CHAPTER - V

GENERALIZED CLOSED SETS WITH RESPECT TO AN IDEAL

After the works of Levine [20] on generalized closed sets in topology, many mathematicians generalized the various concepts of topology by considering generalized closed sets instead of closed sets. In this chapter we generalize the concept of $g$ - closed sets of Levine [20] to $\mathfrak{I}$-g closed sets (generalized closed sets with respect to an ideal) and obtained various properties. We introduce and investigate the concept of generalized locally closed sets with respect to an ideal.

$\mathfrak{I}$-g closed sets

Definition 5.1

Let $(X, \tau)$ be a topological space and $\mathfrak{I}$ be an ideal on $X$. A subset $A$ of $X$ is said to be generalized closed with respect to an ideal (simply written as $\mathfrak{I}$-g closed) if and only if $\text{cl}(A) - O \in \mathfrak{I}$, whenever $A \subseteq O$ and $O \in \tau$.

Remark

(i) Every $\mathfrak{I}$-g closed set is $g$-closed if $\mathfrak{I} = \{\phi\}$.

(ii) Every $g$ - closed set is $\mathfrak{I}$-g closed but converse need not be true, as this may be seen from the following example.

Example 5.2

Let $R$ be the real line with the usual topology and $\mathfrak{I}$ be the ideal of all finite subsets of $R$. Then every open interval in $R$ is $\mathfrak{I}$-g closed but not $g$ - closed.
We characterize $\mathcal{I}$-g closed sets in the following theorem.

**Theorem 5.3**

A set $A$ is $\mathcal{I}$-g closed in $(X, \tau)$ if and only if $F \subseteq \text{cl}(A) - A$ and $F$ is closed in $X$ imply that $F \in \mathcal{I}$.

**Proof**

Assume that $A$ is $\mathcal{I}$-g closed. Let $F \subseteq \text{cl}(A) - A$ and suppose $F$ is closed in $(X, \tau)$. Then $A \subseteq X - F$. By our assumption, $\text{cl}(A) - (X - F) \in \mathcal{I}$. But $F \subseteq \text{cl}(A) - (X - F)$ and hence $F \in \mathcal{I}$.

Conversely, assume that $F \subseteq \text{cl}(A) - A$ and $F$ is closed in $X$ imply $F \in \mathcal{I}$. Suppose $A \subseteq O$ and $O \in \tau$. Then $\text{cl}(A) - O = \text{cl}(A) \cap (X - O)$ is a closed set in $X$, that is contained in $\text{cl}(A) - A$. By assumption $\text{cl}(A) - O \in \mathcal{I}$. This implies $A$ is $\mathcal{I}$-g closed.

**Theorem 5.4**

If $A$ and $B$ are $\mathcal{I}$-g closed in $(X, \tau)$, then $A \cup B$ is $\mathcal{I}$-g closed.

**Proof**

Suppose $A$ and $B$ are $\mathcal{I}$-g closed sets in $(X, \tau)$. If $A \cup B \subseteq O$ and $O$ is open, then $A \subseteq O$ and $B \subseteq O$. By assumption, $\text{cl}(A) - O \in \mathcal{I}$ and $\text{cl}(B) - O \in \mathcal{I}$ and hence $\text{cl}(A \cup B) - O = (\text{cl}(A) \cup \text{cl}(B)) - O = (\text{cl}(A) - O) \cup (\text{cl}(B) - O) \in \mathcal{I}$. Thus is $A \cup B$ is $\mathcal{I}$-g closed.

Generally, the intersection of two $\mathcal{I}$-g closed set is not $\mathcal{I}$-g closed, which can be seen from the following example.
Example 5.5

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. If $A = \{a, b\}$, $B = \{a, c\}$ and $\mathcal{I} = \{\emptyset\}$ then $A$ and $B$ are $\mathcal{I}$-g closed but $A \cap B$ is not $\mathcal{I}$-g closed.

Theorem 5.6

If $A$ is $\mathcal{I}$-g closed and $A \subseteq B \subseteq \operatorname{cl}(A)$ in $(X, \tau)$, then $B$ is $\mathcal{I}$-g closed.

Proof

Suppose $A$ is $\mathcal{I}$-g closed and $A \subseteq B \subseteq \operatorname{cl}(A)$ in $(X, \tau)$. Suppose $B \subseteq O$ and $O$ is open. Then $A \subseteq O$. Since $A$ is $\mathcal{I}$-g closed, we have $\operatorname{cl}(A) - O \in \mathcal{I}$. Now $B \subseteq \operatorname{cl}(A) \Rightarrow \operatorname{cl}(B) \subseteq \operatorname{cl}(A) \Rightarrow \operatorname{cl}(B) - O \subseteq \operatorname{cl}(A) - O \in \mathcal{I} \Rightarrow \operatorname{cl}(B) - O \in \mathcal{I}$. Hence $B$ is $\mathcal{I}$-g closed.

