Sember [44] introduced the concept of semiconservative spaces. These spaces pay a special role in the theory of FK spaces, summability and distinguished spaces. An FK space $X$ ($X \supset \phi$) is called semiconservative if $cs \supset X^f$. We note that $cs = c_S$ with $S$ is an infinite matrix $(s_{nk})$ given by $s_{nk} = 1$, if $k \leq n$; 0, otherwise ($n, k = 1,2,...$). We can replace the matrix $S$ by the Cesàro matrix resulting semi Cesàro-conservative spaces. To extend the idea, $c$ can be replaced by $t_\infty$ (resulting sr spaces) or even by a general $\mu$. This is done in this chapter. The following are also proved in this chapter:

(i) The smallest sr space is $h$.

(ii) If $\lambda^B$ has AK property, then $\lambda$ is $B$-perfect if and only if $\lambda^B$ is the smallest sr($\mu$) space where $\mu^f = \lambda$.

Further, relation with multiplier spaces and sr($\mu$) spaces are examined and an attempt is made on the Wilansky's conjecture.
SEMIREPLETE SPACES

For $\omega \subset X$, let

$$\sigma(X) = \{x \in \omega : \{1/n (x_1 + \ldots + x_n)\} \in X\}$$

Throughout this chapter $X$ is an FK space containing $\emptyset$.

3.1 $X$ is said to be a semi Cesàro-conservative (or simply scc-) space if

$$\sigma(c) \supset X^f$$

3.2 EXAMPLE

$cs$, the space of all convergent series is a scc - space.

3.3 REMARKS

(i) Every sc-space is a scc space.

(ii) A scc space need not be sc. For instance, $cs$ is not sc.

3.4 DEFINITION

$X$ is said to be a semi replete (or, sr) space if $\sigma(\ell_\infty) \supset X^f$.

3.5 REMARKS

(i) Suppose $X$ is replete with $h$, that is, $X \supset h$, then $h^f \supset X^f$. But, $h^f = \sigma(\ell_\infty)$ because $h$ has AK (see [9]) $h^\Omega = h^f$ and $h^\Omega = \sigma(\ell_\infty)$ (see [9]). Hence, we justify the prefix 'semi'.

(ii) Every scc space is a sr space.
(iii) A sr space need not be a scs space. For example, consider the space bv. Since, bv* = bs, bv is a sr space. But, bv is not a scs space. Yet another example (in which a sr space) which is not a scs space is h the Hahn sequence space.

3.6  **LEMMA**

Every sr space contains h.

**Proof**

Let X be sr space. Then

\[ \sigma(l_\infty) \supset X^f. \]

But,

\[ \sigma(l_\infty) = h^f, \]

and so,

\[ h^f \supset X^f. \]

Since h has AK, it has AD. Therefore, by a theorem of Wilansky (see [49, Theo.8.6.1]), X ⊃ h. Thus every sr space is replete with h. This completes the proof of the lemma.

3.7  **COROLLARY**

Every scs space contains h.

3.8  **LEMMA**

\[ z \in \sigma(l_\infty) \text{ if and only if } z^h = \{ u \in W : u(z) \in c_s \}. \]

**Proof**

Suppose \( z \in \sigma(l_\infty) \). Then \( z^h \supset h \). But \( h = (\sigma(l_\infty))^h \) (see [9]), so that

\[ \sigma(l_\infty) = h^h \supset z^h. \]
But, $z^6$ has AK under the sequence of seminorms $p = \{p_n : n = 0,1,\ldots\}$ given by

$$p_0(x) = \|zx\|_{cs}(x \in z^6),$$

$$p_n(x) = |x_n|(x \in z^6) \ (n = 1,2,\ldots)$$

where $\|\cdot\|_{cs}$ is the norm in the space $cs$. Hence,

$$\sigma(l_\infty) \supset z^6.\]

Therefore, $z^6$ is a sr space. Conversely, suppose $z^6$ is a sr space. Then $\sigma(l_\infty) \supset z^6$.

It is known (see [49, Theo.4.3.7]) that $z^6$ is always AK. Therefore,

$$\sigma(l_\infty) \supset z_6.\]

But, $z \in z^6$ and hence $z \in \sigma(l_\infty)$. This completes the proof.

### 3.9 THEOREM

The smallest sr space is $h$.

**Proof**

From Lemma 3.6, we have that every sr space contains $h$. Therefore the intersection $I$ of all sr spaces must contain $h$, that is,

$$I \supset h. \quad (1)$$

Using Lemma 3.8, we have now

$$\bigcap \{z^6 : z \in \sigma(l_\infty)\} \supset I.$$ 

Hence,

$$\sigma(l_\infty) \supset I$$

or,

$$h \supset I. \quad (2)$$

From (1) and (2) we have $I = h$. This completes the proof of the theorem.

