CHAPTER 2
RADIATIVE TRANSFER THEORY

2.1 Introduction

As already discussed in previous chapter, the retrieval of profiles of atmospheric state variables (e.g. temperature, humidity etc) can be done with the help of passive remote sensing. In passive remote sensing, the emission spectrum of a target is used to infer the characteristics of that target. The radiation emitted by any target traverses through the atmosphere to reach up-to space-borne sensors. These sensors measure the top of the atmosphere radiation in terms of radiance or brightness temperature. The study which deals with the interaction of radiation with earth-atmosphere system is known as radiative transfer theory and the governing equation is called radiative transfer equation or model. The RT equation gives the top of the atmosphere radiance or brightness temperature for a given atmospheric state which is generally referred as forward modeling. The process to estimate atmospheric state from the measured radiance or brightness temperatures is known as reverse modeling or retrieval process. The following section describes the fundamentals of RT model.

2.2 Radiative Transfer Model

The fundamental basis of any passive remote sensing application is Radiative Transfer Equation (RTE), first developed by Chandrasekhar (1960), which de-
Chapter 2: Radiative Transfer Theory
scribes the flow of energy or radiation emitted by any target to the top of the atmosphere. This equation can be derived by considering the physical principles of interaction between electromagnetic (EM) radiation and matter. This is described by two processes: extinction (absorption, scattering) and emission. If the intensity of the radiation reduces while traversing in a medium, we have extinction, and if the medium adds energy of its own, we have emission. Both the processes occur simultaneously in this interaction. The brightness of the radiation \( B \) describes the energy flow at a point in a given direction per unit area, per unit solid angle, per unit frequency range \((\nu \text{ to } \nu + d\nu)\) during a time interval of one second. The loss in the brightness \( dB \) (Ulaby et al. (1981)) by extinction due to propagation over the distance \( dr \) is given by

\[
  dB(\text{extinction}) = \kappa_e B dr
\]  

(2.1)

where \( B = \text{brightness, Wm}^2\text{sr}^{-1} \),

\( \kappa_e = \text{extinction coefficient of the medium, nepers m}^{-1} \)

\( \kappa_e \) is also known as the power attenuation coefficient. The loss of energy from the incident radiation may have been caused by absorption in the medium, scattered by the medium or both. Therefore, the extinction coefficient \( \kappa_e \) may be expressed as a sum of both absorption coefficient \( \kappa_a \) and scattering coefficient \( \kappa_s \).

The amount of brightness emitted by the medium while traversing a distance \( dr \) can be given as

\[
  dB(\text{emission}) = \kappa_e J dr
\]  

(2.2)

where \( J = \text{total source function that accounts for thermal emission and scattering by the medium into the propagation direction.} \)

Hence, the total change in the brightness of the incident radiation after traver-
ing a distance of $dr$ in a medium:

$$dB = dB(\text{emission}) - dB(\text{extinction})$$  \hspace{1cm} (2.3)

putting values from equations (2.1) and (2.2), equation (2.3) becomes

$$dB = \kappa_e dr(J - B)$$  \hspace{1cm} (2.4)

Now, the optical depth ($\tau$) of a material along a range $r_1$ to $r_2$ is defined as

$$\tau(r_1, r_2) = \int_{r_1}^{r_2} \kappa_e dr$$  \hspace{1cm} (2.5)

Therefore, from equation (2.5), an increment of optical depth $d\tau$ can be written as

$$d\tau = \kappa_e dr$$  \hspace{1cm} (2.6)

Inserting equation (2.6) into (2.4) and rearranging, equation (2.4) becomes

$$\frac{dB}{d\tau} + B = J$$  \hspace{1cm} (2.7)

This differential equation is known as equation of transfer or RTE.