Theorem 5.7

Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $\mathcal{I}$-g closed in $(X, \tau)$. Then $A$ is $\mathcal{I}$-g closed relative to the subspace $Y$ of $X$.

Proof

Suppose $A \subseteq O \cap Y$ and $O \in \tau$. Then $A \subseteq O$ and since $A$ is $\mathcal{I}$-g closed in $X$, we have $\operatorname{cl}(A) - O \in \mathcal{I}$. Now $(\operatorname{cl}(A) \cap Y) - (O \cap Y) = (\operatorname{cl}(A) - O) \cap Y \in \mathcal{I}$, whenever $A \subseteq O \cap Y$ and $O \in \mathcal{I}$. Hence $A$ is $\mathcal{I}$-g closed relative to the subspace $Y$.

We now give a definition.

Definition 5.8

Let $(X, \tau)$ be a topological space and let $\mathcal{I}$ be an ideal on $X$. If for each $I \in \mathcal{I}$ there exist $J \in \mathcal{I} \cap \tau$ such that $I \subseteq J$, then $\mathcal{I}$ is said to be saturated by $\tau$. 

Example 5.9

Let $X$ be the real line with usual topology and let $\mathcal{I}$ be the ideal of all bounded subsets of $X$. Then $\mathcal{I}$ is saturated by $\tau$.

Theorem 5.10

Suppose that $B \subseteq A \subseteq X$, $B$ is $\mathcal{I}$-g closed relative to $A$ and $A$ is a $\mathcal{I}$-g closed subset of $X$. If $\mathcal{I}$ is saturated by $\tau$ then $B$ is $\mathcal{I}$-g closed relative to $X$.

Proof

Suppose $\mathcal{I}$ is saturated by $\tau$. Let $B \subseteq O$ and $O$ be open in $(X, \tau)$. Then $B \subseteq O \cap A$. Since $B$ is $\mathcal{I}$-g closed relative to $A$, we have $\text{cl}_A(B) \subseteq (A \cap O) \cup I_1$, for some $I_1 \in \mathcal{I}$. By assumption, there exist $J_1 \in \mathcal{I} \cap \tau$ such that $A \cap \text{cl}(B) \subseteq (A \cap O) \cup J_1$. So $A \subseteq (O \cup J_1) \cup [X - \text{cl}(B)]$. Since $A$ is $\mathcal{I}$-g closed, $\text{cl}(A) \subseteq (O \cup J_1) \cup (X - \text{cl}(B)) \cup I_2$, for some $I_2 \in \mathcal{I}$. By assumption, there exist $J_2 \in \mathcal{I} \cap \tau$ such that $\text{cl}(A) \subseteq (O \cup J_1) \cup (X - \text{cl}(B)) \cup J_2$. Since $B \subseteq A$ we have

$$\text{cl}(B) \subseteq \text{cl}(A) \subseteq (O \cup J_1) \cup [X - \text{cl}(B)] \cup J_2$$

$$\Rightarrow \text{cl}(B) \subseteq O \cup (J_1 \cup J_2), \text{ for some } J_1, J_2 \in \mathcal{I}. \text{ This shows that } B \text{ is } \mathcal{I}-\text{g closed relative to } X.$$

Under the assumption "$\mathcal{I}$ is saturated by $\tau$", we have the following corollary.

Corollary 5.11

Let $A$ be a $\mathcal{I}$-g closed set and suppose that $F$ is a closed set then $A \cap F$ is a $\mathcal{I}$-g closed set.
Proof

$A \cap F$ is closed in $A$ and hence it is $\mathcal{S}$-g closed relative to $A$. As $A$ is $\mathcal{S}$-g closed, applying the above theorem, we get $A \cap F$ as a $\mathcal{S}$-g closed set in $X$.

The following theorem shows that the assumption "$\mathcal{S}$ is saturated by $\tau$” can be removed in the corollary (5.11).

**Theorem 5.12**

Let $A$ be a $\mathcal{S}$ -g closed set and suppose that $F$ is closed set, then $A \cap F$ is $\mathcal{S}$-g closed set.

**Proof**

Let $A \cap F \subseteq O$ and $O$ open. Then $A \subseteq O \cup (X-F)$. Since $A$ is $\mathcal{S}$-g closed, we have $\text{cl}(A) - (O \cup (X-F)) \in \mathcal{S}$.

Now $\text{cl}(A \cap F) \subseteq \text{cl}(A) \cap F = (\text{cl}(A) \cap F) - (X-F)$

Therefore $\text{cl}(A \cap F) - O \subseteq (\text{cl}(A) \cap F) - (O \cup (X-F))$

$\subseteq \text{cl}(A) - (O \cup (X-F)) \in \mathcal{S}$

Hence $A \cap F$ is $\mathcal{S}$-g closed.