Note that while proving $h \supset I$ we need not use Lemma 3.8. Because, $h$ is an sr space, it must contain the intersection $I$ of all sr spaces.
The cited results can be generalized in a natural way. In this chapter, hereafter, it is assumed that $\mu$ is an FK-space containing $\phi$ and its $f$-dual is denoted by $\lambda$, that is, $\lambda = \mu^f$.

3.14 DEFINITION

An FK-space $X$ containing $\phi$ is called semireplete with $\mu$ if and only if $\lambda \supset X^f$.

We use the term $sr(\mu)$ space for an FK-space which is semireplete with $\mu$.

Obviously, if $Y$ is an FK space containing $X$ where $X$ is a $sr(\mu)$ space, then $Y$ is also a $sr(\mu)$ space.

3.15 PROPOSITION

Countable intersection of $sr(\mu)$ spaces is a $sr(\mu)$ space.

Proof

Let $\{X_n\}$ be a countable collection of $sr(\mu)$ spaces can be written as $f = \sum_{k=1}^{m} g_k$ for some positive integer $m$ where $g_k \in X_n^f$ for some $n$. But, each $X_n$ is a $sr(\mu)$ space. So, $\lambda \supset X_n^f$ and hence $\{g_k(\delta^k)\} \in \lambda (i = 1,\ldots,m)$. Thus $\{f(\delta^k)\} \in \lambda(f \in X^f)$. Therefore, $\lambda \supset X^f$. Consequently, $X$ is semireplete with $\mu$. This completes the proof of the proposition.

Eventhough the countable intersection of $sr(\mu)$ spaces is semireplete with $\mu$, the intersection of all $sr(\mu)$ spaces fails to be a $sr(\mu)$ space (as in the case $\lambda = cs$). Anyhow, in the case of $B$-perfect spaces (viz., $\lambda^{BB} = \lambda$), with the extra hypothesis that $\lambda^B$ is an AK space, the intersection of all $sr(\mu)$ spaces becomes a $sr(\mu)$ space. This is the content of Theorem 3.18.
3.16 LEMMA

Suppose \( \lambda \) is \( \beta \)-perfect. Then \( z \in \lambda \) if and only if \( z^\beta \) is a \( sr(\mu) \) space.

Proof

It is known that for any sequence \( z \) (and consequently in \( \lambda \)), \( z^\beta \) has AK property. Therefore,

\[
z^\beta = z^{\beta \beta}.
\]  

(1)

If \( z \in \lambda \), then \( z^\beta \supset \lambda^\beta \)

and hence \( \lambda^{\beta \beta} \supset z^{\beta \beta} \).

From (1) and from the fact that \( \lambda \) is \( \beta \)-perfect, we have,

\[
\lambda \supset z^{\beta \beta}.
\]

Hence \( z^\beta \) is a \( sr(\mu) \) space.

Conversely, if \( z^\beta \) is a \( sr(\mu) \) space, that is,

\[
\lambda \supset z^\beta,
\]

then by (1), \( \lambda \supset z^{\beta \beta} \).

But, \( z \in z^{\beta \beta} \). Thus, \( z \in \lambda \). This completes the proof of the lemma.

We note that no topology on \( \lambda \) is used in the proof of Lemma 3.16.

3.17 COROLLARY

If \( \lambda \) is a \( \beta \)-perfect space, then the intersection of all spaces which are semireplete with \( \mu \) is contained in \( \lambda^\beta \).

Proof

Let \( I \) be the intersection of all \( sr(\mu) \) spaces. Then by Lemma 3.16,

\[
\lambda^\beta = \bigcap \{z^\beta : z \in \lambda\} \supset I.
\]
This completes the proof of the corollary.

Some $\beta$-perfect spaces like $I$-possess $\beta$-dual without AK property. There are $\beta$-perfect spaces like $\sigma(I_n)$ or like bs, $\beta$-duals do possess AK property. For those spaces belonging to the latter case we have the following theorem.

**3.18 THEOREM**

Suppose $\lambda$ is a $\beta$-perfect space such that $\lambda^\beta$ is an AK space. Then $\lambda^\beta$ is the smallest $sr(\mu)$ space.

**Proof**

It is enough to prove that the intersection $I$ of all $sr(\mu)$ spaces is $\lambda^\beta$ and $\lambda^\beta$ is a $sr(\mu)$ space. From the hypothesis, $\lambda^\beta = \lambda$, we have, by Corollary 3.17, that

$$\lambda^\beta \supset I.$$  \hfill (1)

On the otherhand, let $X$ be any $sr(\mu)$ space, then

$$\lambda \supset X.$$  \hfill (2)

Now, $\lambda^\beta$ has AK property. Therefore,

$$\lambda^\beta = \lambda^\beta.$$  \hfill (3)

From the hypothesis that $\lambda$ is a $\beta$-perfect space, (3) becomes

$$\lambda^\beta = \lambda.$$  \hfill (4)

Therefore (2) becomes,

$$\lambda^\beta \supset X.$$  \hfill (5)

Again, $X^\beta$ has AK. Consequently, $\lambda^\beta$ has AD property also.