The solution of the above equation (2.7) for a general case of semi-infinite medium (earth-atmosphere system) gives the brightness at any point $r$ propagating in the direction $\hat{r}$ as

$$B(r) = B(0)\exp^{-\tau(0,r)} + \int_{0}^{r} \kappa_e(r')J(r')\exp^{-\tau(r',r)} dr'$$  \hspace{1cm} (2.8)

From the above solution, it can be inferred that the brightness $B(r)$ at any point $r$ propagating in the direction $\hat{r}$ is given by two terms. The first term represents the brightness $B(0)$ at boundary, reduced in magnitude by $\exp^{-\tau(0,r)}$ due to extinction by the material between 0 and $r$. The second terms tells about the emission and scattering by the material along the propagation path. The integral
is the sum of the contributions from infinitesimal thicknesses, each $dr'$ in length; the contribution from a layer at point $r'$ is given by the differential emitted brightness, $\kappa_e(r') J(r') dr'$ according to (2.8), reduced in magnitude by the factor $\exp^{-\tau(r',r)}$ due to extinction by the material between the layer at point $r'$ and the observation point $r$.

### 2.2.1 Brightness Temperature

An ideal body which absorbs or emits all the frequencies incident upon it, is called blackbody. The ratio of emitted radiation by an object to that emitted by a blackbody at the same temperature is known as emissivity of that object. Hence, the emissivity of a black body will be equal to 1. Real materials don’t absorb or emit all the radiation incident upon them. These materials are called Grey bodies. Therefore, their emissivity will always be less than or equal to that of a blackbody. From the Planck’s blackbody radiation law, the brightness of the emitted radiation of an object at a given frequency $\nu$ and temperature $T$ is given as

$$B_\nu(T) = \frac{h\nu^3}{c^2} \cdot \frac{1}{\exp\left(\frac{h\nu}{kT} - 1\right)}$$ (2.9)

where $B_\nu(T)$ is the brightness, $h$ is Planck’s constant and $c$ is the speed of light.

For microwave frequencies $h\nu/kT \ll 1$ for tropospheric temperatures, therefore on expanding from Rayleigh series the above relation becomes

$$B_\nu(T) \approx \frac{2\nu^2 kT}{c^2} = \frac{2kT}{\lambda^2}$$ (2.10)

where $\lambda$ is the wavelength.

From this equation, the brightness temperature ($T_B$) of an object having physical temperature $T$ can be defined as blackbody equivalent radiometric temperature such that it emits brightness $B_\nu$ at $T=T_B$. From the definition of the
emissivity $B_{\nu}(T_B) \leq B_{\nu}(T)$, $0 \leq e \leq 1$. Thus, the brightness temperature ($T_B$) of any object is always less than or equal to its physical temperature ($T$).

### 2.2.2 RTE for Scattering-Free Medium in Microwave

In scattering-free medium the total source function $J$ will consist only of absorption source function $J_a$. Also, from Kirchoff's law, in local thermodynamic equilibrium thermal emission is equal to absorption which leads to that absorption source function $J_a$ is isotropic and can be given by Planck's radiation law (2.9). Therefore, in microwave region the absorption source function can be written using equation (2.10) as

$$J_a(r) = \frac{2k}{\lambda^2} T(r)$$

where $T(r)$ is the kinetic (physical) temperature of the medium at $r$.

Finally, the general solution (2.8) of RTE in terms of brightness temperature for microwave region can be written as

$$T_B(r) = T_B(0) \exp^{-\tau(0,r)} + \int_0^r \kappa_a(r') T(r') \exp^{-\tau(r',r)} dr'$$

(2.12)

where $T_B(r)$ is the brightness temperature at any point $r$.