An important property on compact sets is that every closed subset of a compact set is compact. Levine[20] proved that a g-closed subset of a compact set is compact. But an $\mathcal{S}$-g closed subset of a compact set need not be compact.
Example 5.13
Let $X = [0, 1]$ with usual topology.

Let $A = \bigcup_{n=1}^{\infty} \left( \frac{1}{n+1}, \frac{1}{n} \right)$ and $\mathcal{I}$ be the ideal of all countable subsets of $X$, then $A$ is $\mathcal{I}$-g closed but not compact.

Theorem 5.14
Let $(X, \tau)$ be a $\mathcal{I}$-compact topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}$ is saturated by $\tau$. If $A \subseteq X$ is $\mathcal{I}$-g closed, then $A$ is $\mathcal{I}$-compact.

Proof
Suppose $A$ is $\mathcal{I}$-g closed in $(X, \tau)$. Let $\{U_{\alpha} : \alpha \in \Lambda\}$ be an open cover of $A$.

Since $A$ is $\mathcal{I}$-g closed, we have $\text{cl}(A) \subseteq \bigcup_{\alpha \in \Lambda} U_{\alpha} \cup I$, for some $I \in \mathcal{I}$.

By assumption, there exist $J \in \mathcal{I} \cap \tau$ such that $\text{cl}(A) \subseteq \bigcup_{\alpha \in \Lambda} U_{\alpha} \cup J$.

So $X \subseteq \bigcup_{\alpha \in \Lambda} U_{\alpha} \cup J \cup (X - \text{cl}(A))$ and as $X$ is $\mathcal{I}$-compact, we can find

$\{U_{a1}, U_{a2}, U_{a3}, \ldots, U_{an}\}$ such that

$$X - \bigcup_{i=1}^{n} U_{ai} \cup J \cup (X - \text{cl}(A)) \in \mathcal{I}. $$

Therefore $A - \bigcup_{i=1}^{n} U_{ai} \subseteq \left[ \bigcup_{i=1}^{n} U_{ai} \cup (X - \text{cl}(A)) \right] \in \mathcal{I}$. Hence $A$ is $\mathcal{I}$-compact.

Theorem 5.15
Let $(X, \tau)$ be a $\mathcal{I}$-countably compact space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}$ is saturated by $\tau$. If $A \subseteq X$ is $\mathcal{I}$-g closed then $A$ is $\mathcal{I}$-countably compact.
Proof
Proof is similar to the theorem 5.13.

\[ 3\text{-g open sets} \]

Definition 5.16
Let \((X, \tau)\) be a topological space and \(\mathcal{I}\) be an ideal on \(X\). A subset \(A \subseteq X\) is said to be a generalized open set with respect to an ideal (simply we write \(\mathcal{I}\)-g open) if and only if \(X - A\) is \(\mathcal{I}\)-g closed.

Theorem 3.2
A set \(A\) is \(\mathcal{I}\)-g open in \((X, \tau)\) if and only if \(F - I \subseteq \text{Int}(A)\), whenever \(F \subseteq A\) and \(F\) is closed, for some \(I \in \mathcal{I}\).

Proof
Since \(F\) is closed and \(F \subseteq A\), we have \(X - A \subseteq X - F\) and by assumption,
\[
\text{cl}(X - A) \subseteq (X - F) \cup I, \text{ for some } I \in \mathcal{I}.
\]

This implies \(X - ((X - F) \cup I) \subseteq X - \text{cl}(X - A) \Rightarrow F - I \subseteq \text{Int}(A)\).

Conversely, assume that \(F \subseteq A\) and \(F\) is closed imply \(F - I \subseteq \text{Int}(A)\), for some \(I \in \mathcal{I}\). To prove \(A\) is \(\mathcal{I}\)-g open or \(X - A\) is \(\mathcal{I}\)-g closed, consider an open set \(O\) such that \(X - A \subseteq O\). Then \(X - O \subseteq A\). By assumption,

\[
(X - O) - I \subseteq \text{Int}(A) = X - \text{cl}(X - A)
\]

\[
\Rightarrow X - (O \cup I) \subseteq X - \text{cl}(X - A)
\]

\[
\Rightarrow \text{cl}(X - A) \subseteq O \cup I, \text{ for some } I \in \mathcal{I}.
\]

This shows that \(\text{cl}(X - A) - O \in \mathcal{I}\). Hence \(X - A\) is \(\mathcal{I}\)-g closed.
Generally the union of two $\mathfrak{I}$-g open sets need not be $\mathfrak{I}$-g open, which follows from the example 5.5.

**Theorem 5.18**

If $A$ and $B$ are separated $\mathfrak{I}$-g open sets in $(X, \tau)$, then $A \cup B$ is $\mathfrak{I}$-g open.