Therefore (5) implies

$$X \supset \lambda^\beta.$$  

Thus every $sr(\mu)$ space contains $\lambda^\beta$. Therefore, the intersection $I$ of all $sr(\mu)$ spaces also contains $\lambda^\beta$ that is $I \supset \lambda^\beta$. Therefore, from (1) we have $I = \lambda^\beta$. From (4) we have
that \( \lambda^B \) is a \( sr(\mu) \) space. Thus \( \lambda^B \) is the smallest \( sr(\mu) \) space. This completes the proof of the theorem.

3.19 COROLLARY

Suppose \( \mu \) and \( \lambda^B \) have AK property. Then \( \lambda^B \) is the smallest \( sr(\mu) \) space.

Proof

If \( \mu \) has AK. Then it is known that \( \mu^I = \mu^B \). Now, \( \lambda^B = (\mu^I)^B = \mu^B = \lambda \).

Thus \( \lambda \) is \( B \)-perfect. To get the required result, Theorem 3.18 is applied.

3.20 COROLLARY

Suppose \( \mu \) has AK and \( \beta \)-perfect. Then \( \mu \) is the smallest \( sr(\mu) \) space.

This corollary can be proved easily.

Under some conditions, the converse of Corollary 3.17 is true. In fact we formulate and prove

3.21 PROPOSITION

If the countable intersection \( I \) of all \( sr(\mu) \) spaces is also a \( sr(\mu) \) space such that \( \lambda^B \supset I \) and \( \mu^I \supset I^B \) then \( \lambda \) is \( B \)-perfect.

Proof

We note that, by Theorem 4.2.15 (see [49]) \( I \) is an FK space containing \( \phi \).

From the hypothesis

\[ \lambda^B \supset I \text{ and } \mu^I \supset I^B, \]
we have,
\[ I^f \subseteq I^B \subseteq \lambda^{is}. \] (1)

Again from the hypothesis that \( I \) is \( sr(\mu) \),
\[ \lambda \subseteq I^f. \] (2)

Also, it is always
\[ \lambda^{is} \subseteq \lambda. \] (3)

From (1), (2) and (3), we see that \( \lambda \) is \( B \)-perfect. This completes the proof of the proposition.

3.22 COROLLARY

Suppose \( \lambda^B \) has \( AK \). Then \( \lambda \) is \( B \)-perfect if and only if \( \lambda^B \) is the smallest \( sr(\mu) \) space.

3.23 PROPOSITION

Suppose \( \mu \) has \( AD \) property and \( X \), a \( BK \)-space containing \( \phi \). Then \( X \supseteq \mu X \) if and only if \( M(X) \) is a \( sr(\mu) \) space.

Proof

The proof follows from the observation that \( X \supseteq \mu X \) if and only if \( M(X) \supseteq \mu \) (cf. Lemma 2.4); \( M(X) \supseteq \mu \) if and only if \( \lambda \supseteq M(X)^f \).

3.24 COROLLARY

Suppose \( X \) is a \( BK \) space containing \( \phi \). \( X \supseteq \mu X \) if and only if \( M(X) \) is a \( sr \) space.
Proof

The proof is due to the above proposition because \( h \) has AD property.

There are some interesting applications of sr(\( \mu \)) spaces in summability theory.

**NOTATION**

Let \( A = (a_{nk}) \ (n, k = 1, 2, \ldots) \) be an infinite matrix whose elements are from \( K \).

For \( x \in \omega \), let \( Ax = \{\Sigma_k a_{nk} \ : \ n = k = 1, 2, \ldots\} \), formally \( \Sigma_k a_{nk} x_k \) is defined for each \( n = 1, 2, \ldots \). Let \( \omega \supset X \). The matrix domain of \( A \) (with respect to \( X \)) is given by

\[
X_A = \{x \in \omega : Ax \in X\}.
\]

Let \( A[X] = \{Ax : x \in X\} \).

Being \( Ax \) a sequence, we can treat \( A \) as a linear mapping between sequence spaces. Let \( (X : Y) \) denote the class of all (infinite) matrix mappings each maps \( X \) into \( Y \), that is, if \( A \in (X : Y) \), then \( Ax \in Y \) for every \( x \in X \). Note also that \( A \in (X : Y) \) if and only if \( Y_A \supset X \). We start with

**3.25 PROPOSITION**

If there is an FK space \( X \) such that \( X_A \) is sr(\( \mu \)) and if \( \lambda \) is \( \beta \)-perfect then the rows of \( A \) lie in \( \lambda \) where \( A = (a_{nk}) \ (n, k = 1, 2, \ldots) \).

