CHAPTER 1

DIAGONAL MAPPINGS OF SEQUENCE SPACES

COMPENDIUM

Section : Diagonal mappings of sequence spaces

In this section the properties of $X_\lambda$ in relation to the properties of $X$ have been studied where $X$ is a sequence space, $\lambda$ is a fixed sequence in $\mathbb{K}$ and $X_\lambda = \{x \in \omega : \{\lambda_n x_n\} \in X\}$. The "integrated" space $fX$ and the "differentiated" space $dX$ of S. Goes and G. Goes (see [21]), the space $\Gamma_2(\lambda)$ of Titus [46] and the space $X$ of Kamthan become special cases of this $X_\lambda$. In this connection the author is particularly indebted to Prof. W. Ruckle who has gone through the manuscript and given valuable suggestions and observations.

Section : Difference sequence Spaces

Difference Sequence Spaces have already been used (example, by, the space of all sequences of bounded variation) without being called so. The difference space of a given sequence space $X$ is the space $\{x \in \omega : \{x_n - x_{n+1}\} \in X\}$ and is denoted, in our work, by $X_\Delta$. To study the properties of a difference sequence space, an easy way is to view it as matrix domain of an infinite matrix. This is done in this section and this enables us to prove (by just applying a result of Wilansky [49]) that $X_\Delta$ is a BK space whenever $X$ is so. Define $X_{\Delta \Delta} = (X_\Delta)_\Delta = \{x \in \omega : \{\lambda_n(x_n - x_{n+1})\} \in X\}$. It is proved that if $E$ is a determining set for $X$, then $E_{\Delta \Delta}$ is a determining set for $SX_{\Delta \Delta}$ where $X$ is a BK space and $SX_{\Delta \Delta} = X_{\Delta \Delta} \cap c_0$. 

$\varnothing$
Section: Diagonal Mappings of Sequence Spaces

1.1 Let $\omega$ denote the set of all sequences in $K$ where $K$ may either the field of real or complex numbers. Symbolically,

$$\omega = \{x : x = \{x_n : x_n \in K \ (n = 1, 2, \ldots)\}\}.$$  

A sequence space (also known as a coordinate space, or a $K$-space) is a subset of $\omega$ which is also a linear space (or, a vector space) over $K$ under the coordinatewise addition and the scalar multiplication. Obviously, $\omega$ itself is a sequence space. Let $\phi$ be the linear space of all finite sequences in $K$, that is,

$$\phi = \{x \in \omega : x_n = 0 \text{ for all } n \geq n_0 \text{ for some positive integer } n_0\}.$$  

In the sequel $K$ is equipped with its usual metric topology induced by the 'modulus'.

Further, $\lim_{n \to \infty} x_n$, $\sup \{x_n \}$ and $\inf \{x_n \}$ are indicated respectively by

$$\lim_{n \to \infty} x_n, \sup_n \{x_n \} \text{ and } \inf_n \{x_n \}.$$  

Also, the infinite sum

$$\sum_{n=1}^{\infty} x_n$$  

is written as $\Sigma_n x_n$, or simply as $\Sigma x_n$.

The following are some more examples of sequence spaces:

$$\Gamma = \{x \in \omega : \lim_n |x_n|^{1/n} = 0\}.$$  

$$\ell_p = \{x \in \omega : \Sigma_n |x_n|^p \text{ is convergent}, (1 \leq p < \infty)\}.$$  

as $p \to \infty$.

$$cs = \{x \in \omega : \Sigma_n (-1)^n x_n \text{ is convergent}\}.$$  

$$c_0 = \{x \in \omega : \lim_n x_n = 0\}.$$
\[ b_s = \{ x_{\omega} : \sup_n | \sum_{i=1}^{n} x_i | \text{ exists} \} \]

\[ c = \{ x_{\omega} : \lim_n x_n \text{ exists} \} \]

\[ l_\infty = \{ x_{\omega} : \sup_n | x_n | \text{ exists} \} \]

\[ \Lambda = \{ x_{\omega} : \sup_n | x_n |^{1/n} \text{ exists} \} \]

\[ m_0 = \text{Span} [A] \text{ where } A = \{ x_{\omega} : x_n = 0 \text{ or } 1 \text{ for all } n \} \]

\[ b_v = \{ x_{\omega} : \sum_n | x_n - x_{n+1} | \text{ is convergent} \} \]

