Chapter 2

Set-Indexers of Certain Graphs

The set-indexers of certain graphs are studied in this chapter. The set-indexing numbers of all stars and paths have been computed. Further, the set-indexing numbers of certain classes of cycles and complete $k$-partite graphs have been found. In addition, the set-indexing numbers of all fans are completely determined. Subsequently, the set-indexing numbers of certain related graphs like helms, wheels and suns have also been derived.

2.1 Stars and Related Graphs

This section focusses mainly on the set-indexing numbers of stars and their complements in relation with that of complete graphs. It is observed that among graphs of a given order, star graphs are of smallest set-indexing number.
Theorem 2.1.1. If \( f \) is a set-indexer of a graph \( G \) with indexing set \( X \), then the function \( g \) defined by \( g(v) = X \setminus f(v); \ v \in V \) is also a set-indexer of \( G \) with indexing set \( X \). Moreover \( f(E) = g(E) \).

Proof. Since \( f \) is a set-indexer, \( g(v) = X \setminus f(v); \ v \in V \) are all distinct. Clearly, \[
g(u, v) = (X \setminus f(u)) \oplus (X \setminus f(v)) = ((X \setminus f(u)) \cup (X \setminus f(v))) \setminus (X \setminus f(u)) \cap (X \setminus f(v)) = f(u) \oplus f(v) = f(u, v).
\]

Definition 2.1.2. Let \( f \) be a set-indexer of a graph \( G \) with indexing set \( X \). Then the set-indexer \( g \) defined by \( g(v) = X \setminus f(v); \ v \in V \) is called the dual set-indexer of \( f \) and is denoted by \( f^d \).

Corollary 2.1.3. Let \( G \) be any graph and \( \Theta'_X(G) \) denotes the set of all optimal set-indexers \( f \) of \( G \) with respect to a set \( X \) such that \( f(u) = X \) for some \( u \in V \). Then \( \Theta'_X(G) \) is non-empty.

Proof. Let \( \Theta_X(G) \) denote the set of all optimal set-indexers \( f \) of \( G \) with respect to a set \( X \) such that \( f(u) = \emptyset \) for some \( u \in V \). Then by theorem 1.0.12, \( \Theta_X(G) \neq \emptyset \) and let \( g \in \Theta_X(G) \). Now, the dual set-indexer \( g^d \) of \( g \) belongs to \( \Theta'_X(G) \) so that \( \Theta'_X(G) \neq \emptyset \).

The set-indexing number of any star graph is determined below:

Theorem 2.1.4. If \( G \) is a star graph, then \( \gamma(G) = \lfloor \log_2 o(G) \rfloor \). 


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Proof. By theorem 1.0.8, the result is true for $K_{1,1}$. Now consider $K_{1,n}$; $n \geq 2$. Then there exists $m$ such that $2^m \leq n < 2^{m+1}$. By theorem 1.0.5, $\gamma(K_{1,n}) \geq \lceil \log_2(|E|+1) \rceil = \lceil \log_2(n+1) \rceil = m + 1$. Let $V = \{v_0, \ldots, v_n\}$; $d(v_0) = n$. Employing the following steps a set-indexer $f$ of $K_{1,n}$ with $X = \{x_1, \ldots, x_{m+1}\}$ as the indexing set can be obtained. Assign $\emptyset$ to the vertex $v_0$. Now there remains $n$ vertices and $2^{m+1} - 1$ elements in $2^X$. But, we have $2^m \leq n < 2^{m+1}$. Consequently, these $n$ vertices $v_1, \ldots, v_n$ can be assigned to the distinct nonempty subsets of $X$. Clearly, $f(v_0, v_i) = f(v_i)$ for $1 \leq i \leq n$ and all the vertex labels are distinct. So $f$ is a set-indexer of $G$ with indexing set $X$ and $|X| = m + 1$. Therefore, $\gamma(K_{1,n}) = (m + 1) = \lceil \log_2(n+1) \rceil$.

Corollary 2.1.5. If $G$ is a connected graph of order $n+1$, then $\gamma(G) \geq \gamma(K_{1,n})$.