**Proof**

Suppose $A$ and $B$ are separated $\mathfrak{I}$-g open sets. Let $F$ be a closed subset in $A \cup B$. Then $F \cap \text{cl}(A) \subseteq A$ and $F \cap \text{cl}(B) \subseteq B$. By assumption,

$$(F \cap \text{cl}(A)) - I_1 \subseteq \text{Int}(A) \quad \text{and} \quad (F \cap \text{cl}(B)) - I_2 \subseteq \text{int}(B),$$

for some $I_1, I_2 \in \mathfrak{I}$. This means that $((F \cap \text{cl}(A)) - \text{Int}(A)) \in \mathfrak{I}$ and $((F \cap \text{cl}(B)) - \text{Int}(B)) \in \mathfrak{I}$.

and so $((F \cap \text{cl}(A)) - \text{Int}(A)) \cup ((F \cap \text{cl}(B)) - \text{Int}(B)) \in \mathfrak{I}$.

Hence $(F \cap (\text{cl}(A) \cup \text{cl}(B))) - (\text{Int}(A) \cup \text{Int}(B)) \in \mathfrak{I}$.

Now $F = F \cap (A \cup B) \subseteq F \cap \text{cl}(A \cup B)$, and we have

$$F - \text{Int}(A \cup B) \subseteq (F \cap \text{cl}(A \cup B)) - \text{Int}(A \cup B) \subseteq (F \cap \text{cl}(A \cup B)) - (\text{Int}(A) \cup \text{Int}(B)) \in \mathfrak{I}.$$  

Hence $F - I \subseteq \text{Int}(A \cup B)$, for some $I \in \mathfrak{I}$. This proves that $A \cup B$ is $\mathfrak{I}$-g open.

**Corollary 5.19**

Let $A$ and $B$ be $\mathfrak{I}$-g closed sets and suppose $X - A$ and $X - B$ are separated in $(X, \tau)$. Then $A \cap B$ is $\mathfrak{I}$-g closed.

**Proof**

If $A$ and $B$ are $\mathfrak{I}$-g closed then $X - A$ and $X - B$ are $\mathfrak{I}$-g open. By using the Theorem 5.18, we get $X - (A \cap B)$ is $\mathfrak{I}$-g open and hence $A \cap B$ is $\mathfrak{I}$-g closed.
Corollary 5.20

If $A$ and $B$ are $\mathcal{I}$-g open sets in $(X, \tau)$, then $A \cap B$ is $\mathcal{I}$-g open.

Proof

If $A$ and $B$ are $\mathcal{I}$-g open then $X - A$ and $X - B$ are $\mathcal{I}$-g closed. By theorem 5.4, $X - (A \cap B)$ is $\mathcal{I}$-g closed and hence $A \cap B$ is $\mathcal{I}$-g open.

Theorem 5.21

If $A \subseteq B \subseteq X$, $A$ is $\mathcal{I}$-g open relative to $B$ and $B$ is $\mathcal{I}$-g open relative to $X$, then $A$ is $\mathcal{I}$-g open relative to $X$.

Proof

Suppose $A \subseteq B \subseteq X$, $A$ is $\mathcal{I}$-g open relative to $B$ and $B$ is $\mathcal{I}$-g open relative to $X$. Suppose $F \subseteq A$ and $F$ is closed. Since $A$ is $\mathcal{I}$-g open relative to $B$, by theorem 5.17, $F - I_1 \subseteq \text{Int}_B(A)$, for some $I_1 \in \mathcal{I}$. This implies that there exist an open set $O_1$ such that

$$F - I_1 \subseteq O_1 \cap B \subseteq A,$$

for some $I_1 \in \mathcal{I}$.

Since $B$ is $\mathcal{I}$-g open, $F \subseteq B$ and $F$ is closed, we have $F - I_2 \subseteq \text{Int}(B)$, for some $I_2 \in \mathcal{I}$. This implies that there exist an open set $O_2$ such that $F - I_2 \subseteq O_2 \subseteq B$, for some $I_2 \in \mathcal{I}$.

Now $F - (I_1 \cup I_2) \subseteq (F - I_1) \cap (F - I_2)$

$$\subseteq O_1 \cap O_2 \subseteq O_1 \cap B$$

$$\Rightarrow F - (I_1 \cup I_2) \subseteq O_1 \cap O_2 \subseteq A$$

$$\Rightarrow F - (I_1 \cup I_2) \subseteq \text{Int}(A),$$

for some $I_1 \cup I_2 \in \mathcal{I}$.

$$\Rightarrow A$$

is $\mathcal{I}$-g open relative to $X$. 

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Theorem 5.22

If \( \text{Int} \ A \subseteq B \subseteq A \) and if \( A \) is \( \mathcal{I} \)-g open in \((X, \tau)\), then \( B \) is \( \mathcal{I} \)-g open.

Proof

Suppose \( \text{Int} \ A \subseteq B \subseteq A \) and \( A \) is \( \mathcal{I} \)-g open. Then \( X - A \subseteq X - B \subseteq \text{cl}(X - A) \) and \( X - A \) is \( \mathcal{I} \)-g closed. By theorem 5.6, \( X - B \) is \( \mathcal{I} \)-g closed. That is \( B \) is \( \mathcal{I} \)-g open.