1.2 For an arbitrarily fixed \( \lambda x_{\omega} \), we define

\[ X_\lambda = \{ x_{\omega} : \lambda x_{\omega} \} \]

where \( \omega \subseteq X \) and \( \lambda x = \{ \lambda_n x_n \} \). Suppose \( X \) happens to be a sequence space. Then \( X_\lambda \) is also a sequence space. Sometimes the space \( X_\lambda \) is also denoted by \( X(\lambda) \), for example, see Titus [46, p.360]. The notation \( X_\lambda \) is preferred due to the following reason:

Given an infinite matrix \( A = (a_{nk}) \ (n, k = 1, 2, \ldots) \) of scalars and a sequence space \( X \), the matrix domain of \( A \) (see, for example, [49, p.3]) is given by

\[ X_A = \{ x_{\omega} : Ax_{\omega} \} \]

where \( Ax = \sum_k a_{nk} x_k \). As a special case if we consider the diagonal (infinite) matrix \( A = (a_{nk}) \ (n, k = 1, 2, \ldots) \) formed by a given sequence \( \lambda \), that is,

\[ a_{nk} = \begin{cases} 
\lambda_n, & \text{if } n = k; \\
0, & \text{otherwise},
\end{cases} \]

for \( n, k = 1, 2, \ldots, \) then the matrix transform \( Ax \) is the same as the coordinatewise product \( \lambda x = (\lambda_n x_n) \) of the sequences \( \lambda \) and \( x \). In order to specify the sequence \( \lambda \)
we denote the infinite matrix $A$ by $\text{diag } \lambda$. Hence the matrix domain of $\lambda$ is $X_{\text{diag } \lambda}$ or simply $X_\lambda$.

In the sequel, $\omega \supset X$, $\omega \supset Y$, $\lambda \in \omega$ are arbitrarily fixed. We denote the sequence $\{1/\lambda_n\}$ by $\lambda^{-1}$ provided $\lambda_n \neq 0$ for all $n = 1, 2, \ldots$. Let $\lambda(X) = \{\lambda x : x \in X, x = \{x_n\}\}$ and let $XY = \{xy : x \in X \text{ and } y \in Y\}$. The following results can easily be verified:

1.3 For $\lambda \in \omega$ with $\lambda_n \neq 0$ for all $n$,

(i) $X_\lambda = \lambda^{-1}(X)$,

(ii) $X \supset X_\lambda$ if and only if $X_{\lambda^{-1}} \supset X$.

1.4 EXAMPLES

Several interesting examples are available of which a few are given below:

(i) Obviously, $as = cs\lambda$ where $\lambda = \{(-1)^n\}$. For further properties of $as$ and $cs$ one can refer, for example [27, p.048].

(ii) The spaces

\[ \Gamma(\lambda) = \{xe \Gamma : \lambda xe \Gamma\} \]

of Titus (see [46, p.360]) will become $\Gamma \cap \Gamma_\lambda$, by our notation.

(iii) The space $\chi$ is defined as

\[ \chi = \{xe\omega : \lim_n [n! \mid x_n \mid]^{1/n} = 0\}. \]

Clearly, $\chi = \Gamma_\lambda$ with $\lambda = \{n!\}$.

It is known that $\Gamma \supset \chi$. So, it is natural to ask under what conditions $X_\lambda$ is contained in $X$, or vice-versa.
1.5 **DEFINITION**

A sequence space $X$ is said to be **solid** if $y \in X$ whenever $y \in \omega$ with

$$|y_n| \leq |x_n| \quad (n = 1,2,...)$$

for some $x \in X$.

Some authors refer to a solid space as a **normal space** (see [27, p.48] and also see [13, p.278]). The following result (see [27, p.49]) characterizes solid spaces:

$X$ is a solid space if and only if $X \supset \ell_\infty$.

For $x \in \omega$ and $t \in \omega$ with $t_n = 0$ for all $n$, the notation $x_n = O(t_n)$ or, $x = O(t)$ is to mean $xt^{-1}$ is bounded.

1.6 **PROPOSITION**

Let $X$ be a solid sequence space and let $\lambda_n \neq 0$ for all $n$.

(i) If $\inf_n |\lambda_n| = \theta > 0$, then $X \supset X_{\lambda}$;

(ii) If $\mu = O(\lambda)$, then $X_{\mu} \supset X_{\lambda}$ where $\mu \in \omega$.

**Proof** (i) Let $x \in X_{\lambda}$. Then $\lambda x \in X$. Now,

$$|\theta x_n| \leq |\lambda_n x_n| \quad (n = 1,2,...).$$

Using solidity of $X$, we have $x \in X$. Thus $X \supset X_{\lambda}$.