Proof. For any natural number $n$, there exists an integer $m$ such that $2^m \leq n < 2^{m+1}$. Then by theorem 2.1.4, $\gamma(K_{1,n}) = m + 1$. But by theorem 1.0.5, $\gamma(G) \geq \lceil \log_2(|E|+1) \rceil \geq \lceil \log_2(n+1) \rceil = (m + 1) = \gamma(K_{1,n})$.

Though simple, the following is a useful result.

Theorem 2.1.6. $\gamma(K_{1,n}) = \gamma(N_{n+1})$.

Proof. Given any positive integer $n$, there exists an integer $m$ such that $2^m \leq n < 2^{m+1}$. The indexing set corresponding to $\gamma(N_{n+1})$ should contain at least $m+1$ elements so that $\gamma(N_{n+1}) \geq m + 1$. Let $X = \{x_1, \ldots, x_{m+1}\}$. Since $n + 1 \leq 2^{m+1}$, distinct elements of $2^X$ can be assigned to the distinct vertices of $N_{n+1}$ to get a set-indexer for $N_{n+1}$ with indexing set $X$. Consequently,
$\gamma(N_{n+1}) = n + 1$. Now, the desired result follows from theorem 2.1.4.

It is already seen that if $G$ is a connected graph of order $n \geq 2$, then $\gamma(G) \geq \gamma(K_{1,n-1})$. But even if $G$ is disconnected, this inequality holds as is seen in the following corollary:

**Corollary 2.1.7.** If $G$ is a graph of order $n \geq 2$, then $\gamma(G) \geq \gamma(K_{1,n-1})$.

**Proof.** By theorem 2.1.6, $\gamma(K_{1,n-1}) = \gamma(N_n)$. But $N_n$ is a subgraph of $G$ and therefore by theorem 1.0.6, $\gamma(N_n) \leq \gamma(G)$ so that $\gamma(K_{1,n-1}) \leq \gamma(G)$. □

**Remark 2.1.8.** There are graphs for which equality as well as strict inequality holds in the above result. For instance, $\gamma(K_{1,2}) = 2 = \gamma(K_3)$ whereas $2 = \gamma(K_{1,3}) < \gamma(K_4) = 3$.

**Remark 2.1.9.** In general, $\gamma(K_n) \leq \gamma(K_{1,n})$ by theorem 1.0.6. The following theorem characterizes the situation of equality.

**Theorem 2.1.10.** $\gamma(K_{1,n}) > \gamma(K_n)$ if $n < 3$ and $\gamma(K_{1,n}) = \gamma(K_n)$ for $n \geq 3$.

**Proof.** It can be easily noted that $\gamma(K_{1,1}) = 1$ and $\gamma(K_1) = 0$. Also $\gamma(K_{1,2}) = 2$ and $\gamma(K_2) = 1$ and further, $\gamma(K_{1,3}) = 2$ and $\gamma(K_3) = 2$. Now suppose $n > 3$. Then we have $|V(K_n)| < |E(K_n)|$.

Let $f$ be any set-indexer of $K_n$ and let $A = f(V(K_n))$ and $B = f(E(K_n))$. Clearly, $|A| < |B|$. Now, by assigning an element of the nonempty set $B \setminus A$ to the isolated vertex of $K_{1,n}$ we can extend $f$ to be a set-indexer of $K_{1,n}$. Consequently, $\gamma(K_{1,n}) \leq \gamma(K_n)$. Now, the result follows from theorem 1.0.6. □
Theorem 2.1.11. For any graph \( G \), \( \lceil \log_2 |V| \rceil \leq \gamma(G) \leq \gamma(G \vee K_1) \leq \gamma(G) + 1 \).

Proof. Let \( G \) be of order \( n \). Then by applying theorem 1.0.5 and corollary 2.1.7, \( \gamma(G) \geq \gamma(K_{1,n-1}) \geq \lceil \log_2(|E(K_{1,n-1})| + 1) \rceil = \lceil \log_2(n - 1 + 1) \rceil = \lceil \log_2(n) \rceil = \lceil \log_2 |V| \rceil \). The other inequalities are obvious.

The following corollary which supplies upper bounds for the set-indexing numbers of complete graphs recursively, is an easy consequence of the above theorem.

