CHAPTER 2
LITERATURE REVIEW

2.1 General

The finite element method today, has become a powerful tool in engineering analysis and design. The ease of application, reliability of solutions and ability to model complex geometries seem to be the main reasons for its popularity. It encompasses many diverse fields like structural mechanics, fluid mechanics, solid mechanics, electromagnetism etc.

Like in all original developments it is difficult to pin point an exact date for the initiation of finite element method. Basic ideas of the FEM originated from advances in aircraft structural analysis. The development of FEM mainly attributed to three separate groups- mathematicians (Courant 1943, Collatz 1950, Courant and Hilbert 1953), physicists (Synge 1957) and engineers (Turner et al.1956). Courant's paper, which used piecewise polynomial interpolation over triangular sub regions to model torsion problems, appeared in 1943. The term finite element was first used by Clough (1960). In the late 1960's and early 1970's, finite element analysis was applied to non-linear problems and large deformations. Mathematical foundations were laid in the 1970's. New element development, convergence studies and other related areas fall in this category.

A comprehensive study on the historical development of the family of finite element can be made by broadly classifying it in to conforming and nonconforming elements.
2.2 Development of conforming finite elements

Conforming elements are formulated by strictly adhering to the three cardinal principles, known as "convergence criteria" (Zienkiewicz and Taylor 1989). Conventionally, to ensure convergence to the exact solution, the interpolation function should satisfy certain criteria. They are formally defined as given below.

a. The displacement function must be continuous within the element. One way of satisfying this condition is by choosing complete polynomials for the displacement model.

b. The displacement function must be capable of representing rigid body displacement of the element. The constant terms in the displacement polynomial would ensure this condition.

c. The displacement function must be capable of reproducing the exact strain states defined by the respective elasticity equations within the element. In one, two and three dimensional elasticity problems, the linear terms in the assumed displacement function would satisfy this requirement. In the case of beam, plate and shell element the displacement function should be capable of representing constant curvature states.

Conventional finite elements also satisfy compatibility conditions. The displacements must be compatible between adjacent elements. When the elements deform, there must not be any discontinuity like overlap or separation between the elements. In the case of beam, plate and shell element, this requirement would ensure that there should not be any discontinuity or sudden changes in slope across the inter-element boundaries. Elements that satisfy the convergence requirements and compatibility conditions are called Conforming elements. And elements, which violate
compatibility conditions, but satisfy the convergence requirements, are termed as Nonconforming elements.

It will be Herculean to list all the conforming elements developed so far. Hence only a few of them, which can be treated as milestones in the development of conforming elements, are considered here. The literature review is presented here under the subtitle two-dimensional and three-dimensional finite elements and plate elements.

2.2.1 Two dimensional and three dimensional finite elements

From early days, major research efforts have been made to develop simpler and lower order finite elements with translational degrees of freedom only, like four node quadrilateral used in two-dimensional analysis, eight node brick element and simple tetrahedral element in three dimensional analysis. The four node quadrilateral element QUAD4 (Zienkiewicz and Taylor 1989, Cook et al. 1989) which is a bilinear isoparametric element used for plane stress analysis is one of the simplest element. Hence this element is used as a ‘work horse’ element in most of the applications. The only draw back is that a very fine mesh is required to get reasonably accurate solutions. This is especially true when problems with in-plane bending loads are modelled. Moreover shear stress predicted across the element oscillates enormously.

Even prior to this element, a quadrilateral element built with four constant strain triangles had been developed (Cook 1969, Cook et al. 1989). The internal node was condensed out and stress was evaluated at the centre of the quadrilateral using an averaging scheme. However this element was also affected with parasitic shear and the performance was slightly inferior to the QUAD4 (Desai and Abel 1972).

The first formulation of simple tetrahedral element was done by Gallagher et al. (1962) and they used it for stress analysis of heated complex shapes. Early elaborations
of tetrahedral elements were by Melosh (1963 a.) and Clough (1969). An extensive numerical study was done by Rashid et al. (1969 b. and 1970). These elements used volume co-ordinates similar to triangular elements that used area co-ordinates, and were simple generalisation of the later. Amongst them the first (Gallagher et al. 1962) was a \( C^0 \) continuous, 4 node, 12 dof constant strain tetrahedron. It used linear shape functions along the three orthogonal cartesian directions.