(ii) Let $x \in X_{\lambda}$. Then $\lambda x \in X$. Put $a = \sup_n |\mu_n/\lambda_n|$. Now,

$$|\mu_n x_n| \leq |\lambda_n y_n| \quad (n = 1,2,...)$$

where $y_n = ax_n (n = 1,2,...)$ so that $y \in X_{\lambda}$. By the solidity of $X$, $\mu x \in X$. Consequently, $X_{\mu} \supset X_{\lambda}$. This completes the proof of the proposition.
1.7 COROLLARY

Let $X$ be a solid sequence space. If $\lambda$ is a bounded sequence, then $X_\lambda \supset X$.

Proof

In Proposition 1.6 (ii), replace $\mu$ by $\lambda$ and $\lambda$ by $e$ where $e$ is the constant sequence $\{1\}$.

The converse of the above corollary is not true. That is, for a solid space $X$ if $X_\lambda \supset X$, then $\lambda$ need not be bounded. As an example, consider the subspace $X = O_2 \cap \ell_\infty$ where $O_2 = \{x_{2n} = 0, n = 1,2,...\}$ (see [13, p.274]). Let

$$
\lambda_k = \begin{cases} 
1, & \text{if } k = 2n-1, \quad (n = 1,2,...) \\
n, & \text{if } k = 2n, \quad (n = 1,2,...)
\end{cases}
$$

for $k = 1,2,...$. Then $\lambda$ is unbounded and $X_\lambda = X$.

1.8 COROLLARY

$\Gamma \supset X$.

Proof

$\Gamma$ is solid. Further, \( \inf_n \left| \lambda_n \right| = \inf_n \left| n! \right| = 1 > 0 \) and apply Proposition 1.6.(i).

In the cited results, the solidity of $X$ is essential. To illustrate this, consider the space $c$ which is not solid. Let $\lambda = e$ and $\mu = \{(1)^n\}$. Then $c_\lambda = c$. But, $\{(1)^n\} \in c \setminus c_\mu$ and $e \in c \setminus c_\mu$. 
1.9 DEFINITIONS

A sequence space $X$ is said to be **convergence free** when, if $x \in X$, and if $y_n = 0$ whenever $x_n = 0$, then $y \in X$.

Thus, $\omega, \phi, 0_1 = \{x \in \omega: x_{2n+1} = 0 (n = 1,2,...)\}$ are convergence-free. For more details see [13, pp.280-282]. For a linear space $X$ over $K$, let $X^\#$ be the set of all linear functions on $X$. Suppose $\tau$ is a Hausdorff topology on a vector space $X$ such that the vector operations are continuous with respect to $\tau$. Then $X$ (together with $\tau$) is said to be a **topological vector space**. For a topological vector space $X$, let $X'$ denote the set of all continuous linear functionals on $X$. An $F$-space is a topological vector space in which the Hausdorff topology $\tau$ is induced by a complete invariant metric $d$. A locally convex $F$-space is known as a Fréchet space (see [48,p.13]). An $FK$ (= Fréchet koordinat) space is a Fréchet space which is a sequence space having the property that the coordinate functionals given by $f_n(x) = x_n (n = 1,2,...)$ are continuous.

A sequence $\{b_n\}$ in a topological vector space $X$ is called a basis for $X$ if each $x$ in $X$ can be uniquely expressed as

$$x = \tau - \lim_n \sum_{i=1}^n \alpha_i b_i$$

where $\{\alpha_n\}$ is the sequence of associated coordinate (linear) functionals (= s.a.c.f.). If $\alpha_n = f_n(x) (n = 1,2,...)$ where each $f_n$ is a continuous linear functional, then $\{b_n\}$ is called a **Schauder base**. To emphasize the s.a.c.f. attached to a Schauder base $\{x_n\}$, it is unambiguously, sometimes, written as $\{x_n, f_n\}$. We shall call a Schauder base, simply a basis.
An FK-space $X$ is said to have AK, or be an AK-space if $X \supset \phi$ and $\{\delta^n = (0, \ldots, 0, 1, 0, \ldots) \text{ where } 1 \text{ occurs in the } n\text{-th place and zero elsewhere } (n = 1, 2, \ldots)\}$ is a basis for $X$. Thus, if $x^n = (x_1, x_2, \ldots, x_n, 0, \ldots)$ ($n = 1, 2, \ldots$), then in the case of AK-space $X$, $x^n \to x$ as $n \to \infty$, otherwise expressed $x = \sum x_n \delta^n$.