Theorem 5.23

A set \( A \) is \( \mathcal{I} \)-g closed in \((X, \tau)\) if and only if \( \text{cl}(A) - A \) is \( \mathcal{I} \)-g open.

Proof

Assume that \( A \) is \( \mathcal{I} \)-g closed. Let \( F \subseteq \text{cl}(A) - A \) and \( F \) be closed. Then \( F \in \mathcal{I} \). This implies \( F - I = \emptyset \), for some \( I \in \mathcal{I} \).

Clearly \( F - I \subseteq \text{Int}(\text{cl}(A) - A) \).

By theorem 5.17, \( \text{cl}(A) - A \) is \( \mathcal{I} \)-g open.

Conversely, assume that \( \text{cl}(A) - A \) is \( \mathcal{I} \)-g open and suppose \( A \subseteq O \) and \( O \) is open. Then

\[
\text{cl}(A) \cap (X - O) \subseteq \text{cl}(A) \cap (X - A) = \text{cl}(A) - A.
\]

By assumption, \([\text{cl}(A) \cap (X - O)] - I \subseteq \text{Int}(\text{cl}(A) - A) = \emptyset\)

\[
\Rightarrow \text{cl}(A) \cap (X - O) \subseteq I \in \mathcal{I}.
\]

\[
\Rightarrow \text{cl}(A) - O \in \mathcal{I}. \text{ Then we have } A \text{ is } \mathcal{I}-\text{g closed.}
\]

Levine [20] proved that the image of a \( g \)-closed set under a continuous closed mapping is \( g \)-closed. We generalize this in the following theorem for \( \mathcal{I} \)-g closed sets.
Theorem 5.24

Let $f : X \to Y$ be continuous and closed. If $A \subseteq X$ is $\mathcal{I}$-g closed in $X$, then $f(A)$ is $f(\mathcal{I})$-g closed in $Y$.

Proof

Suppose $A \subseteq X$ and $A$ is $\mathcal{I}$-g closed. Suppose $f(A) \subseteq O$ and $O$ is open in $Y$. Then $A \subseteq f^{-1}(O)$. By definition of $\mathcal{I}$-g closed sets,

$$\text{cl}(A) - f^{-1}(O) \in \mathcal{I} \Rightarrow f(\text{cl}(A)) - O \in f(\mathcal{I}).$$

But $\text{cl}(f(A)) \subseteq f(\text{cl}(A)) = f(\text{cl}(A))$, because $f$ is closed.

So $\text{cl}(f(A)) - O \subseteq f(\text{cl}(A)) - O \in f(\mathcal{I})$. This proves that $f(A)$ is $f(\mathcal{I})$-g closed.

A subset $A$ of a topological space $(X, \tau)$ is locally closed if it is the intersection of an open set and a closed set [2]. By using the concept of a locally closed set, Balachandran, Sundram and Maki [17] introduced generalized locally closed set and GLC- continuous functions and discussed properties of them. We introduce and investigate the concept of generalized locally closed sets with respect to an ideal.

Definition 5.25

Let $(X, \tau)$ be a topological space and let $\mathcal{I}$ be an ideal on $X$. A subset $S$ of $(X, \tau)$ is said to be generalized locally closed with respect to an ideal (simply we write $\mathcal{I}$-g locally closed ) if there exist an $\mathcal{I}$-g open set $G$ and $\mathcal{I}$-g closed set $F$ such that $S = G \cap F$.

The collection of all $\mathcal{I}$-g locally closed sets will be denoted by $\mathcal{I}\text{GLC}(X, \tau)$. 
Remark

A subset $S$ of $(X, \tau)$ is $\mathcal{I}$-g locally closed if its compliment $X-S$ is a union of a $\mathcal{I}$-g closed set and a $\mathcal{I}$-g open set.

The following two collections of subset of $(X, \tau)$ i.e., $\mathcal{I}$GLC$^*$$(X, \tau)$ and $\mathcal{I}$GLC$^**$(X, \tau), are defined as follows:

Definition 5.26

For a subset $S$ of $(X, \tau)$, $S \in \mathcal{I}$GLC$^*$$(X, \tau)$ if there exist a $\mathcal{I}$-g open set $G$ and closed set $F$ of $(X, \tau)$ respectively, such that $S = G \cap F$.

Definition: 5.27

For a subset $S$ of $(X, \tau)$, $S \in \mathcal{I}$GLC$^**$(X, \tau) if there exist open set $G$ and a $\mathcal{I}$-g closed set $F$ of $(X, \tau)$ respectively, such that $S = G \cap F$.

Now we state the following lemma. The proofs are omitted.