For a earth-looking radiometer at a range $r$ from the earth's surface, the first term $T_B(0)$ in equation (2.12) is the brightness temperature of the earth's surface, reduced by a factor $\exp^{-\tau(0,r)}$, which accounts for the atmospheric absorption between the earth's surface and the radiometer. While the second term represents the net upwelling atmospheric self-emission in the direction of the radiometer. This second terms further comprises of two terms: upwelling atmospheric radiation and down-welling atmospheric radiation.
2.2.3 Upwelling Atmospheric Radiation

The atmosphere between the earth’s surface and the radiometer emits the radiation in all the directions. The part of this emitted radiation which goes in the upward direction i.e. towards the direction of the radiometer is known as the upwelling radiation. From equation (2.12), the net upwelling radiation emitted by the atmosphere at a point \((r, \theta)\) at a height \(H\) in the direction \(\theta\) relative to the surface normal can be given as

\[
T_{UP}(\theta; H) = \sec \theta \int_0^H \kappa_a(z') T(z') \exp^{-\tau(z', H) \sec \theta} dz'
\]  

(2.13)

where \(r' = z' \sec \theta\)

If \(H\) is sufficiently higher than the physical extent of the earth’s atmosphere, it can be taken as \(\infty\), since \(\kappa_a(z) = 0\) for free space.

2.2.4 Down-welling Atmospheric Radiation

The some part of the radiation emitted by the atmosphere also propagates towards down to the earth’s surface. This downwards radiation further gets reflected by the earth’s surface and finally gets received by the radiometer at the top of the atmosphere. This radiation is known as down-welling atmospheric radiation. By analogy to the previous case, the down-welling radiation emitted by the atmospheric layer at \(z'\) and of vertical thickness \(dz'\) can be given as

\[
T_{DN}(\theta) = \sec \theta \int_0^\infty \kappa_a(z') T(z') \exp^{-\tau(0, z') \sec \theta} dz'
\]  

(2.14)

where \(T_{DN}(\theta) = \text{Downwelling Radiation}\)

2.2.5 Absorption Models of Gases

From equation (2.12) it is clear that to calculate the brightness temperature at top of the atmosphere we need to have total absorption coefficient \(\kappa_a\). The total
absorption coefficient depends upon concentrations as well as emission or absorption spectra of various atmospheric gases. The radiation emitted from the earth gets absorbed by various atmospheric gases e.g. oxygen, water vapour, CO$_2$ etc. This absorption gives rise to the transition of the gaseous atoms or molecules in different electronic states. These transition are governed by specific selection rules as given by quantum theory of gases. According to selection rules, gases can absorb or emit radiation of discrete frequencies only. These discrete frequencies are known as line spectra of that gas and are determined by the difference of quantised energy levels of the transitions. Depending upon their atomic or molecular structure, the line spectra can occur in different regions of EM spectrum. For example, atomic gases radiate EM energy in visible or ultraviolet parts of the EM spectrum while molecular gases have their emission spectra in infrared and microwave wavelengths. A detailed description of these processes and the coupling that can occur between them is given in Rosenkranz (1993).

2.2.5.1 Line broadening

As stated above, the absorption (or emission) spectrum of a molecule consists of sharply defined frequency lines corresponding to transitions between sharply defined (quantized) energy levels of the molecule. Such a spectrum would be characteristic of an isolated, undisturbed, and stationary molecule system. In reality, however, the molecules are in constant motion, interacting and colliding with one another, and colliding with other material objects (such as dust particles). These disturbances cause the energy levels to vary in width, which result in spectral lines with finite width. The increase in line-width is called line broadening. Among various sources of spectral line broadening, (Townes and Schawlow (1955); Gordy and Cook (1970)), pressure broadening, which arises from collisions between molecules, is the most important for atmospheric
absorption in the microwave region of the spectrum. Hence, by exploiting the pressure broadening of spectral lines we can get information of the species’ vertical distribution needed to retrieve profiles of temperature and humidity with any practical microwave radiometer (with finite bandwidths).

