1), S_\mu (x_2)\}$

Hence $S_\mu f^\dagger$ is a fuzzy meet semi L-ideal of $A$ corresponding to $S_\mu'$ of $A'$.
**Definition : 4.1.5**

Let $A$ and $A'$ be two fuzzy meet semilattices. Let $f$ be any function from a fuzzy meet semi L-ideal of $A$ onto a fuzzy meet semi L-ideal of $A'$. Then $S_\mu$ is called $f$-invariant if $f[S_\mu(x)] = f[S_\mu(y)]$ then $S_\mu(x) = S_\mu(y)$, where $x, y \in A$.

**Theorem : 4.1.6**

Let $A$ and $A'$ be two fuzzy meet semilattices. Let $f$ be a fuzzy meet semi L-ideal isomorphism from a fuzzy meet semi L-ideal of $A$ onto a fuzzy meet semi L-ideal of $A'$. Let $S_\mu$ and $S_\mu'$ be fuzzy meet semi L-ideals of $A$ and $A'$ respectively and let $S_\mu$ be $f$-invariant. Let $t = S_\mu(x) = S_\mu'[f(x)]$. Then the following statements are true.

(i) $F_{f(S_\mu)} = \{ f(S_{\mu_t}) / t \in \text{Im } S_\mu \}$ and

(ii) $F^{-1}_{f(S_\mu')} = \{ f^{-1}(S_{\mu_t'}) / s \in \text{Im } S_\mu' \}$

where $F_{S_\mu}$ denotes the family of fuzzy level meet semi L-ideals of $S_\mu$ in $A$.

**Proof :**

For (i): $t \in \text{Im } S_\mu \iff S_\mu(x) = t$, for some $x \in A$.  

$\iff [f^{-1}(S_\mu)](x) = t$  

$\iff [f(S_\mu)]f(x) = t$  

$\iff [f(S_\mu)](y) = t$  

$\iff t \in \text{Im } f[S_\mu] = \text{Im } S_\mu'$.  

82
Therefore $\text{Im } S_\mu = \text{Im } S_{\mu'}$.

Claim: $f[S_{\mu}] = [f(S_\mu)]_t$

Let $y \in f[S_{\mu}]$

$\Rightarrow y = f(x)$, for some $x \in S_{\mu}$, $x \in A$

$\Rightarrow y = f(x)$ and $S_\mu(x) \geq t$

$\Rightarrow \sup\{S_\mu(z) : y = f(z)\} \geq t$

$\Rightarrow [f(S_\mu)](y) \geq t$

$\Rightarrow y \in f[S_{\mu}]$

Therefore, $f[S_{\mu}] \subseteq f[S_{\mu}]$

Also, $f[S_{\mu}] \subseteq f[S_{\mu}]$, since $y \in f[S_{\mu}]$, $[f(S_\mu)](y) \geq t$

$\Rightarrow [f(S_\mu)](x) \geq t$, since $y = f(x)$, for some $x \in A$.

$\Rightarrow f^{-1}[f(S_\mu)](x) \geq t$

$\Rightarrow S_\mu(x) \geq t$

$\Rightarrow x \in S_{\mu}$

$\Rightarrow y = f(x) \in f(S_{\mu})$

Hence $F_{f \mid S_{\mu}} = \{[f(S_\mu)]_t : t \in \text{Im } f[S_{\mu}]\}$

$= \{f[S_{\mu}] : t \in \text{Im } f[S_{\mu}]\}$

For (ii):

Let $s \in f^{-1}[S_{\mu}] \Leftrightarrow$ there exists $x \in A$ such that $[f^{-1}(S_{\mu})](x) = s$
\( \Rightarrow S'_\mu \cdot [f(x)] = s \)

\( \Rightarrow [S'_\mu] (y) = s \)

\( \Rightarrow s \in \text{Im } S'_\mu \)

Next, \( x \in \{ f^{-1} (S'_\mu) \} \), \( \Rightarrow \{ f^{-1} (S'_\mu) \} (x) \geq s \)

\( \Rightarrow S'_\mu \cdot [f(x)] \geq s \)

\( \Rightarrow f(x) \in [S'_\mu]_s \)

\( \Rightarrow x \in f^{-1} [S'_\mu]_s \)