Corollary 2.1.12. \( \gamma(K_{n+1}) \leq \gamma(K_n) + 1 \).

The line graph of \( K_{1,n} \) is \( K_n \). The natural question, does there exist any relationship between their set-indexing numbers, leads us to observe that \( \gamma(K_n) \leq \gamma(K_{1,n}) \) for \( n < 5 \). But, for \( n \geq 5 \), the situation changes the other way as shown in the following:

Theorem 2.1.13. \( \gamma(K_n) > \gamma(K_{1,n}); n \geq 5 \).

Proof. The proof is by induction on \( n \).

When \( n = 5, 6, 7, 8 \), the result is true by theorem 1.0.8, theorem 2.1.4 and theorem 2.1.11.

Now suppose \( \gamma(K_m) > \gamma(K_{1,m}) \) where \( m \geq 8 \). Let \( \gamma(K_{1,m}) = l \) and \( \gamma(K_m) = k \). Then by theorem 2.1.4, \( l \geq 4 \).

Again by theorem 2.1.4, \( l = \lceil \log_2 (m + 1) \rceil \)

\[ \Rightarrow 2^{l-1} + 1 \leq m + 1 \]
\[ \Rightarrow 2^{l-1} \leq m \]
\[ \Rightarrow 2^{l-2} \leq \frac{m}{2} \]
\[ \Rightarrow 2^{l-2} (m - 1) \leq \frac{m}{2} (m - 1) \]
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\[ 2^{\frac{n-2}{2}}(2^{l-1} - 1) \leq \frac{m}{2}(m-1), \text{ since } 2^l - 1 \leq m < 2^l. \]

\[ \Rightarrow \log_2(2^{\frac{n-2}{2}}(2^{l-1} - 1)) \leq \log_2(\frac{m}{2}(m-1)) \]

\[ \Rightarrow [\log_2(2^{\frac{n-2}{2}}(2^{l-1} - 1))] \leq [\log_2(\frac{m}{2}(m-1))] \]

\[ \Rightarrow 2l - 3 \leq \gamma(K_m), \text{ by theorem 1.0.5} \]

\[ \Rightarrow 2l - 3 \leq \gamma(K_m), \text{ since } l - 3 \geq 1. \]

Thus, the result is true for \( n = m + 1 \) also. \( \square \)

By combining theorem 2.1.10 and theorem 2.1.13, \( \gamma(K_{i,n}^c) > \gamma(K_{1,n}); n \geq 5. \) But for \( n < 5, \) it can be easily noted that \( \gamma(K_{i,n}^c) = \gamma(K_{1,n}). \) We record it in the following:

**Theorem 2.1.14.** \( \gamma(K_{i,n}^c) = \gamma(K_{1,n}) \) if \( n \leq 4 \) and \( \gamma(K_{i,n}^c) > \gamma(K_{1,n}) \) otherwise.

**Corollary 2.1.15.** \( \gamma(K_{n+1}) < \gamma(K_{1,n}) + \gamma(K_{i,n}^c) \leq 2\gamma(K_{i,n}^c). \)

**Proof.** The result is trivial for \( n = 1. \) So assume that \( n \geq 2. \) By corollary 2.1.12,

\[ \gamma(K_{n+1}) \leq \gamma(K_n) + 1 \]

\[ \leq \gamma(K_{i,n}^c) + 1, \text{ by theorem 1.0.6} \]

\[ < \gamma(K_{i,n}^c) + \gamma(K_{1,n}), \text{ by theorem 2.1.4}. \]

The other inequality follows from theorem 2.1.14. \( \square \)

In [3], B. D. Acharya obtained the set-indexing number of \( K_n \) for \( n \leq 7. \) We have computed this number for \( K_n; 8 \leq n \leq 15. \)

The results are stated in the following theorem.

**Theorem 2.1.16.** \( \gamma(K_n) = \begin{cases} 
6 & \text{if } 8 \leq n \leq 9 \\
7 & \text{if } 10 \leq n \leq 12 \\
8 & \text{if } 13 \leq n \leq 15
\end{cases} \)
Remark 2.1.17. Evidently, if $|V| \geq 2^n$, then $\gamma(G) \geq n$. But $|V| = 2^n \Rightarrow \gamma(G) = n$. For example, $|V(K_4)| = 2^2$ but $\gamma(K_4) = 3$. However, there are graphs $G$ such that $|V| = 2^n$ and $\gamma(G) = n$. For instance, $|V(K_{1,3})| = 2^2$ but $\gamma(K_{1,3}) = 2$.