The absorption spectrum for transitions between energy states \( \varepsilon_1 \) and \( \varepsilon_2 \) may be given as

\[
\kappa_a(f, f_{lm}) = \frac{4\pi f}{c} \cdot S_{lm} F(f, f_{lm})
\]  

(2.15)

where \( \kappa_a = \) power absorption coefficient, Np m\(^{-1}\),
\( f = \) frequency, Hz,
\( f_{lm} = \) molecular resonance frequency for transition between energy states \( \varepsilon_1 \) and \( \varepsilon_2 \), Hz,
\( c = \) velocity of light, \( 3 \times 10^8 \) ms\(^{-1}\),
\( S_{lm} = \) line strength of the \( lm \) line, Hz,
\( F = \) line-shape function, Hz\(^{-1}\).

The line strength \( S_{lm} \) of the \( lm \) line of a specific atmospheric gas is governed by the number of absorbing molecules of that gas per unit volume, the temperature of the gas, and the molecular parameters associated with that transition. The line shape function \( F(f, f_{lm}) \) describes the shape of the absorption spectrum with respect to the resonance frequency \( f_{lm} \). Several different line-shape functions, based on different models for the nature of the collision between the molecules, have been derived and used in connection with gaseous microwave spectra. The simplest is the Lorentzian function which is valid for sharp lines whose values of line-width parameter \( \gamma \) are much smaller than the transition frequencies \( f \). The line-width parameter \( \gamma \) is defined as half frequency width at half peak intensity. For atmospheric pressure conditions, where \( \gamma \) is comparable in magnitude to \( f_{lm} \), Vleck and Weisskopf (1945) developed the shape function.
At the line center, \( f = f_{lm} \), equation (2.16) reduces to \( F_{vw} \approx 1/(\pi \gamma) \). Measurements have shown good agreement with calculations based on the above line-shape function for frequencies near the resonant frequency \( f_{lm} \), but have significant deviations in the far wings. For better agreement on far wings side, Gross (1955) derived a line-shape function that follows same functional form as \( F_{vw}(f, f_{lm}) \) near resonance but provide good agreement with measurements on far wings. Gross's line-shape formula, which was independently derived by Zhevakin and Naumov (1963) and is sometimes referred to as the kinetic line shape, is given by

\[
F_G(f, f_{lm}) = \frac{1}{\pi} \frac{4ff_{lm}\gamma}{(f_{lm} - f)^2 + 4f^2\gamma^2} \tag{2.17}
\]

Again, at the line center, \( F_G = 1/(\pi \gamma) \). It has been shown that Gross line-shape function provides a better fit to absorption data. But due to small difference between Gross and Van Vleck-Weisskopf line-shape function in microwave absorption lines, the latter enjoys wide use in microwave absorption calculations.

Figure (2.1) shows the atmospheric absorption of microwave frequencies. In troposphere, oxygen and water vapour dominate the absorption of microwave frequencies. These will be discussed in details in the following sections. However, minor species are also radiatively active.

### 2.2.5.2 Oxygen Absorption Model

The absorption spectrum of oxygen in microwave region (1-300 GHz) consists of a large number of absorption lines spread out over the 50-70 GHz (known
as 60 GHz oxygen complex) and an additional line at 118.75 GHz as shown in figure (2.1).

Figure (2.2) shows the absorption coefficient of 60 GHz oxygen complex for four different altitudes e.g. 0, 3, 10 and 20 km. It can be seen that the absorption coefficient increases as altitude decreases. This can be attributed to the dependency of absorption coefficient over gas concentration and concentration of oxygen gas decreases with altitude. The regions of very low or zero absorption are called window regions. The window regions are used to exploit the information about the surface characteristics of the earth. In lower part of the earth’s atmosphere pressure broadening causes the complex of oxygen lines to blend together, forming a continuous absorption band around 60 GHz. Although Zeeman splitting of the fine structure in this band can occur due to earth’s magnetic field, this is significant only in the upper atmosphere.
To compute the absorption spectrum of the 60-GHz complex, the traditional approach of adding the absorption coefficient due to individual lines (Meeks and Lilley (1963); Tolbert and Straiton (1963); Carter et al. (1968)) using the Vleck and Weisskopf (1945) line-shape function has been used.