Lemma 5.28

Let $S$ be a subset of $(X, \tau)$

(i) If $S$ is locally closed, then $S \in \mathcal{I}$GLC$^*$$(X, \tau)$ and $S \in \mathcal{I}$GLC$^**$(X, \tau).

(ii) If $S \in \mathcal{I}$GLC$^*$$(X, \tau)$ or $S \in \mathcal{I}$GLC$^**$(X, \tau), then $S$ is $\mathcal{I}$-g locally closed.

Example 5.29

Let $R$ be the real line with the usual topology. If $\mathcal{I}$ is the ideal of all finite subsets of $R$ and $S = (0, 1]$ then $S$ is $\mathcal{I}$-g locally closed but not locally closed.
The following theorem is a characterization of $\mathcal{I}GLC^*(X, \tau)$.

**Theorem 5.30**

Let $(X, \tau)$ be a topological space and let $\mathcal{I}$ be an ideal on $X$. Then for any subset $S$ of $(X, \tau)$, the following are equivalent.

(i) $S \in \mathcal{I}GLC^*(X, \tau)$.

(ii) $S = P \cap \text{cl}(S)$, for some $\mathcal{I}$-g open set $P$.

(iii) $\text{cl}(S) - S$ is $\mathcal{I}$-g closed.

(iv) $S \cup (X - \text{cl}(S))$ is $\mathcal{I}$-g open.

**Proof**

(i) $\Rightarrow$ (ii) 

Let $S \in \mathcal{I}GLC^*(X, \tau)$. Then there exist a $\mathcal{I}$-g open set $G$ and a closed set $F$ such that $S = G \cap F$. We have $S \subseteq G \cap \text{cl}(S)$.

Since $S \subseteq F$, $\text{cl}(S) \subseteq F$, because $F$ is closed, so $G \cap \text{cl}(S) \subseteq G \cap F = S$ which implies that $S = G \cap \text{cl}(S)$.

(ii) $\Rightarrow$ (iii): 

Assume that $S = P \cap \text{cl}(S)$, for some $\mathcal{I}$-g open set $P$. Clearly $\text{cl}(S) - S = \text{cl}(S) \cap (X - P)$. Let $(\text{cl}(S) \cap (X - P)) \subseteq G$, for some open set $G$. we have $\text{cl}(\text{cl}(S) \cap (X - P)) - G \subseteq (\text{cl}(S) \cap \text{cl}(X - P)) - G$.

Since $P$ is $\mathcal{I}$-g open and $X - P \subseteq G \cup (X - \text{cl}(S))$, we have

$$\text{cl}(X - P)(G \cup (X - \text{cl}(S))) \in \mathcal{I}.$$

Now $(\text{cl}(S) \cap \text{cl}(X - P)) - G = (\text{cl}(S) \cap \text{cl}(X - P) - (G \cup (X - \text{cl}(S))))$

$$\subseteq \text{cl}(X - P) - (G \cup (X - \text{cl}(S))) \in \mathcal{I}.$$

This implies $\text{cl}(S) \cap (X - P)$ is $\mathcal{I}$-g closed i.e., $\text{cl}(S) - S$ is $\mathcal{I}$-g closed.
Let $F = \text{cl}(S) - S$. Then $F$ is $\mathcal{I}$-g closed by the assumption and $X - F = X - (\text{cl}(S) - S) = (X - \text{cl}(S)) \cup S$ is $\mathcal{I}$-g open.

$\Rightarrow$ (iii): Let $U = S \cup (X - \text{cl}(S))$. Then $U$ is $\mathcal{I}$-g open. This implies that $X - U$ is $\mathcal{I}$-g closed and $X - U = X - (S \cup (X - \text{cl}(S))) = (X - S) \cap \text{cl}(S) = \text{cl}(S) - S$.

Thus $\text{cl}(S) - S$ is $\mathcal{I}$-g closed.

$\Rightarrow$ (ii): Let $U = X - (\text{cl}(S) - S)$ be $\mathcal{I}$-g open. Therefore $U \cap \text{cl}(S) = ((X - \text{cl}(S)) \cup S) \cap \text{cl}(S) = S$. Thus $S = U \cap \text{cl}(S)$, for some $\mathcal{I}$-g open set $U$.

$\Rightarrow$ (i): Since $P$ is $\mathcal{I}$-g open and $\text{cl}(S)$ is closed, we have $S = P \cap \text{cl}(S) \in \mathcal{I}\text{GLC}^*(X, \tau)$.

Ganster and Reilly (5) [proposition 1 (v)] proved that $S$ is locally closed if and only if $S \subset \text{Int}(S \cup (X - \text{cl}(S)))$. But this result is not true for $\mathcal{I}$-locally closed set and hence not true for $\mathcal{I}$-g locally closed set.

Example 5.31
Let $X = [0,1]$ with usual topology and $S = (0,1]$ then $\text{Int}(S \cup (X - \text{cl}(S))) = (0,1) \not\subset S$ and $S \in \mathcal{I}\text{GLC}^*(X, \tau)$, where $\mathcal{I}$ is the ideal of finite subsets of $X$. 