An FK-space $X$ is said to have AD, or be an AD-space if $\phi$ is dense in $X$. Initials come from Abschnittskonvergenz (sectional convergence) and Abschnittsdichte (sectional dense). We note that every AK-space is an AD-space but not conversely (see [49, p.59]).

A BK-space is a special case of foregoing in which the FK-space is a Banach space. In this case the FK-topology is due to a norm. Some examples of BK space are $(\ell_p, \| \cdot \|_p)$ ($1 \leq p < \infty$), $(c_0, \| \cdot \|_\infty)$, $(c, \| \cdot \|_\infty)$, and $(\ell_\infty, \| \cdot \|_\infty)$ where

\[
\| x \|_p = \left( \sum_n |x_n|^p \right)^{1/p} \quad (x \in \ell_p), \\
\| x \|_\infty = \sup_n |x_n| \quad (x \in \ell_\infty).
\]

The spaces cs and bs are also BK-spaces with the norm

\[
\| x \| = \sup_m \left| \sum_{n=1}^m x_n \right| \quad (x \in bs).
\]

Assuming that $\lambda_n \neq 0$ ($n = 1, 2, \ldots$), it is proved (see [49, Theo. 4.3.12]) that, as a special case, $X$ is FK (BK) if and only if $X_\lambda$ is FK (BK). Suppose the FK-topology of $X$ is generated by a sequence $\{p_n\}$ of seminorms. Then the corresponding FK-topology of $X_\lambda$ is generated by the sequence $\{p_n \circ \lambda\}$ of seminorms. In the case that $X$ is a BK-space with norm $\| \cdot \|$, then the corresponding norm on $X_\lambda$ is given by $\| \cdot \|_\lambda$ where $\| x \|_\lambda = \| \lambda x \| (x \in X_\lambda)$. 
A BK-space $X$ is said to have a monotone norm if $\| x^{[m]} \| \leq \| x^{[n]} \|$ for $m \leq n$ and $\| x \| = \sup_n \| x^{[n]} \| \ (x \in X)$. $c_0$, $c$, $\ell_\infty$, $cs$ and $bs$ are some BK-spaces having monotone norm. The monotone property of norm plays an important role in summability theory and spaces having monotone norm are of special importance.

Wilansky (see [49, p.64]), has proved the following result: If $Y$ is a closed subspace of $X$, then $Y_\lambda$ is a closed subspace of $X_\lambda$. In a similar way we have the following proposition.

1.10 PROPOSITION

(a) Let $X$ be a sequence space.

(i) If $X$ is convergence-free, then $X_\lambda$ is convergence-free.

(ii) If $X$ is solid, then $X_\lambda$ is solid.

(b) Let $X$ be a BK-space and $\lambda_n \neq 0 \ (n = 1,2,...)$.

(i) If $X$ has monotone norm, then $X_\lambda$ has monotone norm.

(ii) If $X$ has AD, then $X_\lambda$ has AD.

(iii) If $X$ has AK, then $X_\lambda$ has AK.

(iv) If $X$ is c-like, then $X_\lambda$ is also c-like.

(c) Let $X$ be a Banach sequence space. If $X$ possesses Radon-Riesz property, then $X_\lambda$ also possesses Radon-Riesz property.

Proof

(a) (i) Suppose $X$ is convergence-free. Let $y$ be such that $y_n = 0$ whenever $x_n = 0$ for some $x \in X_\lambda$. Then $\lambda_n y_n = 0$ whenever $\lambda_n x_n = 0$ for some $\lambda x \in X$. By hypothesis, $\lambda y \in X$. Consequently, $y \in X_\lambda$ and hence $X_\lambda$ is convergence-free.
(ii) Suppose $X$ is solid. Let $y$ be such that
\[ |y_n| \leq |x_n| \quad (n = 1, 2, \ldots) \]
for some $x \in X$. Then
\[ |\lambda_n y_n| \leq |\lambda_n x_n| \quad (n = 1, 2, \ldots) \]
for some $\lambda \in X$. But $X$ is solid. Hence $\lambda y \in X$, so that $y \in X$. Thus $X$ is solid.