Hence \( f^{-1} ([S'_\mu, \cdot]) = \{ f^{-1} (S'_\mu) \} / s \in \text{Im } f^{-1} (S'_\mu) \}

\[ = \{ f^{-1} (S'_\mu)_s \} \}

**Theorem : 4.1.7**

Let \( A \) and \( A' \) be two fuzzy meet semilattices. Let \( f \) be a fuzzy meet semi L-ideal homomorphism from a fuzzy meet semi L-ideal of \( A \) onto a fuzzy meet semi L-ideal of \( A' \). If \( S'_\mu \) and \( S'_{\varphi} \) are any two fuzzy meet semi L-ideals of \( A' \) then

\[ [S'_\mu \wedge S'_{\varphi}] (f^{-1}) \subseteq [S'_\mu] (f^{-1}) \wedge [S'_{\varphi}] (f^{-1}) \]

**Proof :**

Let \( x \in A \) and let \( \epsilon > 0 \) be given

Let \( S_a = \{ [S'_\mu] f^{-1} \wedge [S'_{\varphi}] f^{-1} \} (y) \)

and \( S_b = \{ [S'_\mu \wedge S'_{\varphi}] f^{-1} \} (y) \)

Then \( S_a \cdot \epsilon > \min \{ \max \{ [S'_\mu] f^{-1} (y_1), [S'_{\varphi}] f^{-1} (y_2) \} \} \forall y_1, y_2 \in A' \)

\[ = \min \{ \max \{ S'_\mu \cdot [f^{-1} (y_1)], S'_{\varphi} \cdot [f^{-1} (y_2)] \} \} \text{ for some } y_1, y_2 \in A' \Rightarrow y = y_1 \wedge y_2 \]
\[
\leq [S_{\mu'} \wedge S_{\nu'}] (f^1(y)) = S_{\beta}
\]

\Rightarrow S_{\alpha} \leq S_{\beta} \text{ since } \epsilon \text{ is arbitrary.}

Hence \([S_{\mu'} \wedge S_{\nu'}] (f^1) \subseteq [S_{\mu'} f^1] \wedge [S_{\nu'} f^1] \).

**4.2 FUZZY MEET SEMI L-QUOTIENT IDEAL**

**Definition : 4.2.1**

Let \(S_{\mu}\) be any fuzzy meet semi L-ideal of a fuzzy meet semilattice \(A\). Then the fuzzy meet semi L-ideal \(S_{\mu^*}\) of \(A\), where \(x \in A\) defined by \(S_{\mu^*}(y) = S_{\mu}(y \wedge x)\) for all \(y \in A\) is termed as the fuzzy meet semi L-coset ideal determined by \(x\) and \(S_{\mu}\).

**Remark : 4.2.2**

If \(S_{\mu}\) is constant, then \(A_{S_{\mu}^*} = S_{\mu^*}(0)\)

**Theorem : 4.2.3**

Let \(S_{\mu}\) be any fuzzy meet semi L-ideal of a fuzzy meet semilattice \(A\). Then \(S_{\mu^*}\) for all \(x \in A\), the fuzzy meet semi L-coset ideal of \(S_{\mu}\) in \(A\) is also a fuzzy meet semi L-ideal of \(A\).

**Proof :**

Given \(S_{\mu}\) is any fuzzy meet semi L-ideal of \(A\) and \(S_{\mu^*}\) is a fuzzy meet semi L-coset of \(x\) in \(A/S_{\mu}\).
To prove that $S_{\mu_*}$ is a fuzzy meet semi L-ideal.

For all $y, z \in A$, $S_{\mu_*} (y \land z) = S_{\mu} [(y \land z) \land x]$, by definition

$$= S_{\mu} [(y \land x) \land (z \land x)]$$

$$\geq \max \{S_{\mu} (y \land x), S_{\mu} (z \land x)\}$$

$$\geq \max \{S_{\mu_*} (y), S_{\mu_*} (z)\}$$

Hence $S_{\mu_*}$ is a fuzzy meet semi L-ideal of $A$.

**Lemma 4.2.4**

If $S_{\mu}$ is a fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$, then the following holds:

$$S_{\mu} (x) = S_{\mu} (0) \iff S_{\mu_*} = S_{\mu_0_*}, \forall x \in A.$$

**Proof:**

Let $S_{\mu} (x) = S_{\mu} (0)$  \hspace{1cm} (1)

$\forall y \in A, S_{\mu} (y) \leq S_{\mu} (0)$  \hspace{1cm} (2)

From (1) and (2), we have $S_{\mu} (y) \leq S_{\mu} (x)$.