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**Definition 5.32**

A topological space \((X, \tau)\) is called \(\mathcal{I}\)-g submaximal if every dense subset is \(\mathcal{I}\)-g open.

**Example 5.33**

Let \(X\) be the uncountable set and let \(\tau\) be the cocountable topology in \(X\). Let \(\mathcal{I}\) be the ideal of all countable subsets of \(X\). Let \(A\) be uncountable set and \(F \subseteq A\), where \(F\) is closed. If \(A \neq X\), \(F\) is countable and \(F \in \mathcal{I}\). Clearly \(\overline{A} = X\) and \(F \cap (X - \overline{A}) \in \mathcal{I}\). Hence \(X\) is \(\mathcal{I}\)-g submaximal.

We now give the necessary and sufficient condition for a topological space to be \(\mathcal{I}\)-g submaximal.

**Theorem 5.34**

A topological space \((X, \tau)\) is \(\mathcal{I}\)-g submaximal if and only if \(\wp(X) = \mathcal{I}\text{GLC}^*(X, \tau)\).

**Proof**

**Necessity:** Let \(S \in \wp(X)\) and let \(V = S \cup (X - \overline{S})\). Then \(\overline{V} = \overline{S \cup (X - \overline{S})} = X\). This implies that \(V\) is dense subset of \(X\): But by assumption, \(V\) is \(\mathcal{I}\)-g open. By the above theorem 5.30, \(S \in \mathcal{I}\text{GLC}^*(X, \tau)\). Therefore \(\wp(X) = \mathcal{I}\text{GLC}^*(X, \tau)\).

**Sufficiency:** Let \(S\) be a dense subset of \((X, \tau)\). As \(\mathcal{I}\text{GLC}^*(X, \tau) = \wp(X)\), \(S \cup (X - \overline{S})\) is \(\mathcal{I}\)-g open. But \(S = S \cup (X - \overline{S})\) as \(\overline{S} = X\). i.e., every dense subset is \(\mathcal{I}\)-g open. This proves that \(X\) is \(\mathcal{I}\)-g submaximal.
Theorem 5.35

Let A and B be subset of \((X, \tau)\).

(i) If \(A, B \in \mathcal{GLC}'(X, \tau)\), then \(A \cap B \in \mathcal{GLC}'(X, \tau)\).

(ii) If \(A \in \mathcal{GLC}''(X, \tau)\) and B is open or closed, then \(A \cap B \in \mathcal{GLC}''(X, \tau)\).

(iii) If \(A \in \mathcal{GLC}(X, \tau)\) and B is \(\mathcal{S}\)-open or closed, then \(A \cap B \in \mathcal{GLC}(X, \tau)\).

(iv) If \(A \in \mathcal{GLC}'(X, \tau)\) and B is \(\mathcal{S}\)-closed, then \(A \cap B \in \mathcal{GLC}(X, \tau)\).

Proof

(i) Let \(A \text{ and } B \in \mathcal{GLC}'(X, \tau)\). Then there exist \(\mathcal{S}\)-open sets \(P\) and \(Q\) such that \(A = P \cap \text{cl}(A)\) and \(B = Q \cap \text{cl}(B)\). Therefore \(A \cap B = (P \cap \text{cl}(A)) \cap (Q \cap \text{cl}(B)) = (P \cap Q) \cap (\text{cl}(A) \cap \text{cl}(B))\),

where \(P \cap Q\) is \(\mathcal{S}\)-open and \(\text{cl}(A) \cap \text{cl}(B)\) is closed. This shows that \(A \cap B \in \mathcal{GLC}'(X, \tau)\).

(ii) Suppose B is open. If \(A \in \mathcal{GLC}''(X, \tau)\), then there exist an open set \(G\) and a \(\mathcal{S}\)-closed set \(F\) such that \(A = G \cap F\). So \(A \cap B = (G \cap F) \cap B = (G \cap B) \cap F\), where \(G \cap B\) is open and \(F\) is \(\mathcal{S}\)-closed. This implies that \(A \cap B \in \mathcal{GLC}''(X, \tau)\). Next suppose that B is closed. Then \(A \cap B = (G \cap F) \cap B = G \cap (F \cap B)\), where \(F \cap B\) is \(\mathcal{S}\)-closed. Hence \(A \cap B \in \mathcal{GLC}''(X, \tau)\).
(iii) Suppose \( B \) is \( \mathcal{I}_g \)-open. Let \( A \in \mathcal{J}_GL(X, \tau) \). Then \( A = G \cap F \), where 
\( G \) is \( \mathcal{I}_g \)-open and \( F \) is \( \mathcal{I}_g \)-closed. So \( A \cap B = (G \cap F) \cap B = (G \cap B) \cap F \), where 
\( G \cap B \) is \( \mathcal{I}_g \)-open and \( F \) is \( \mathcal{I}_g \)-closed. This proves that \( A \cap B \in \mathcal{J}_GL(X, \tau) \).