(b) (i) The BK-part is already proved. Suppose $X$ has monotone norm $\| \cdot \|$. Then, the corresponding norm on $X$ is given by
\[ \| x \|_\lambda = \| \lambda x \| \quad (x \in X). \]
It remains to prove that $\| \cdot \|_\lambda$ is a monotone norm. For $m < n$, and $x \in X$,
\[ \| x^{[m]} \|_\lambda \leq \| (\lambda x)^{[m]} \| \leq \| (\lambda x)^{[n]} \|, \]
since $\| \cdot \|$ is monotone and $\lambda x \in X$. Hence,
\[ \| x^{[m]} \|_\lambda \leq \| (\lambda x)^{[n]} \| \leq \| \lambda x^{[n]} \| = \| x^{[n]} \|_\lambda. \]
Further,
\[ \sup_n \| x^{[n]} \|_\lambda = \| x \|_\lambda. \]
Thus, $\| \cdot \|_\lambda$ is also monotone.

(ii) Suppose $X \supset Y$. Then it is easy to see that $X \supset Y$. Suppose $Y$ is dense in $X$. Let $x' \in X$ be arbitrary. Then $x = \lambda x' \in X$. But $Y$ is dense in $X$. Therefore, there exists a sequence $\{u^n\}$ in $Y$ such that
\[
\lim_{n} \| u^n - x \| = 0,
\]
where \( \| . \| \) is the norm defined on \( X \). Then we have
\[
\lim_{n} \| u^n - \lambda x' \| = 0.
\]
Consequently,
\[
\lim_{n} \| v^n - x' \|, = 0
\]
where \( v^n = u^n \lambda^{-1} (n = 1, 2, \ldots) \). It is easy to verify that \( u^n \in Y \) if and only if \( u^n \lambda^{-1} \in Y \lambda \) for each \( n \). Therefore to each \( x' \in X_\lambda \) there exists a sequence \( \{ v^n \} \) in \( Y_\lambda \). Thus, it is proved that if \( Y \) is dense in \( X \), then \( Y_\lambda \) is dense in \( X_\lambda \). To prove \( X_\lambda \) has AD, it suffices to prove that \( \phi \) is dense in \( X_\lambda \). This follows from the fact that \( \phi = \phi_\lambda \). Thus \( X_\lambda \) has AD.

(iii) Clearly, \( X_\lambda \supset \phi \). Now,
\[
\| \sum_{n=1}^{m} x_n \delta^n - x \| = \| \lambda (\sum_{n=1}^{m} x_n \delta^n - x) \| = \| \sum_{n=1}^{m} \lambda x_n \delta^n - \lambda x \| \rightarrow 0 (m \rightarrow \infty),
\]
since \( X \) has AK. Therefore, \( X_\lambda \) has AK.

(c) Let \( X_\lambda \supset \{ u^n \} \) be such that
\[
u^n \rightarrow u \quad (\text{weakly}) \quad (n \rightarrow \infty)
\]
and
\[
\| u^n \|, \rightarrow \| u \|, (n \rightarrow \infty).
\]
As \( u^n \rightarrow u \) (weakly),
\[
g(u^n) \rightarrow g(u)
\]
for all \( g \in (X_\lambda)' \). But, if \( g \in (X_\lambda)' \) then
\[
g(u) = f(Au)
\]
for some \( f \in X' \) and conversely. Therefore,
\[
f(Au^n) \rightarrow f(Au), (n \rightarrow \infty)
\]
for all \( f \in X' \). Thus
\[ \lambda u^n \rightharpoonup \lambda u \text{ (weakly) } (n \to \infty). \]

Further, \[ \| u^n \|_A \to \| u \|_A \text{ (n \to \infty)} \]

implies \[ \| \lambda u^n \| \to \| \lambda u \| \text{ (n \to \infty)} \]

with \( X \supset \{ \lambda u^n \}. \) But \( X \) has Radon-Riesz Property.

\[ \therefore \| \lambda u^n - \lambda u \| \to 0 \text{ (n \to \infty)} \]

Consequently,

\[ \| u^n - u \|_A \to 0 (n \to \infty) \]

resulting that \( X_A \) has Radon-Riesz Property.

This completes the proof of the proposition.

In the above results the converse parts also hold.

\[ \text{1.11 DEFINITION} \]

A subset \( E \) of \( \phi \) is called a \textbf{determining set} for \( X \) if \( D(X) \) is the absolute convex hull of \( E \) where \( D(X) = \{ x \in \phi : \| x \| \leq 1 \} \). Here, it is not assumed that \( X \supset \phi \) (see [49, p.112]).

The concept of determining set of a BK-space plays an important role in the summability theory.

\[ \text{1.12 PROPOSITION} \]

Suppose \( E \) is a determining set of a BK-space \( X \). Let \( \lambda_n \neq 0 \) for all \( n \). Then \( E_A \)

is the corresponding determining set for \( X_A \).
Proof

Obviously $X_\lambda$ is a BK-space. To prove the required result we need the following two lemmas.