Clough (1969) used a \( C^0 \), 10 node, 30 dof linear strain tetrahedron by adding mid-side nodes. This tetrahedron used complete quadratic polynomials in the three directions. Rashid et al. (1969 a.) used a \( C^0 \), 16 node, 48 dof tetrahedron. Hughes and Allik (1969) have formulated and used a 4 node 48 dof tetrahedron. They used four vertex nodes and dof of \( u, v, w \) and their derivatives in \( x, y, z \), directions at each node. Being a higher order element with derivative degree of freedom, it required higher order continuity. As it could be expected "this is the most advantageous tetrahedron introduced" (Yang 1986). Initial work on conforming hexahedral elements were restricted to rectangular ones. Since the faces and sides of the rectangular elements are orthogonal to one another, these elements can be formulated using non-dimensional local co-ordinate systems. Many such elements are available. Amongst the first was a \( C^0 \), 8 node, 24 dof, linear displacement, rectangular tetrahedron (Melosh 1963 b., Clough 1969). The element used tri-linear displacement interpolation functions in the three orthogonal directions. The addition of one node to midpoint of each side gives a \( C^0 \), 20 node, 60 dof, and quadratic displacement hexahedron. Like tri-linear element used incomplete cubic polynomials, this element used incomplete quadratic ones.

The addition of four facial nodes and eight interior nodal points yield a 54 node, 192 dof, \( C^0 \) hexahedral element first used by Argyris and Fried (1968). Here the
interpolation functions are obtained by taking the product of three complete cubic polynomials in three directions. Another commonly used rectangular Lagrangian element is from the use of the product of quadratic polynomials in the three orthogonal directions. It is a 27 node, 54 dof hexahedron with a centroidal node, the degree of freedom corresponding to this could be statically condensed. The Lagrangian element has a disadvantage that the interpolation functions require the use of large degrees of polynomial. Solid finite elements of shape other than tetrahedron or hexahedron are also available. Some of them are wedge shaped and pentagonal elements. For wedge shaped elements (triangular prisms) the interpolation functions are obtained as the product of Lagrange approach and Serendipity approach.

In the elements described above the number of nodes has to be increased to increase the order of the interpolation polynomial. Alternatively, the elements with higher derivatives of displacements as nodal degrees of freedom can also be used. Another means of generating interpolating functions is to use hierarchic approximations. Here one needs to associate the monomial term in each interpolating polynomial with just a parameter and not to one with an obvious physical meaning. Further hierarchic functions need to have zero values at the end of the range (on the nodal points along each edge under consideration). Using these polynomials one can arrive at a variety of interpolation functions for elements of different geometries.

2.2.2 Plate elements

Problems involving thick plates consist of complete set of three-dimensional differential equations and have to be tackled with solid finite elements. Thin plates with small deflections can be dealt with noncompatible finite elements based on Kirchhoff's theory of thin plates in which the transverse shear deformations are
neglected. Several attempts have been made in the past to develop simple and efficient plate elements using displacement models satisfying the \( C^0 \) continuity requirement. These models are based on Mindlin theory, which considers shear deformation in plates. In \( C^0 \) continuous elements, three independent displacement quantities namely \( w, \theta_x \) and \( \theta_y \) are to be considered for the inclusion of shear deformation. Hence for the finite element formulation, three shape functions are chosen to represent the variation of \( w, \theta_x \) and \( \theta_y \). Such elements showed promise for application to thick or thin plates, with curved boundaries. However, main difficulty experienced in the use of such elements was that they experienced over stiff locking behaviour in thin plate situations. Zienkiewicz et al (1971) proposed an eight node isoparametric element with reduced integration, capable of using in the thin plate situations.

Another approach to the development of elements for thin plates involves the use of discrete Kirchhoff theory (Bathe et al. 1980, Bathe 1982). In this approach, the independent displacement quantities were assumed for the finite element formulation of \( w, \theta_x \) and \( \theta_y \), and only \( C^0 \) continuity requirements need to be satisfied. The transverse shear energy is neglected and Kirchhoff hypothesis is introduced in a discrete way along the edges of the element to relate the rotations to the transverse displacements. Hence the constraint of zero shear strain \((\gamma_{xz} = \gamma_{yz} = 0)\) is imposed at the discrete number of points along the edges of the element to represent the behaviour of the thin plates. Each constraint removes one degree of freedom and thus yields a flexible mesh. This property makes it possible to avoid element locking associated with the lower order elements applied to very thin plates. The implementation of DKT (Discrete Kirchhoff Triangle) is complicated and the elements predict stresses relatively poorly.
2.3 Development of nonconforming finite elements

Elements mentioned so far were derived by strictly adhering to the convergence criteria. The behaviour of these elements in situations, such as bending or near incompressibility limit (especially the lower order elements) are not very encouraging. The reasons for the poor performance of these elements are mainly due to parasitic locking and incompressibility locking. The term locking is used to denote a definite decay of accuracy in displacement recovery. Other common problems encountered are "violent stress oscillations" (Prathap 1992) and delayed convergence. Various new elements formulated lately, address themselves to tackle these problems. Since the early days, the development of such elements has been the source of both challenge and motivation for new developers.