**Case (1):**

If $S_{\mu} (x) < S_{\mu} (y)$, then

$$S_{\mu} (y \land z) \geq \max \{S_{\mu} (y), S_{\mu} (x)\}$$

$$= S_{\mu} (x)$$
Case (ii):

If $S_{\mu}(y) = S_{\mu}(x)$, then $x, y \in S_{\mu}$, Where $t = S_{\mu}(0)$

Hence $S_{\mu}(y \land x) \geq \max\{S_{\mu}(y), S_{\mu}(x)\}$

$= S_{\mu}(x)$

$= S_{\mu}(0)$

Therefore, $S_{\mu}(y \land x) = S_{\mu}(0) = S_{\mu}(x) = S_{\mu}(y)$

Thus in either case,

$S_{\mu}(y \land x) = S_{\mu}(y), \forall \ y \in A$

(ie) $S_{\mu}^* (y) = S_{\mu}(y) = S_{\mu}^* (y)$

Therefore, $S_{\mu}^* = S_{\mu}^*$

The converse is straight forward.

Definition: 4.2.5

Let $S_{\mu}$ be any fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$. The fuzzy meet semi L-quotient ideal $S_{\mu}^*$ of $A_{S_{\mu}} (= A / S_{\mu})$ is defined by $S_{\mu}^* (x \land y_r) = S_{\mu}(x)$.

$\forall x \in A$, where $S_{\mu}(x) = \{x / S_{\mu}(x) = S_{\mu}(0) = t\}$.

Theorem: 4.2.6

If $S_{\mu}$ is a fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$, then $A/S_{\mu} \equiv A_{S_{\mu}}$, where $t = S_{\mu}(0)$. 

87
Proof:

Given $S_\mu$ is any fuzzy meet semi $L$-ideal of a fuzzy meet semi lattice $A$.

To prove $f:A \to A_{S_\mu}$ is a map defined by $f(x) = S_\mu'$, for all $x \in A$ is an onto fuzzy meet semi $L$-ideal homomorphism.

(ie) to prove $f(x \wedge y) = S_\mu'$.

$$f(x \wedge y) = S_\mu' = S_\mu'(z)$$

$$= [S_\mu(x \wedge y) \wedge z]$$

$$= S_\mu(x \wedge z) \wedge S_\mu(y \wedge z)$$

Therefore, $f$ is a fuzzy meet semi $L$-ideal homomorphism.

Now, $f(x) = S_\mu'$ $\iff$ $S_\mu' = S_\mu'^*$

$\Rightarrow S_\mu(x) = S_{\mu}(0)$, by lemma 4.2.4.

This shows that Kernel of $f$ equal $S_{\mu}$.

Therefore $A/S_{\mu} \equiv A_{S_{\mu}}$.

Theorem: 4.2.7

Let $A$ and $A'$ be two fuzzy meet semilattices. Let $f$ be a fuzzy meet semi $L$-ideal homomorphism from a fuzzy meet semi $L$-ideal of $A$ onto a fuzzy meet semi $L$-ideal of $A'$ and let $S_\mu$ be any $f$-invariant fuzzy meet semi $L$-ideal of $A$, then $A_{S_\mu} \equiv A'_{S_{\mu}}$. 
Proof:

Since $S_\mu$ is $f$-invariant, $K_f \subseteq S_\mu$, where $t = S_\mu(0)$

Now, $f[S_\mu(0')] = t$, because

$$f[S_\mu(0')] = \sup_{x \in f^{-1}(0')} S_\mu(x),$$

$$= S_\mu^*(0), \text{ since } f(0) = 0' \text{ and } S_\mu(x) \leq S_\mu(0), \forall x \in A.$$

Now, $f[S_\mu] = f[S_\mu']$, since

$$f(x) \in f[S_\mu] \iff f(S_\mu(f(x))) \geq t$$

$$\iff f^{-1}(f(S_\mu))(x) \geq t$$

$$\iff S_\mu(x) \geq t, \text{ as } f^{-1}[f(S_\mu)] = S_\mu$$

$$\iff x \in S_\mu'$$

$$\iff f(x) \not\in f[S_\mu'], \text{ because } k_t \subseteq S_\mu'$$

Therefore, by theorem 4.2.6, $A_{S_\mu} = A/S_\mu'$ and $A_{f(S_\mu)}' \cong A' / f(S_\mu)$$$

Also, note that $A/S_\mu' \cong A' [f(S_\mu)]$

From this it can be shown that,

$$A_{S_\mu} = A/S_\mu \cong A'/f(S_\mu),$$

$$A_{S_\mu} \cong A'/f(S_\mu)$$

Hence $A_{S_\mu} \cong A'/f(S_\mu)$
Theorem 4.2.8

If $S_{\mu}$ is any fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$, then the fuzzy meet subset $S_{\mu^*}$ of $A_{S_{\mu}}$ by $S_{\mu^*}(x \wedge S_{\mu}) = S_{\mu}(x)$, where $x \in A$, is a fuzzy meet semi L-ideal of $A_{S_{\mu}}$.

