Chapter 1

Preliminaries

In this chapter we collect the basic definitions and theorems which are needed for the subsequent chapters. For graph theoretic terminology, we refer to [1, 10, 11].

Definition 1.1 A graph $G$ is a finite non-empty set of objects called vertices together with a set of unordered pairs of distinct vertices of $G$, called edges. The vertex set and the edge set of $G$ are denoted by $V(G)$ or simply $V$ and $E(G)$ or simply $E$ respectively.

The number of vertices in $G$, denoted by $n$, is called the order of $G$, while the number of edges in $G$, denoted by $m$, is called the size of $G$. A graph of order $n$ and size $m$ is called a $(n, m)$-graph.

If $e = \{u, v\}$ is an edge of a graph $G$, written $e = uv$, we say that $e$ joins the vertices $u$ and $v$; $u$ and $v$ are adjacent vertices; $u$ and $v$ are incident with $e$.

If two vertices are not joined, then we say that they are non-adjacent. If two distinct edges $e$ and $f$ are incident with a common vertex $v$, then $e$ and $f$ are said to be adjacent to each other.

A set of vertices in a graph is independent if no two vertices in the set are adjacent.
Similarly, a set of edges in a graph is *independent* if no two edges in the set are adjacent.

If two or more edges join the same pair of (distinct) vertices, then these edges are called *parallel edges*. If an edge $e$ joins a vertex $v$ to itself, then $e$ is called to be a *loop*. A graph $G$ without loops and parallel edges is called a *simple graph*.

**Definition 1.2** The *degree* of a vertex $v$ in a graph $G$ is the number of edges of $G$ incident with $v$ and is denoted by $\deg_G(v)$ or $\deg(v)$.

A vertex of degree 0 in $G$ is called an *isolated vertex* and a vertex of degree 1 is called a *pendant vertex* or an *end vertex* of $G$.

A graph is said to be *$k$-regular* if every vertex of $G$ has degree $k$.

**Definition 1.3** A graph $H$ is called a *subgraph* of $G$, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then $H$ is a *proper subgraph* of $G$.

A *spanning subgraph* of $G$ is a subgraph $H$ with $V(H) = V(G)$.

For any set $S$ of vertices of $G$, the *induced subgraph* $< S >$ is the maximal subgraph of $G$ with vertex set $S$. Thus two vertices of $S$ are adjacent in $< S >$ if and only if they are adjacent in $G$.

Let $v$ be a vertex of a graph $G$. The induced subgraph $< V(G) - \{v\} >$ is denoted by $G - v$; it is the subgraph of $G$ obtained by the removal of $v$ and edges incident with $v$. Similarly, if $e$ is an edge of a graph $G$, then $G - e$ is the subgraph of $G$ having the same vertex set as $G$ and whose edge set consists of all edges of $G$ except $e$. 
**Definition 1.4** A graph $G$ is *complete* if every two distinct vertices of $G$ are adjacent. A complete graph of order $n$ is denoted by $K_n$.

**Definition 1.5** A *bipartite graph* $G$ is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with $V_2$; $(V_1, V_2)$ is called a *bipartition* of $G$. If $G$ contains every edge joining $V_1$ and $V_2$, then $G$ is called a *complete bipartite graph*. The complete bipartite graph with bipartition $(V_1, V_2)$ such that $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r,s}$.

A *star* is a complete bipartite graph $K_{1,s}$.

**Definition 1.6** Let $u$ and $v$ be vertices of a graph $G$. A $u-v$ *walk* of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, e_n, u_n = v$ of vertices and edges in $G$ beginning with vertex $u$ and ending with vertex $v$ such that $e_i = u_{i-1}u_i$, $i = 1, 2, \ldots, n$. The number $n$ is called the *length* of the walk. The walk is said to be *open* if $u$ and $v$ are distinct vertices; it is *closed* otherwise. A walk $u_0, e_1, u_1, e_2, u_2, \ldots, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_n$ of its vertices and hence we specify this walk by $W : u_0, u_1, u_2, \ldots, u_n$.