2.2 Complete K-Partite Graphs

The set-indexing numbers of certain classes of complete $k$-partite graphs for $k = 2, 3, 4, 5$ are discussed here.

In the following two theorems, the set-indexing numbers of two different families of complete bipartite graphs have been computed.

Theorem 2.2.1. $\gamma(K_{2^m-1,n}) = m + l; m \geq l = \lceil \log_2 n \rceil$ and $m > 1$.

Proof. By theorem 1.0.5, $\gamma(K_{2^m-1,n}) \geq \lceil \log_2 n (2^m - 1) + 1 \rceil = m + l$. Let $V = \{u_1, \ldots, u_{2^m-1}, v_1, \ldots, v_n\}$ with $d(u_i) = n$ for $1 \leq i \leq 2^m-1$ and $d(v_j) = 2^m - 1$ for $1 \leq j \leq n$. Now consider the sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_l\}$. Define a set-indexer $f$ of $K_{2^m-1,n}$ with $X \cup Y$ as the indexing set as follows: Assign $2^m - 1$ distinct nonempty subsets of $X$ to the vertices $u_1, \ldots, u_{2^m-1}$ and $n$ distinct subsets of $Y$ to the vertices $v_1, \ldots, v_n$ so that all the vertex labels are distinct. Then $f(u_i, v_j) = f(u_i) \oplus f(v_j) = f(u_i) \cup f(v_j)$ for $1 \leq i \leq 2^m - 1$ and $1 \leq j \leq n$. Consequently, all the edge labels are also distinct. Hence, $f$ is a set-indexer of $K_{2^m-1,n}$ with $X \cup Y$ as the indexing set, as desired. \qed

Theorem 2.2.2. $\gamma(K_{2^m,n}) = m + l; l = \lfloor \log_2 n \rfloor + 1$. 
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**Proof.** By theorem 1.0.5, \( \gamma(K_{2^m,n}) \geq \lceil \log_2(n2^m + 1) \rceil \geq m + l \), since \( 2^{l-1} \leq n < 2^l \). Let \( V = \{u_1, \ldots, u_{2^m}, v_1, \ldots, v_n\} \) with \( d(u_i) = n \) for \( 1 \leq i \leq 2^m \) and \( d(v_j) = 2^m \) for \( 1 \leq j \leq n \). Now consider the sets \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_l\} \). A set-indexer \( f \) of \( K_{2^m,n} \) with \( X \cup Y \) as the indexing set can be found as follows: Assign \( 2^m \) distinct subsets of \( X \) to the vertices \( u_1, \ldots, u_{2^m} \) and \( n \) distinct nonempty subsets of \( Y \) to the vertices \( v_1, \ldots, v_n \) in any order. Then, \( f(u_i, v_j) = f(u_i) \oplus f(v_j) = f(u_i) \cup f(v_j) \) for \( 1 \leq i \leq 2^m \) and \( 1 \leq j \leq n \). Since all the vertex labels as well as the edge labels are distinct, \( f \) is a set-indexer of \( K_{2^m,n} \) with \( X \cup Y \) as the indexing set. Since \( \gamma(K_{2^m,n}) \geq m + l \) where \( 2^{l-1} \leq n < 2^l \), the result follows.

For the remaining complete bipartite graphs, an upperbound for the set-indexing numbers is derived:

**Theorem 2.2.3.** If \( m \) is not a power of \( 2 \), then \( \gamma(K_{m,n}) \leq \lceil \log_2 m \rceil + \lceil \log_2 n \rceil \).