2.2.5.3 Water Vapour Absorption Model

In the microwave region, water vapour has rotational absorption lines at 22.235 and 183.31 GHz as shown in figure (2.1). However, numerous other lines at frequencies just above this region also contribute to the microwave absorption spectrum.

In addition to these absorption lines, it has been found that to match the model computed absorption with the observations, a non-resonant contribution by a
water vapour continuum especially in window regions between absorption lines should necessarily be included in the computation. Many theories have been developed to explain the process responsible for the water vapour continuum. But, it is usually represented as an empirical term used to the theoretically based resonant terms. It is generally divided into two contributions, which are self- and foreign-broadened, due to collisions of water vapour molecules with molecules of the same species or other gases, respectively (Rosenkranz (1998)).

2.2.5.4 Other Gas constituents Absorption Model

In addition to oxygen and water vapour, other atmospheric gases e.g. O$_3$, SO$_2$, NO$_2$, and N$_2$O etc. also have absorption lines in the microwave region. But their relative concentrations at sea level are so small that their contribution in microwave gaseous absorption spectrum is negligible in comparison to the contribution of oxygen and water vapour.

2.2.6 Available Absorption Models

Many researchers have developed the absorption models for various atmospheric gases for microwave region of EM spectrum. The details about some of the widely used absorption models have been given in the following sections:

2.2.6.1 MPM87 [Liebe and Layton (1987)]

The clear air absorption part of the Millimeter-wave Propagation Model, MPM87 includes 30 water vapour lines and 44 oxygen lines all in the range 20-1000 GHz, based on theoretical values and a Van-Vleck Weisskopf shape function. These are supplemented by an empirically derived water vapour continuum, fitted to laboratory observations at 138 GHz. However, these observations were limited to 282-316 K, and must be extrapolated for typical atmo-
spheric conditions. Additional terms represent the non-resonant absorption due to the Debye spectrum of oxygen below 10 GHz and the pressure-induced nitrogen absorption above 100 GHz, which can become a significant contribution to the overall absorption in low humidity.

2.2.6.2 MPM89 [Liebe (1989)]

The 1989 revision of MPM modified the parameters describing the 183 GHz water vapour line, fitting the pressure broadened line width with four parameters, instead of one. Other components are the same as MPM87.

2.2.6.3 MPM93 [Liebe et al. (1993)]

This version of MPM, has 34 water vapour lines between 20-1000 GHz, defined in a slightly different manner from MPM89. The 183 GHz line is 8.5% wider and 5% stronger than in MPM89. The water vapour continuum absorption is formulated as a pseudo-line near 2 THz, and has a different temperature dependence, based on newer observations. Like its predecessors, MPM93 includes 44 oxygen lines with the same line strengths, but 5% greater widths and 15% stronger mixing than MPM89. The non-resonant nitrogen absorption is essentially the same as MPM89 at the frequencies in this study.

2.2.6.4 Ros98 [Rosenkranz (1998)]

Ros98 uses 15 water vapour line parameters, which are very similar to the strongest lines used in MPM89. The other half of the lines have been omitted as they were judged to have negligible impact. Rosenkranz’s investigations suggested a range of observations could be best modeled by using a water vapour continuum with a combination of MPM87’s foreign-broadened component, and MPM93’s self-broadened component. However, the water vapour lines used
were truncated at ±750 GHz, so the foreign- and self- broadened parts of the water vapour continuum were increased 15% and 3%, respectively to compensate. This model uses the same oxygen line parameters as MPM93, except at sub-millimeter frequencies, where values from the HITRAN database were used. It also uses a different form of non-resonant absorption due to pressure broadening by nitrogen.