Proof:

Given that $S_{\mu}$ is a fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$.

To show that the fuzzy meet semi L-ideal $S_{\mu^*}$ of $A_{S_{\mu}}$ defined by $S_{\mu^*}(x \wedge S_{\mu}) = S_{\mu}(x)$, where $x \in A$ is a fuzzy meet semi L-ideal of $A$.

For this, let $x,y \in A$.

Then $S_{\mu^*}[(x \wedge S_{\mu}) \wedge (y \wedge S_{\mu})] = S_{\mu^*}(x \wedge y \wedge S_{\mu})$

$= S_{\mu}(x \wedge y)$

$\geq \max \{S_{\mu}(x), S_{\mu}(y)\}$

$\geq \max \{S_{\mu^*}(x \wedge S_{\mu}), S_{\mu^*}(y \wedge S_{\mu})\}$

Therefore $S_{\mu^*}$ is a fuzzy meet semi L-ideal of $A_{S_{\mu}}$.

Theorem 4.2.9

(i) Let $S_{\mu}$ be any fuzzy meet semi L-ideal of a fuzzy meet semilattice $A$ and let $t = S_{\mu}(1)$. Then the fuzzy meet semi L-ideal $S_{\mu^*}$ of $A|S_{\mu}$ defined by $S_{\mu^*}(x \wedge S_{\mu}) = S_{\mu}(x)$ for all $x \in A$ is a fuzzy meet semi L-ideal of $A|S_{\mu}$. 

90
(ii) If \( B \) is a fuzzy meet semilattice of \( A \) and \( S_\emptyset \) is a fuzzy meet semi \( L \)-ideal of \( A \mid B \) such that \( S_\emptyset (x \land A) = S_\emptyset (A) \) only when \( x \in A \), then there exists a fuzzy meet semi \( L \)-ideal \( S_\mu \) of \( A \) such that \( S_\mu = B, [t = S_\mu (1)] \) and \( S_\emptyset = S_\mu \).

Proof:

For (i):

Since \( S_\mu \) is a fuzzy meet semi \( L \)-ideal of \( A \), \( S_\mu \) is an fuzzy level meet semi \( L \)-ideal of \( A \).

Now, \( x \land S_\mu = y \land S_\mu \)

\[ \Rightarrow x \land y \in S(\mu) \]

\[ \Rightarrow S_\mu (x \land y) = t = S_\mu (0) \]

\[ \Rightarrow S_\mu (x) = S_\mu (y) \]

\[ \Rightarrow S_\mu \ast (x \land S_\mu) = S_\mu \ast (y \land S_\mu) \]

Therefore, \( S_\mu \ast \) is well defined.

Next, for all \( x, y \in A \),

\[ S_\mu \ast [(x \land S_\mu) \land (y \land S_\mu)] = S_\mu \ast [(x \land y) \land S_\mu] \]

\[ = S_\mu (x \land y) \]

\[ \geq \max \{ S_\mu (x), S_\mu (y) \} \]

\[ = \max \{ S_\mu \ast [x \land S(\mu)], S_\mu \ast [y \land S(\mu)] \} \]

For (ii):

Define \( S_\mu : A \rightarrow [0, 1] \) by \( S_\mu (x) = S_\emptyset (x \land B) \)

\( S_\mu (x \land y) = S_\emptyset [(x \land y) \land B] \)
\[ \geq \max \{ S_\theta(x \land B), S_\theta(y \land B) \} \]
\[ \geq \max \{ S_\mu(x), S_\mu(y) \} \]

Therefore, \( S_\mu \) is a fuzzy meet semi L-ideal.

Also, \( S_{\mu, \gamma} = B \)

\[ x \in S_{\mu, \gamma} \iff S_\mu(x) = S_\mu(0) \]
\[ \iff S_\theta(x \land B) = S_\theta(B) \]
\[ \iff x \in B \]

Now, \( S_{\mu, \gamma}(x \land S_{\mu, \gamma}) = S_\mu(x) \)
\[ = S_\theta(x \land B) \]
\[ = S_\theta(x \land S_{\mu, \gamma}) \]

Hence \( S_{\mu, \gamma} = S_\theta \)

**Theorem : 4.2.10**

Let \( A \) be any fuzzy meet semilattice. Let \( S_{\mu, \gamma} \) be any fuzzy meet semi L-ideal of the quotient fuzzy meet semilattice \( A/K \), where \( K \) is any fuzzy meet subset of \( A \). Then corresponding to \( S_{\mu, \gamma} \) in \( A/K \), there exists a fuzzy meet semi L-ideal in \( A \).