**Lemma (i)** if $B = \phi \cap \bar{U}$, then $B_\lambda = \phi \cap \bar{U}_\lambda$ where $\bar{U}$ is the closed unit ball in $X$.

**Proof**

Since $B = \phi \cap \bar{U}$, $x \in B$ if and only if $x \in \phi$ and $\|x\| \leq 1$. Therefore, $\lambda x \in B$ if and only if $\lambda x \in \phi$ and $\|\lambda x\| \leq 1$. In other words, if $x \in B_\lambda$, then $x \in \phi$ and $x \in \bar{U}_\lambda$ and conversely. That is, $B_\lambda = \phi \cap \bar{U}_\lambda$. Here, we have used the fact that $\phi = \phi_\lambda$ for $\lambda_n \not= 0$ for all $n$.

**Lemma (ii)** $H(E) = \lambda H(E_\lambda)$ where $H(E)$ denotes the absolute convex hull of $E$.

This can be proved analogously.

**The main proof**: Given that $E$ is a determining set of $X$. Now, from Lemma (i), $B_\lambda = \{x \in \phi: x \in \phi \text{ and } x \in \bar{U}_\lambda\}$. That is, $B_\lambda = \{x \in \phi: \lambda x \in \phi \cap \bar{U}\}$. By the hypothesis that $E$ is a determining set, $B_\lambda = \{x \in \phi: \lambda x \in H(E)\}$.

From Lemma (ii) $B_\lambda = \{x \in \lambda x \in H(E_\lambda)\}$

$= H(E_\lambda)$

which is the required result. This completes the proof of the proposition.

From the cited propositions, we have seen that many of the properties of $X_\lambda$ agree with the corresponding properties of the space $X$. Before proceeding further consider the following definition and example.
A sequence space $X$ is said to be symmetric if, $y \in X$ whenever $x \in X$ and the coordinates of $y$ are those of $x$ but in a different order, that is, $y_n = x_{\pi(n)}$ ($n = 1,2,...$) for some permutation $\pi$ on the set of all natural numbers (see [27, p.48]).

Let $\lambda$ be such that

$$
\lambda_n = \begin{cases} 1, & \text{if } n \text{ is even;} \\
+1, & \text{if } n \text{ is odd}, 
\end{cases}
$$

for $n = 1,2,...$. If $x = \{x_n\}$ with

$$
x_n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\
1/n, & \text{if } n \text{ is even}, 
\end{cases}
$$

for $n = 1,2,...$, then $\lambda x = x$ and $x \in c_0$ so that $x \in (c_0)_\lambda$. Now, consider the permutation $\{x_{m_n}\}$ given by

$$
x_{m_n} = \begin{cases} 1/(n+1), & \text{if } n \text{ is odd}; \\
0, & \text{otherwise}, 
\end{cases}
$$

for $n = 1,2,...$. But then, $\lambda\{x_{m_n}\} = \{1,0,1,0,...\} \notin c_0$ so that $\{x_{m_n}\} \notin (c_0)_\lambda$. Hence $(c_0)_\lambda$ is not symmetric even though $c_0$ is symmetric.

This example suggests that the symmetric property of $X$ does not behave well like other properties do. The next result deals with symmetric property. Here, $\rho$ denotes the set of all permutations $\pi$ on the set $\mathbb{N}$ of all positive integers. Here also, we assume that $\lambda_n \neq 0$ for all $n$.

1.14 PROPOSITION

Let $X$ be a solid sequence space which is also symmetric. If $\{\lambda_n / x_{\pi(n)} : \pi \in \rho\}$ is a subset of $l_\infty$, then $X_\lambda$ is symmetric.
Proof

Let $x \in \mathcal{X}_\lambda$. Then $\lambda x \in \mathcal{X}$. But, $\mathcal{X}$ is symmetric. So $\{(\lambda x)_{\pi(n)}\} \in \mathcal{X}$ for all $\pi \in \mathcal{P}$. Now,

$$|\lambda_n x_{\pi(n)}| = |\lambda_n x_{m_n}|$$

for all $n = 1, 2, \ldots$, for some $\{m_n\}$ with $\pi(n) = m_n$ ($n = 1, 2, \ldots$). But $\{\lambda_n/\lambda_{m_n}\} \in \ell_\infty$. Therefore, there exists $M > 0$ such that $|\lambda_n/\lambda_{m_n}| \leq M$ for all $n$. Consequently, from (1),

$$|\lambda_n x_{\pi(n)}| \leq M |\lambda_n x_{m_n}| = M |(\lambda x)_{\pi(n)}|$$

with $\{(\lambda x)_{\pi(n)}\} \in \mathcal{X}$. By the solidity of $\mathcal{X}$, $\{(\lambda x)_{\pi(n)}\} \in \mathcal{X}$. Thus $\{x_{\pi(n)}\} \in \mathcal{X}_\lambda$ for all permutations $\pi$, resulting $\mathcal{X}_\lambda$ is a symmetric space. This completes the proof of the proposition.