2.3 RT Models available

In the field of atmospheric remote sensing, there is an increasing tendency to use so-called physical retrieval methods which involves the RT computation for each retrieval. And RT models involve the transmission or absorption models of atmospheric gases. Hence, depending upon the different procedure of computation of transmission coefficients, RT models are broadly classified into two categories: line-by-line models and fast models. The details of these models are given in the following sections.

2.3.1 line-by-line RT model

In line-by-line models, the absorption for a single frequency is computed as described in the section 2.2.5 by considering the influence of all the lines in their data base. Because of this, it is computationally rather expensive for the calculation of radiance or brightness temperature. The situation becomes very problematic when one has to do the calculation repeatedly at a grid point to minimize the difference between simulated and observed radiance in physical retrieval methods of retrieving the atmospheric state.

Therefore, to speed up the calculation of absorption or transmission coefficient a number of approximations and parametrization has been made in the past.
The RT models incorporating the fast calculation of transmission coefficient are known as fast RT models. The brief description of fast model is given below.

### 2.3.2 Fast RT model

To generate a fast transmittance model, one generally requires accurate line-by-line transmission computation for a limited number of diverse atmospheric profiles. These profiles must cover the total dynamic ranges of the atmospheric variables so as to represent all possible atmospheric conditions. These transmission coefficient are then used with a set of predictors from atmospheric profile variables to compute regression coefficients. These regression coefficient can be used to compute transmission coefficient for any given atmospheric profile. More details about this fast predictor model can be found in [McMillin et al. (1979); Eyre and Woolf (1988); Eyre (1991)]. The examples of fast RT models available are RTTOV, PFAST etc. Since our study is concerned with microwave only, we will be discussing only microwave RT model.

### 2.3.3 Model Used in the Study

For our study, we have used a scattering based fast RT model for microwave frequencies developed by Liu (1998). In equation (2.8) the source function \( J_r \) consists of both thermal emission as well as scattering by the medium into the direction of propagation. The source function for scattering-free atmosphere have already been discussed in the section 2.2.2. For scattering atmosphere, the source function \( J_s(\hat{r}) \) accounts for radiation scattered in the direction \( \hat{r} \) in terms of radiation incident from all directions. If \( \hat{r}_i \) be the direction of incident radiation, then \( J_s(\hat{r}) \) can be expressed as

\[
J_s(\hat{r}) = \frac{1}{4\pi} \int \int_{4\pi} P(\hat{r}, \hat{r}') B(\hat{r}_i) d\omega_i
\]

(2.18)
where $B(r_i)$ is the brightness of radiation incident from the direction $\hat{r}_i$ and $P(\tau, r')$ is called the phase function. The phase function tells about the portion of energy scattered from direction $\hat{r}_i$ into $\hat{r}$.

Therefore, for a plane-parallel and azimuthally-symmetric scattering atmosphere with spherical particles (e.g., cloud and/or rain drops) the RTE can be expressed by (Tsang and Kong (1977)):

$$
\mu \frac{d}{d\tau} \begin{bmatrix} I_V(\tau, \mu) \\ I_H(\tau, \mu) \end{bmatrix} = \begin{bmatrix} I_V(\tau, \mu) \\ I_H(\tau, \mu) \end{bmatrix} - \frac{\omega_0}{2} \int_{-1}^{1} \begin{bmatrix} P_{VV} & P_{VH} \\ P_{HV} & P_{HH} \end{bmatrix} \begin{bmatrix} I_V(\tau, \mu') \\ I_H(\tau, \mu') \end{bmatrix} d\mu' 
- (1 - \omega_0) B(\tau) \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

(2.19)

where $I_p(\tau, \mu)$ is the radiance at optical depth $\tau$ ($\tau = 0$ at the top of the layer) in direction $\mu$ (the cosine of zenith angle) for polarization $p(H or V)$, $\omega_0$ is the single-scattering albedo, and $B(\tau)$ is Planck function at $\tau$. And $P_{pp}$ is the scattering phase function for polarization $p(H or V)$.