2.3.1 Two dimensional and Three dimensional elements

Many techniques do exist in the literature to tackle the above-mentioned problems. Many of these techniques are categorised as "adhoc", for their success in some problems does not necessarily imply the same when extrapolated to other problems. These techniques are the "milestones" of progress of FEM and are called extra variational techniques (MacNeal 1992). A few of them worked very well in certain situations but failed in other situations. The important techniques that developed over the years for the improvement of the performance of these finite elements are described in subsequent sections.

2.3.1.1 Reduced or selective integration

This method is applicable to all types of finite elements such as two dimensional, three-dimensional and plate elements. Here the strain energy is not exactly integrated.
An ‘n’ point rule in one dimension can be used to integrate a polynomial of the order $2n - 1$ exactly (Conte and de Boor 1980). Usage of a lower Gauss point rule than that is required for exact integration of the strain energy will result in reduced integration and faster convergence to the exact solution. In selective integration, the different strain energy terms are integrated with different order of integration (Hughes, Taylor and Kanoknukulchai 1977).

These rules need to be used with care. A very low order integration can lead to mechanisms, while the use of a very high order leads to delayed convergence. One common mechanism encountered during reduced integration is the presence of hourglass modes. Zeinkiewicz and Taylor (1989) proved that for success of this method the gauss points selected should be exactly the optimal points for the stress recovery.

**2.3.1.2 Addition of bubble modes**

The technique involves the addition of certain degrees of freedom not associated with any node (Wilson 1973). This brought into use, incompatible elements, where the displacement fields are not continuous across element boundaries. The variables associated with the nodeless degrees of freedom are later condensed out. MacNeal (1987) proved analytically that it is impossible for a rectangular element with only four nodes to be able to model in-plane bending satisfactorily and also to pass patch test. This difficulty has to be overcome either by introducing incompatible modes or by increasing the number of degrees of freedom in the transverse direction on the model. Wilson *et al.* (1973) introduced the incompatible element Q6. This element has additional degrees of freedom, which are incompatible and popularly known as ‘bubble modes’. Static condensation is necessary and it is found that the
resulting element has to be further manipulated with selective integration of the incompatible or bubble modes in order for it to pass the patch test. Taylor et al. (1976) developed the approach for making the Q6 to pass the patch test and called the element QM6. The element QM6 is not susceptible to parasitic shear and also performs well in other situations. However this element also experiences difficulties when modelling in-plane bending using increasingly distorted or quadrilateral meshes. The eight-node brick element when used in tandem with reduced integration, gives very good results. It has found its way into many commercial finite element packages. Unfortunately use of these techniques requires expertise. The polynomial functions so chosen to represent the nodeless degrees of freedom should be the exact ones required for eliminating the required type of locking. For example, in the eight-node brick element, the incompatible modes selected alleviate the parasitic shear.

2.3.1.3 Using unequal order of interpolation

This is a simple technique in use, especially for one and two-dimensional elements. Here the order of the interpolation functions used for the rotational degrees of freedom are one less than that used for the translational ones. Its success could be attributed to the terms dropped from the interpolation functions of the rotational degrees of freedom. They are exactly the one, which if present will cause locking. Unequal order of interpolation has been used in the formulation of the many finite elements. For example, Tessler and Dong (1981) formulated one such Timoshenko beam element.

2.3.1.4 Assumed strain method

This technique involves the use of interpolation functions of lower order and smoothening them in some least square sense (MacNeal 1982). The method has an advantage that the procedure can be used to obtain the interpolation function.
2.3.1.5 Residual energy balancing

Here certain constraints contributing to locking are identified. They are then artificially removed by using a constant, which the designer of the element sets an arbitrarily small value (Fried 1975, Cook 1977). The value of the constant is problem dependent and it is difficult to choose one value for a set of elements or problems. The constant is also mesh dependent, by that increasing the confusion. Stresses predicted by this method are “very unreliable” (Prathap 1992) and grossly depend on the scaling constant chosen.