1.15 DEFINITIONS

Let $X$ be an FK-space containing $\phi$. The f-dual of $X$ is denoted by $X'$ and is the linear space $\{\{f(\delta \%): f \in X'\}$.

Sometimes f-duals (see [49, p.105]) are also referred to as sequential duals. An FK-space containing $\phi$ is called semiconservative (= sc, for short) if $cs \supset X'$ (see [49, p.141]).

1.16 PROPOSITION

Let $X$ be an FK-space containing $\phi$. If $cs \supset X'$, then $\mathcal{X}_\lambda$ is an sc-space.
Proof

To each \( f \in (X_\lambda)^f \),

\[
\sum_n f(\delta^n) = \sum_n \lambda_n g(\delta^n) \text{ for some } g \in X'
\]

\[
= \sum_n \lambda_n u_n, \ u \in X'.
\]

Now, \( cs_\lambda \supset X' \) implies \( \lambda u \in cs \). Therefore \( \sum_n \lambda_n u_n \) is convergent. Consequently, \( \sum_n f(\delta^n) \) is convergent. Thus \( X_\lambda \) is sc. This completes the proof of the proposition.

Another result whose proof is simple.

1.17 PROPOSITION

Let \( X \) be an FK-space containing \( \phi \) and let \( \lambda_n \neq 0 \) for all \( n \). Then \( (X_\lambda)^f = AX^f = (X^f)_\lambda = \lambda^{-1} \).

We define, for \( \omega \supset X \) and \( \lambda \in \omega \),

\[
X(\lambda) = X \cap X_\lambda = \{ x \in X : \lambda x \in X \}.
\]

The proof of the following can easily be obtained.

1.18 PROPOSITION

Let \( X \) be a sequence space.

(i) If \( X \) is convergence-free, then \( X(\lambda) \) is convergent-free.

(ii) If \( X \) is solid, then \( X(\lambda) \) is solid.

(iii) Let \( X \) be solid. If \( \lambda \in \ell_\infty \), then \( X = X(\lambda) \).

(iv) Let \( X \) be solid. If \( \mu = O(\lambda) \) then \( X(\mu) \supset X(\lambda) \).
Section: Difference sequence spaces

Difference sequence spaces have already been used (example, by the space of all sequences of bounded variation) without being called so.

The difference (sequence) space of a given sequence space $X$ is the space
\[ \{x \in X : \{x_n - x_{n+1}\} \in X\} \]
and is denoted, in our work, by $X_\Delta$. It seems that the nomenclature is first used by Kizmaz (see [28]) in 1980 in which the notation $X(\Delta)$ was used. Ahmad and Mursaleen (see [1]) used the notation $\Delta X$. Kizmaz actually introduced the spaces $c_0\Delta$, $c_\Delta$ and $l_\infty\Delta$ while Ali Sarogol [2] made a generalization of these spaces resulting $c_0q\Delta$, $c_q\Delta$ and $\ell_{\infty q}\Delta$ with $q < 1$. Ahmad and Mursaleen discussed the properties of $(c_0(p))\Delta$, $(c(p))\Delta$ and $l_{\infty p}\Delta$. Here, we make an attempt on further generalization.

1.19 EXAMPLES

(i) $bv = l_\Delta$.

Proof
\[
\begin{align*}
l_\Delta &= \{x \in \ell : \Delta x \in \ell\} \\
       &= \{x \in \ell : \{x_n - x_{n+1}\} \in \ell\} \\
       &= \{x \in \ell : \Sigma_n |x_n - x_{n+1}| \text{ exists}\} \\
       &= bv.
\end{align*}
\]

(ii) The spaces $c_0(\Delta)$, $c(\Delta)$ and $l_{\infty(\Delta)}$ introduced by Kizmaz in [28]. These spaces (in terms of our present notation) are $X_\Delta$ with $X = c_0$, $c$ and $l_{\infty}$ respectively.
(iii) The spaces \( \Delta c_0(p), \Delta c(p) \) and \( \Delta \ell_{\infty}(p) \) which are discussed by Ahmad and Mursaleen in [1] are given by \( X_\Delta \) with \( X = c_0(p), c(p) \) and \( \ell_{\infty}(p) \) respectively.