In order to solve equation (2.19), Liu (1998) used discrete ordinate method (DOM) with sufficient number of streams to compute scattering source term and calculate radiance or brightness temperature at any required direction. He also considered $B(\tau)$ as a linear function of $\tau$, i.e., $B(\tau) = B_0 + B_1 \tau$, where $B_0 = B(0)$ is the Planck function at the top of the layer. The DOM with sufficient number of streams has been proven to be an accurate and stable way to solve RT problem (Liou (1973); Stamnes and Swanson (1981)). However, this procedure is computationally intensive, primarily because (i) the phase functions are expressed as summations of infinite number (in practice a large number) of terms and need to be calculated for many directions, and (ii) the eigenvalue problem for general solutions of equation (2.19) needs to be solved numerically for stream number 4 (Liou (1973); Stamnes and Swanson (1981)). Hence, to develop a fast and accurate microwave RT model, Liu used two approximations. First, cross polarization ($P_{VH} and P_{HV}$) scattering in the scatter-
ing source term is assumed to be negligible and the scattering phase function can be expressed by Henyey and Greenstein (HG) formula which can be expanded by Legendre polynomials as

\[ P(\mu, \mu') = \sum_l A_l p_l(\mu) p_l(\mu') \]  

(2.20)

where \( A_l = (2l + 1)g^l \) and \( g^l \) is asymmetry factor as given by Bohren and Huffman (1983), \( p_l(\mu) \) is the \( l^{th} \) order Legendre polynomial.

The formal solution of equation (2.19) without polarization can be given by (Stamnes and Swanson (1981)):

\[ I(\tau, \pm \mu) = I(\tau^*, \pm \mu) e^{-(\tau^*-\tau)/\mu} + \int_{\tau}^{\tau^*} J(t, \pm \mu) e^{-(t-\tau)/\mu} \frac{dt}{\tau} \]  

(2.21)

\[ I(\tau, -\mu) = I(0, +\mu) e^{-\tau/\mu} + \int_0^\tau J(t, -\mu) e^{-(\tau-t)/\mu} \frac{dt}{\tau} \]  

(2.22)

where \( \mu \) and \(-\mu\) represent upward and downward directions, respectively. \( \tau^* \) is the optical depth of the layer. The source term \( J(\tau, \mu) \) with the discrete ordinate approximation is

\[ J(\tau, \mu) = \frac{1}{2\omega_0} \sum_{l=0}^{2n-1} A_l p_l(\mu) \sum_{j=-n}^{n} a_j p_l(\mu_j) I(\tau, \mu_j) + (1 - \omega_0) B(\tau) \]  

(2.23)

where \( 2n \) is the stream number in discrete ordinate approximation and \( a_j \) is the quadrature weight for \( j^{th} \) quadrature point.

Now, for 4-stream approximation i.e., \( n=2 \), the solution at \( i^{th} \) quadrature point can be given as (Liou (1973))
Chapter 2: Radiative Transfer Theory

\[ I(\tau, \mu_i) = \sum_{j=-n}^{n} L_j W_j(\mu_i)e^{-k_j \tau} + B_1 \tau \quad (2.24) \]

where \( k_j \) and \( W_j(\mu_i) \) are eigenvalue and eigenvector, and \( L_j \) can be determined from continuity of radiance between layers and boundary condition. \( q(\mu_i) \) could be calculated by solving:

\[ \sum_j b_{ij} q(\mu_j) = -(1 - \omega_0)B_0/\mu_i - B_1 \quad (2.25) \]

where \( b_{ij} \)s have the same definition as those in Liou (1992). Substituting 2.24 into 2.23 yields

\[ J(\tau, \mu) = \sum_{j=-n}^{n} (1 + k_j \mu)L_j W_j(\mu)e^{-k_j \tau} + Z_0 + B_1 \tau \quad (2.26) \]

where

\[ W_i(\mu) = \frac{1}{2}\omega_0 \sum_{l=0}^{2n-1} A_l p_l(\mu) \sum_{j=-n}^{n} a_j p_l(\mu_j)W_i(\mu_j), \quad (2.27) \]

\[ Z_0 = \frac{1}{2}\omega_0 \sum_{l=0}^{2n-1} A_l p_l(\mu) \sum_{j=-n}^{n} a_j p_l(\mu_j)q(\mu_j) + (1 - \omega_0)B_0 \quad (2.28) \]