2.3.1.6 Other general methods

Amongst the most popular formulations are the one in which compatible displacements and equilibrating stresses are independently formulated. Stress parameters are eliminated at the element level (Pian 1973). These formulations are known as hybrid/mixed formulations. Many three dimensional hybrid/mixed stress elements have been developed (Zienkiewicz and Taylor 1989). Here too, extravariational techniques like reduced integration and introduction of bubble modes can be used. Other elements formulated using these principles are eight node elements (Irons 1972), 20-node element (Ahmad and Iron 1974), special purpose three dimensional elements for thick plate analysis (Spilker 1981). Tang and Chen (1982) proposed a series of nonconforming stress based elements. Chen and Cheung (1987) derived a new functional (a functional with displacements, stresses and strains as independent variables) to obtain a series of isoparametric elements. Sze and Ghali (1993) started with assumed stress element and identified the strain components that cause locking and selectively scale them down to obtain an incompatible element.
There are several other ingenious techniques used to obtain better elements. Some of them are synthesis using Fourier components (Park 1984), use of trigonometric interpolation functions (Heppler and Hanson 1987) etc.

Another technique developed by Prathap (1986 and 1992), says that in-plane bending involving Kirchhoff constraint of zero shear energy in very thin beams is a constrained minimal problem. It is not possible to model these constraints directly to a displacement based finite element. Instead, a displacement field that is consistent with the constraints has to be used. This is accomplished by identifying the term in the expression for shear strain that absorbs parasitic energy and ignoring the contribution of that term during shear energy computation. The consistent field principle is a technique that is applicable to any constrained minimal problem such as plate bending and near incompressible three-dimensional analysis. Unfortunately this technique cannot be applied to any arbitrary quadrilateral (Prathap 1992). However, this technique throws more light on the locking problem.

All the techniques summarised above are artifices, attempted to primarily deal with the locking effect. Moreover, each technique has its own risk and inadequacies and often needs further manipulation of elemental functions to enable it to perform well in all loading situations.

2.3.2 Plate elements

A variety of plate elements have been proposed since the early days of finite element. The development of plate bending element based on Kirchhoff's theory of thin plates lead to either incompatible elements or involved complicated formulation and programming (Zienkiewicz and Taylor 1991). A rectangular plate element with 12 dof proposed by the Melosh (1963 c.) is one of the oldest and best known element. This
element has three dof w, \( \partial w/\partial x \) and \( \partial w/\partial y \) per node and is not fully compatible. However the performance of the element is reasonably good and is widely used (Zienkiewicz and Taylor 1991). Bogner, Fox and Schmit (1965) developed a sixteen degrees of freedom element (LCCT-12). Clough and Felippa (1968) proposed a refined quadrilateral element in which a sub domain approach is used. Although the LCCT-12 element employs an optimum compatible displacement field, its midside node and rotational degrees of freedom complicate the analysis. A special version of this element designated as LCCT-11 is developed by avoiding the midside node, employs the static condensation of the internal degrees of freedom. This element is a fully compatible element and gave good results in the analysis of plate bending.

2.4 Test problems for element performance comparisons

All the elements mentioned above, have tested with standard test problems and the results were compared with that obtained with other similar elements. Comparisons are generally made against standard problems proposed in the literature (MacNeal and Harder 1985, White and Abel 1989). They include a variety of problems such as patch tests, problems with in-plane bending, problems with stress concentration, curved shell tests, three dimensional tests etc.

These tests were considered to indicate the performance of the element in general. It has been observed that no single element is capable of performing well in all these problems. The most common failing of these occurs when increasingly distorted meshes are used in models involving in-plane bending.

2.5 Scope of the present work

After detailed review of the literature, definite need is felt for exploring the possibilities of finite element that will not have the deficiency of locking and related
drawbacks. This thesis addresses the development of field equilibrium finite elements and proposing them for the stress analysis of membranes, solids and thin plates. Field consistent approach is based on displacement functions that satisfies stress equilibrium equations and hence combines the simplicity of displacement formulation and accuracy in stress prediction.

A new family of elements within which the displacement functions satisfy the differential equations of stress field equilibrium are proposed for the plane stress, three dimensional and plate bending analysis. The objectives of the thesis are listed below.

• Development of a general procedure to construct displacement polynomials which satisfy the differential equations of stress field equilibrium for plane stress, three dimensional and plate bending elements.

• Generation of elemental stiffness matrices of simple plane stress, three dimensional and plate bending elements using the above procedure.

• Development of Software for implementation of these elements.

• Testing the performance of these elements on standard test problems and comparing the results with theoretical closed form solutions and results obtained with other existing elements.