A walk in which all the vertices are distinct is called a *path*. A closed walk $u_0, u_1, u_2, \ldots, u_n$ in which $u_0, u_1, u_2, \ldots, u_{n-1}$ are distinct is called a *cycle*. A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. Given a path $P$ in a graph $G$ and two vertices $x, y$ on $P$, we use $P[x, y]$ to denote the portion of $P$ between $x$ and $y$, inclusive of $x$, and $y$. 

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Definition 1.7  A graph $G$ is said to be *connected* if any two distinct vertices of $G$ are joined by a path. A maximal connected subgraph of $G$ is called a *component* of $G$.

Definition 1.8  A *cut-vertex* of a graph $G$ is a vertex whose removal increases the number of components. A *non-separable graph* is connected, non-trivial and has no cut-vertices.

A *block* of a graph is a maximal non-separable subgraph. A graph in which each block is complete is called a *block graph*.

For a cut-vertex $v$ in a connected graph $G$ and a component $H$ of $G - v$, the subgraph $H$ and the vertex $v$ together with all edges joining $v$ and $V(H)$ in $G$ is called a *branch* of $G$ at $v$. An end-block of $G$ is a block containing exactly one cut-vertex of $G$. Thus every end-block is a branch of $G$.

Theorem 1.9  [11] A vertex $v$ of a connected graph $G$ is a cut-vertex of $G$ if and only if there exists vertices $u$ and $w$ distinct from $v$ such that $v$ lies on every $u - w$ path of $G$.

Definition 1.10  For a vertex $v$ in a connected graph $G$, $N(v)$ denotes the set of all neighbors of $v$, and $N[v] = N(v) \cup \{v\}$.

A vertex $v$ in $G$ is an *extreme vertex* if the subgraph induced by $N(v)$ is complete.
Definition 1.11  A graph $G$ is called acyclic if it has no cycles. A connected acyclic graph is called a tree. A non-trivial path is a tree with exactly two end vertices. A graph $G$ with exactly one cycle is called a unicyclic graph.

A caterpillar is a tree of order 3 or more, for which the removal of all end-vertices leaves a path.

Definition 1.12  For vertices $u$ and $v$ in a connected graph $G$, the distance $d(u, v)$ is the length of a shortest $u$–$v$ path in $G$. A $u$–$v$ path of length $d(u, v)$ is called a $u$–$v$ geodesic.

The eccentricity $e(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ and a vertex of $G$. The minimum eccentricity among the vertices of $G$ is the radius, $\text{rad } G$ and the maximum eccentricity is its diameter, $\text{diam } G$ of $G$.

It is known that the distance $d$ in a connected graph $G$ is a metric on the vertex set of $G$. It is also known that the radius and diameter of a connected graph $G$ satisfy the relation $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$. It is proved in [15] that given positive integers $a, b$ such that $a \leq b \leq 2a$, there is a connected graph $G$ whose radius is $a$ and diameter is $b$.

Two vertices $u$ and $v$ of $G$ are antipodal if $d(u, v) = \text{diam } G$. A vertex $v$ is a peripheral vertex of $G$ if $e(v) = \text{diam } G$. A double star is a tree of diameter 3.

Theorem 1.13  [15] For every connected graph $G$, $\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$.

Definition 1.14  For vertices $u$ and $v$ in a connected graph $G$, the closed interval $I[u, v]$ consists of all vertices lying on some $u$ - $v$ geodesic of $G$, while for $S \subseteq V$,
$I[S] = \bigcup_{u,v \in S} I[u,v]$. A set $S$ of vertices of $G$ is a geodetic set if $I[S] = V$, and a geodetic set of minimum cardinality is a minimum geodetic set or a $g$-set of $G$. The cardinality of a minimum geodetic set of $G$ is the geodetic number $g(G)$ of $G$.

For the graph $G$ given in Figure 1.1, no 2-element subset is a geodetic set. The set $S_1 = \{v_1, v_2, v_6\}$ is a geodetic set of $G$. Also, $S_2 = \{v_1, v_3, v_6\}$ and $S_3 = \{v_2, v_4, v_6\}$ are minimum geodetic sets of $G$ so that $g(G) = 3$. This example shows that there can be more than one minimum geodetic set for a graph.