Combining diagonal mappings the difference sequence spaces can further be generalized. We define

\[
X_{\lambda\Delta} = (X_{\lambda})_{\Delta} = \{x \in X : (\lambda \Delta) x \in X\} \quad \text{and} \\
X_{\Delta\lambda} = (X_{\Delta})_{\lambda} = \{x \in X : (\Delta \lambda) x \in X\}.
\]

We note that the matrix domains \( X_{\lambda\Delta} \) and \( X_{\Delta\lambda} \) are not equal in general. The associative matrix multiplications \( \lambda(\Delta x) = (\lambda \Delta) x \) and \( \Delta(\lambda x) = (\Delta \lambda) x \) are done here using Theorem 4.4 of Wilansky [49].

(iv) The spaces \( c_0(\Delta q), c(\Delta q) \) and \( \ell_{\infty}(\Delta q) \) discussed by Ali Sarigöl in [2] can be obtained by \( X_\Delta \) with \( \lambda = \{n^q\}, q < 1 \) where \( X = c_0, c \) and \( \ell_{\infty} \).

(v) The Hahn sequence space, discussed by Chandrasekhara Rao in [9], is the sequence space \( h \) given by

\[
h = \{x \in c_0 : \Sigma_n |x_n - x_{n+1}| \text{ exists}\}.
\]

Then \( h = h_1 \cap c_0 \)

where \( h_1 = \{x \in c_0 : \Sigma_n |x_n - x_{n+1}| \text{ exists}\} \).

Clearly, \( h_1 = \ell_{\lambda \Delta} \)

with \( \lambda = \{n\} \) and hence,

\[
h = \ell_{\lambda \Delta} \cap c_0.
\]
1.20 THEOREM

Let $X$ be a BK-space and let $SX_{\lambda} = X_{\lambda} \cap c_0$. Suppose $E$ is a determining set for $X$. Then $E_{\lambda} = \lambda_{\lambda}$ is a determining set for $SX_{\lambda}$. Here $\lambda_{n} \neq 0$ for all $n$.

Proof

Let $B$ and $B_{\lambda}$ be the closed unit balls in $X$ and $X_{\lambda}$ respectively. Let $D = \phi \cap B_{\lambda}$. It is required to prove $D = H(E_{\lambda})$ where $H(A)$ denotes the absolute convex hull of the set $A$. Let $x \in D$. Then $x \in \phi_{\lambda} \cap B_{\lambda}$ because $\phi_{\lambda} \supset \phi$. Therefore $(\lambda\Delta)x \in \phi\cap B$. But $E$ is the determining set for $X$ so that $(\lambda\Delta)x \in H(E)$. Also, $x \in H(E_{\lambda})$ if and only if $(\lambda\Delta)x \in H(E)$. Thus, $x \in H(E_{\lambda})$. Consequently, $H(E_{\lambda}) \supset D$. Conversely, let $x \in H(E)$. Then $x \in (\lambda\Delta)^{-1} H(E)$. Being $E$, a determining set for $X$, $x \in (\lambda\Delta)^{-1} \phi \cap B$ or $(\lambda\Delta)x \in \phi\cap B$. Thus, $x \in \phi_{\lambda} \cap B_{\lambda}$. Since $SX_{\lambda}$ does not contain $e$, $\phi = \phi_{\lambda}$. Therefore, $x \in \phi \cap B_{\lambda}$. Consequently, $D \supset H(E_{\lambda})$. Thus $D = H(E_{\lambda})$. This completes the proof of the theorem.

1.21 COROLLARY

$\{e^{[nl]}\}$ is a determining set for $bv_{o} = c_{o} \cap bv$.

Proof

This follows from the fact that $\{d^{n}\}$ is a determining set for $\ell$ (see [49, p.112]) and from the fact that $bv_{o} = c_{o} \cap \ell_{\lambda}$.

1.22 COROLLARY

$\{(1/n)^{[nl]}\} = \{1,1/2,..., 1/n, 0, 0,...\}$

is a determining set for $h = \{x \in c_{o} : \Sigma_{n} n | x_{n} - x_{n+1} | \text{ is convergent}\}$

The Corollary 1.22 is a particular case of Proposition 1.20. Thus a result of K.C. Rao (see [9, p.76]) becomes a special case.

*