Proof

Let \( r \) be an arbitrary row of \( A \). Then

\[
r^\beta \supset \omega_A \supset X_A.
\]

Here \( \omega_A = \{x \in \omega : Ax = \{\Sigma_k a_{nk} x_k \in \omega\} \} \).

But \( X_A \) is sr(\( \mu \)). Consequently, \( r^\beta \) is sr(\( \mu \)). To get the required result, Lemma 3.16 is applied.
3.26 LEMMA

The following are equivalent:

(i) $X_A$ is a sr($\mu$) space whenever $A \in (X:X)$. \hfill (S_1)

(ii) $X$ is a sr($\mu$) space. \hfill (S_2)

Proof

Suppose (i) holds: Set $A = I$. Then obviously, $I \in (X:X)$ and hence, $X_I$ is sr($\mu$) which is (ii). Suppose (ii) holds and let $A \in (X:X)$. Then $X_A \supset X$. By our present hypothesis $X$ is sr($\mu$). Consequently, $X_A$ is sr($\mu$). This completes the lemma.

3.27 LEMMA

Let $\lambda$ be a $\beta$-perfect space with $\lambda^\beta$ having AK property. Suppose $X$ is an FK space satisfying

"$A \in (X:X)$ whenever $X_A$ is a sr($\mu$) space". \hfill (S_3)

Then $\lambda^\beta \supset X$. \hfill (S_4)

Proof

If possible $\lambda^\beta \not\supset X$. Then $X^\beta \not\supset \lambda$ (for, $X^\beta \supset \lambda$ implies $\lambda^\beta \supset X^\beta \supset X$ and always it is true that $X^\beta \supset X$). Let $u \in \lambda \setminus X^\beta$ and let $0 \neq v \in X$. Put

$$a_{nk} = v_n u_k \ (n, k = 1, 2, \ldots).$$

Then $X_A = \omega_A = u^\beta$. By Lemma 3.16, $u^\beta$ is sr($\mu$). Hence $X_A$ is a sr($\mu$) space. But $A \not\in (X:X)$ (since, $\omega_A \not\in X$). This completes the proof of the lemma.
Wilansky conjectured that the converse of the analogous result for $sc$ spaces is true cf [49, p.151]). But the conjecture has not been decided. However the converse of Lemma 3.27 holds if $X = \lambda^B$ as seen below:

3.28 THEOREM

Suppose $\lambda$ is a $B$-perfect space. Suppose its $B$-dual has AK property. Then $\lambda^B$ is the only space such that $X = \lambda^B$ with

$$X_A \text{ is sr(\mu) if and only if } A \in (X:X).$$

(S$5$)

Proof

First we shall show that $\lambda^B$ satisfies (S$5$). By Lemma 3.27, S$3$ implies S$4$. We shall prove S$5$ implies S$3$ for the case $X = \lambda^B$. Suppose $\lambda^B A$ is sr(\mu), then by Theorem 3.18, $\lambda^B A \supset \lambda^B$. Thus $A \in (\lambda^B : \lambda^B) = (X:X)$. Therefore, under the present hypothesis converse of Lemma 3.27 is also true and consequently the following are equivalent, by Lemma 3.26

"$X_A$ is sr(\mu) space whenever $A \in (X:X)$", 

(S$1$)

"$X$ is a sr(\mu) space". 

(S$2$)

"$A \in (X:X)$ whenever $X_A$ is a sr(\mu) space". 

(S$3$)

"$\lambda^B = X$". 

(S$4$)

Or in otherwords,

"$\lambda^B_A$ is a sr(\mu) space whenever $A \in (\lambda^B : \lambda^B)$" if and only if

"$\lambda^B_A$ is sr(\mu) if and only if $A \in (\lambda^B : \lambda^B)$" satisfying S$5$ in the case of $X = \lambda^B$

Secondly, it remains to prove that the only space which sr(\mu) satisfying S$5$ is $\lambda^B$. If $X$ were an FK space containing $\phi$ such that
\[ X \uplus \lambda^\delta \] (1)
satisfying \( S_5 \), then
\[ A \in \epsilon(X:X) \text{ whenever } X_A \text{ is } \text{sr}(\mu). \]

Therefore by Lemma 3.27, we have
\[ \lambda^\delta \subseteq X. \] (2)

But by Theorem 3.18, we have
\[ X \supseteq \lambda^\delta. \] (3)

From (2) and (3) \( X = \lambda^\delta \), contradicting (1). This completes the proof of the theorem.

\( \Box \)