Next suppose that \( B \) is closed. Then \( A \cap B = (G \cap F) \cap B = G \cap (F \cap B) \), where \( F \cap B \) is \( \mathcal{I}_g \)-closed. Hence \( A \cap B \in \mathcal{J}_GL(X, \tau) \).

(iv) Let \( A \in \mathcal{J}_GLC^*(X, \tau) \). Then \( A = G \cap F \) where \( G \) is \( \mathcal{I}_g \)-open and \( F \) is closed. Therefore \( A \cap B = (G \cap F) \cap B = G \cap (F \cap B) \), where \( G \cap B \) is \( \mathcal{I}_g \)-open and \( F \cap B \) is \( \mathcal{I}_g \)-closed. Hence \( A \cap B \in \mathcal{J}_GLC(X, \tau) \).

**Theorem 5.36**

Let \( A \) and \( Z \) be subsets of \((X, \tau)\) and let \( A \subseteq Z \). If \( Z \) is \( \mathcal{I}_g \)-open in \((X, \tau)\) and \( A \in \mathcal{J}_GLC^*(Z, \tau/Z) \), then \( A \in \mathcal{J}_GLC^*(X, \tau) \).

**Proof**

Let \( A \in \mathcal{J}_GLC^*(Z, \tau/Z) \). Then there exist \( \mathcal{I}_g \)-open set \( G \) of \((Z, \tau/Z)\) such that \( A = G \cap cl_{Z}(A) \). But \( cl_{Z}(A) = Z \cap cl(A) \). Therefore \( A = G \cap Z \cap cl(A) \), where \( G \cap Z \) is \( \mathcal{I}_g \)-open. Thus \( A \in \mathcal{J}_GLC^*(X, \tau) \).

**Theorem 5.37**

Assume that the collection of all \( \mathcal{I}_g \)-open subsets of \((X, \tau)\) is closed under finite unions. Let \( A \in \mathcal{J}_GLC^*(X, \tau) \) and \( B \in \mathcal{J}_GLC^*(X, \tau) \). If \( A \) and \( B \) are separated i.e., \( A \cap cl(B) = \emptyset \) and \( cl(A) \cap B = \emptyset \), then \( A \cup B \in \mathcal{J}_GLC^*(X, \tau) \).

**Proof**

Let \( A \) and \( B \in \mathcal{J}_GLC^*(X, \tau) \). Then there exist \( \mathcal{I}_g \)-open sets \( G \) and \( S \) such that \( A = G \cap cl(A) \) and \( B = S \cap cl(B) \). Put \( U = G \cap (X - cl(B)) \) and \( V = S \cap (X - cl(A)) \). Then \( U \) and \( V \) are \( \mathcal{I}_g \)-open sets and \( A = U \cup cl(A) \) and
B = V \cap \text{cl}(B). Also U \cap \text{cl}(B) = \emptyset \text{ and } V \cap \text{cl}(A) = \emptyset. Therefore A \cup B = (U \cap \text{cl}(A)) \cup (V \cap \text{cl}(B)) = (U \cup V) \cap (\text{cl}(A) \cup \text{cl}(B)). Here U \cup V \text{ is 3-g open by assumption. Thus } A \cup B \in \mathcal{J}\text{-GLC}^*(X, \tau).

We conclude this chapter by bringing the following theorem.

**Theorem 5.38**

Let \{Z_i : i \in \Lambda\} be an finite \(\mathcal{J}\)-g open cover of topological space \((X, \tau)\) and let \(A \subseteq X\). If \(A \cap Z_i \in \mathcal{J}\text{GLC}^*(Z_i, \tau/Z_i)\), for each \(i \in \Lambda\), then \(A \in \mathcal{J}\text{GLC}^*(X, \tau)\).

**Proof**

Suppose \{Z_i : i \in \Lambda\} be an finite \(\mathcal{J}\)-g open cover of \((X, \tau)\), i.e., \(X = \bigcup\{Z_i : i \in \Lambda\}\). Let \(A \cap Z_i \in \mathcal{J}\text{GLC}^*(Z_i, \tau/Z_i)\). Then \(A \cap Z_i = V_i \cap \text{cl}(A \cap Z_i)\), where \(V_i\) is \(\mathcal{J}\)-g open in \(Z_i\) and \(V_i \cap Z_i = V_i\). Therefore \(V_i \cap \text{cl}(A) = V_i \cap Z_i \cap \text{cl}(A) \subseteq V_i \cap \text{cl}(A \cap Z_i) = A \cap Z_i\). If \(V = \bigcup\{V_i : i \in \Lambda\}\), then \(V \cap \text{cl}(A) = A\). This shows that \(A \in \mathcal{J}\text{GLC}^*(X, \tau)\).