![Figure 1.1](image_url)

Note that $I(u, v)$ consists of all vertices lying on some $u - v$ geodesic of $G$ except $u$ and $v$.

The geodetic number of a graph was introduced in [1, 12] and further studied in [2, 7]. It was shown in [12] that determining the geodetic number of a graph is an NP-hard problem.
Theorem 1.15  [5] Each extreme vertex of a connected graph $G$ belongs to every geodetic set of $G$. In particular, if the set of all extreme vertices $S$ is a geodetic set of $G$ then $S$ is the unique $g$-set of $G$.

Theorem 1.16  [5] Let $G$ be a connected graph with a cut-vertex $v$. Then every geodetic set of $G$ contains at least one vertex from each component of $G - v$.

Theorem 1.17  [7] For the complete graph $K_n$, $g(K_n) = n$ if and only if $G = K_n$.

Theorem 1.18  [7] For any tree $T$, the geodetic number $g(T)$ equals the number of end vertices in $T$. In fact, the set of all end vertices of $T$ is the unique minimum geodetic set of $T$.

Theorem 1.19  [1] For a connected graph $G$, $g(G) = 2$ if and only if there exist peripheral vertices $u$ and $v$ such that every vertex of $G$ is on a diametral path joining $u$ and $v$.

Theorem 1.20  [1] For a connected graph $G$, no cut-vertex belongs to any $g$-set of $G$.

Theorem 1.21  [7] For the complete bipartite graph $K_{r,s}$ ($r, s \geq 2$), $g(K_{r,s}) = \min\{r, s, 4\}$.

Definition 1.22  [7] A connected geodetic set of $G$ is a geodetic set $S$ such that the subgraph $< S >$ induced by $S$ is connected. The minimum cardinality of a connected geodetic set of $G$ is the connected geodetic number of $G$ and is denoted by $g_c(G)$. A
connected geodetic set of cardinality $g_c(G)$ is called a $g_c$-set of $G$.

For the graph $G$ in Figure 1.2, $S = \{v_1, v_2, v_3\}$ is the unique minimum geodetic set of $G$ so that $g(G) = 3$. Since, the induced subgraph $< S >$ is not connected, $S$ is not a connected geodetic set of $G$. It is clear that $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is a minimum connected geodetic set of $G$ and so $g_c(G) = 5$.

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**Theorem 1.23** [14] For a connected graph $G$, $g_c(G) \geq 1 + \text{diam } G$.

**Theorem 1.24** [16] For any non-trivial tree $T$ of order $n$, $g_c(T) = n$.

**Theorem 1.25** [16] For any connected graph $G$ of order $n \geq 2$, $g_c(G) = 2$ if and only if $G = K_2$.

**Theorem 1.26** [16] Every cut-vertex of a connected graph $G$ belongs to every connected geodetic set of $G$. 
**Definition 1.27**  Let $S$ be a minimum geodetic set of $G$. A subset $T$ of $S$ is called a **forcing subset** for $S$ if $S$ is the unique minimum geodetic set containing $T$. A forcing subset for $S$ of minimum cardinality is a **minimum forcing subset** of $S$. The **forcing geodetic number** of $S$, denoted by $f(S)$, is the cardinality of a minimum forcing subset of $S$. The **forcing geodetic number** of $G$, denoted by $f(G)$, is $f(G) = \min\{f(S)\}$, where the minimum is taken over all minimum geodetic sets $S$ in $G$.

![Figure 1.3](image_url)

For the graph $G$ given in Figure 1.3, $S_1 = \{v_1, v_6, v_7\}$ and $S_2 = \{v_1, v_6, v_8\}$ are the only two minimum geodetic sets of $G$. It is clear that $f(S_1) = f(S_2) = 1$ so that $f(G) = 1$.

The forcing geodetic number of a graph was introduced and studied in [3].

Throughout the thesis, $G$ denotes a connected graph with at least two vertices.