**Proof**

Let \( S_{\mu, \gamma} \) be any fuzzy meet semi L-ideal of \( A/K \).

Define the fuzzy meet subset \( S_\theta \) of \( A \) by \( S_\theta(x) = S_{\mu, \gamma}(x \land k) \), \( \forall \ x \in A \).

To prove, \( S_\theta \) is a fuzzy meet semi L-ideal of \( A \).

\[ S_\theta(x \land y) = S_{\mu, \gamma}[(x \land y) \land k] \]
\[ S_{\mu^*} [(x \land k) \land (y \land k)] \]
\[ \geq \max \{ S_{\mu^*} (x \land k), S_{\mu^*} (y \land k) \} \]
\[ = \max \{ S_{\phi} (x), S_{\phi} (y) \} \]

Therefore, \( S_{\phi} (x \land y) \geq \min \{ S_{\phi} (x), S_{\phi} (y) \} \)

Hence \( S_{\phi} \) is a fuzzy meet semi L-ideal of \( A \).

Theorem: 4.2.11

Let \( f \) be a fuzzy meet semi L-ideal homomorphism from a fuzzy meet semi L-ideal of \( A \) onto a fuzzy meet semi L-ideal of \( A' \) and let \( S_{\mu} \) be any fuzzy meet semi L-ideal of \( A \) such that \( S_{\mu} \subseteq K_t \), where \( t = S_{\phi}(0) \). Then there exists a unique fuzzy meet semi L-ideal homomorphism \( f' \) from \( A_{\mu} \) onto \( A' \) with the property that \( f' = f \circ g \), where \( g(x) = S_{\mu^*}, x \in A \).

Proof:

Define a function \( f' : A_{\mu} \rightarrow A' \) by \( f'(S_{\mu^*}) = f(x), \forall x \in A \).

Now, \( S_{\mu^*} = S_{\mu^*} \)

\[ \Rightarrow S_{\mu^*} = S_{\mu^*} \]
\[ \Rightarrow S_{\mu^*} (x \land y) = S_{\mu^*} (0) = t \]
\[ \Rightarrow x \land y \in S_{\mu^*} \subseteq K_t \]
\[ \Rightarrow f'(S_{\mu^*}) = f'(S_{\mu^*}) \]
Therefore $f'$ is well defined.

Since $f$ is onto, $f'$ is also onto.

Therefore, $f'$ is a fuzzy meet semi $L$-ideal homomorphism.

Now, $f(x) = f' \left(S_{\mu}; \right)$

$= f'(g(x))$

$= [f \circ g](x)^x, \forall x \in A$.

Finally, to show that this factorization of $f$ is unique.

Suppose, that $f = h \circ g$ for some function $h : A_{S_{\mu}} \to A'$.

Then $f'(S_{\mu}) = f(x)$

$= [h \circ g](x)$

$= h[g(x)]$

$= h \left[ S_{\mu} \right]^x, \forall x \in A$

Hence, there is a unique fuzzy meet semi $L$-ideal $f'$ from $A_{S_{\mu}}$ onto $A'$ with the property that $f = f' \circ g$, where $g(x) = S_{\mu}^x, \forall x \in A$.

**Corollary: 4.2.12**

The induced $f'$ is an fuzzy meet semi $L$-ideal isomorphism if $S_{\mu}$ is $f$-invariant.

**Proof:**

Let $f'$ be one - one.

Claim: $S_{\mu}$ is $f$-invariant.
Let \( x, y \in A \)

\[ f(x) = f(y) \]

\[ \Rightarrow f'[S_{\mu^*}] = f'[S_{\nu^*}] \]

\[ \Rightarrow S_{\mu^*} = S_{\nu^*}, \text{ since } f \text{ is invariant} \]

\[ \Rightarrow S_{\nu^*} = S_{\mu^*} \]

\[ \Rightarrow S\mu(x, \nu y) = S\mu(0) \]

\[ \Rightarrow S\mu(x) = S\mu(y) \]

On the otherhand, let \( S\mu \) be \( f \)-invariant

Claim: \( f' \) is one-one.

\[ S\mu(x) = S\mu(y) \]

\[ \Rightarrow f'[S\mu(x)] = f'[S\mu(y)] \]

\[ \Rightarrow f'[S_{\mu^*}] = f'[S_{\nu^*}] \]

\[ \Rightarrow S_{\mu^*} = S_{\nu^*}, \text{ since } f \text{ is invariant} \]

\[ \Rightarrow f \text{ is one-one.} \]