Now substituting equation (2.26) into equations (2.21) and (2.22) and performing the integral operations, the upward and downward radiances at direction \( \mu \) can be given as

\[ I(\tau, +\mu) = I(\tau^*, +\mu)e^{-(\tau^*-\tau)/\mu} + \sum_{j=-n}^{n} L_j W_j(\mu)(e^{-k_j \tau} - e^{-[k_j \tau^* + (\tau^*-\tau)/\mu]}) \\
+ Z_0(1 - e^{-(\tau^*-\tau)/\mu}) - B_1[(\tau^* - \tau) - \mu(1 - e^{-(\tau^*-\tau)/\mu})] \quad (2.29) \]
\[ I(\tau, -\mu) = I(0, -\mu)e^{-\tau/\mu} + \sum_{j=-n}^{n} L_j W_j(-\mu)(e^{-k_j \tau} - e^{-\tau/\mu}) + Z_0(1 - e^{-\tau/\mu}) - B_1[\tau - \mu(1 - e^{-\tau/\mu})] \] (2.30)

For numerical calculations the whole atmosphere is divided into many layers and it is assumed that all micro-physical properties (e.g., particle concentration, liquid/ice water content, etc.) are uniform within each layer, but temperature varies linearly with optical depth. \( L_j \)'s in 2.24 are first computed by solving a linear equation system that is resulted from applying boundary conditions and the continuity of radiance between layers to 2.24. Thus, the radiances at quadrature angles \( (\mu_q) \) are obtained. Now, the radiance at any desired angle \( \mu \) can be calculated from eqs. 2.29 and 2.30. Since the downward radiation from space is known (3 K), the calculation of downward radiation starts from the top layer using 2.30. The downward radiation at the bottom of the layer can be calculated by putting \( \tau = \tau^* \) in eq. 2.30 given the downward radiance at the top of the layer. Using the continuity of radiation at the boundary of neighbored layers, this operation continues until the downward radiation at the surface is derived. A boundary condition is required to calculate the upward radiation at surface given the surface emissivity and temperature. A specular boundary is used in this RT model. The same analogy of calculating downward radiation from top layer to surface by 2.30, upward radiation can be continuously calculated from the bottom of a layer to the top of a layer using 2.29 and letting \( \tau = 0 \) until the radiance at the top of the layer is solved. The ‘satellite observed’ radiance at the top of the atmosphere is then obtained. Brightness temperature can be calculated from radiance using Planck function. Horizontally- and vertically polarized radiances or brightness temperatures are calculated separately due to different surface emissivity.
2.4 Summary of RT Modeling

This chapter described the RTM, which is the back bone of the forward model used to transform from state space to observation space, i.e. to obtain radiance or brightness temperature at the top of the atmosphere corresponding to the given atmospheric state. This is required for all physical retrieval methods, along with estimates of its uncertainties. In absence of satellite measurements RTM is also needed to simulate the radiances to make the training data set for statistical regression based retrieval methods.

The RTE for scattering-free atmosphere and brightness temperature were first defined. Then the emission mechanism for various atmospheric gases like oxygen, water vapour etc. were discussed to provide a theoretical basis of RT modeling for microwave. Finally, the solution of scattering based microwave RTM using discrete ordinate method was discussed in details. The absorption model of Rosenkranz (1998) was used in this microwave RTM.