**Proof.** Let \( V = \{u_1, \ldots, u_m, v_1, \ldots, v_n\} \); \( d(u_i) = n \) for \( 1 \leq i \leq m \) and \( d(v_j) = m \) for \( 1 \leq j \leq n \). Consider the sets \( X = \{x_1, \ldots, x_p\} \) and \( Y = \{y_1, \ldots, y_q\} \) where \( p = \lceil \log_2 m \rceil \) and \( q = \lceil \log_2 n \rceil \). Assigning any \( m \) distinct nonempty subsets of \( X \) to the vertices \( u_1, \ldots, u_m \) and any \( n \) distinct subsets of \( Y \) to the vertices \( v_1, \ldots, v_n \) in any order we get a set-indexer of \( K_{m,n} \) with indexing set \( X \cup Y \). Consequently, \( \gamma(K_{m,n}) \leq p + q \).

**Remark 2.2.4.** If both \( m \) and \( n \) are powers of \( 2 \), the above upper bound need not be true. For example, take \( K_{m,n} = K_{4,16} \). Then, \( 6 = \lceil \log_2 m \rceil + \lceil \log_2 n \rceil < \gamma(K_{m,n}) = 7 \).

**Theorem 2.2.5.** \( \gamma(K_{1,2^{m-1},n}) = m + l; l = \lceil \log_2 n \rceil + 1 \).
Proof. By theorem 1.0.5, \( \gamma(K_{1,2^m - 1,n}) \geq \lceil \log_2 2^m(n + 1) \rceil = m + l \), since \( 2^{l-1} \leq n < 2^l \)

Let \( V = \{u, v_1, \ldots, v_{2^m - 1}, w_1, \ldots, w_n\} \) with \( d(u) = 2^m + n - 1 \), \( d(v_i) = n + 1 \) for \( 1 \leq i \leq 2^m - 1 \) and \( d(w_j) = 2^m \) for \( 1 \leq j \leq n \).
Now consider the sets \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_l\} \). The following steps provide a set-indexer \( f \) of \( K_{1,2^m - 1,n} \) with \( X \cup Y \) as the indexing set: Assign \( 2^m - 1 \) distinct nonempty subsets of \( X \) to the vertices \( v_1, \ldots, v_{2^m - 1} \) and the \( n \) distinct nonempty subsets of \( Y \) to the vertices \( w_1, \ldots, w_n \). Finally, assign the empty set to the vertex \( u \). Then,

\[
\begin{align*}
f(u, v_i) &= f(u) \oplus f(v_i) = f(v_i); 1 \leq i \leq 2^m - 1 \\
f(u, w_j) &= f(u) \oplus f(w_j) = f(w_j); 1 \leq j \leq n \text{ and} \\
f(v_i, w_j) &= f(v_i) \oplus f(w_j) = f(v_i) \cup f(w_j) \text{ for } 1 \leq i \leq 2^m - 1 \text{ and } 1 \leq j \leq n.
\end{align*}
\]

Clearly, this defines a set-indexer \( f \) of \( K_{1,2^m - 1,n} \) with \( X \cup Y \) as the indexing set. Now the result follows from (*). \( \square \)

Theorem 2.2.6. \( \gamma(K_{1,1,2^m-1,2^m-1}) = m + n + 1. \)

Proof. By theorem 1.0.5, \( \gamma(K_{1,1,2^m-1,2^m-1}) \geq \lceil \log_2(|E| + 1) \rceil \geq \lceil \log_2(2^{m-n} + 2^m + 2^n - 1) \rceil = m + n + 1. \) Let \( V = \{u_1, u_2, v_1, \ldots, v_{2^m - 1}, w_1, \ldots, w_{2^m - 1}\} \) with \( d(u_1) = d(u_2) = 2^m + n - 1 \), \( d(v_i) = 2^n + 1 \) for \( 1 \leq i \leq 2^m - 1 \) and \( d(w_j) = 2^m + 1 \) for \( 1 \leq j \leq 2^n - 1 \). Now consider the sets \( A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\} \) and \( C = \{c\}. \)

Through the following process, a set-indexer \( f \) of \( K_{1,1,2^m-1,2^m-1} \) with \( A \cup B \cup C \) as the indexing can be obtained. Assign \( \emptyset \) to \( u_1 \) and \( C \) to \( u_2 \). Again assign the \( 2^m - 1 \) distinct nonempty subsets of \( A \) to the vertices \( v_1, \ldots, v_{2^m - 1} \) and the \( 2^n - 1 \) distinct nonempty subsets of \( B \) to vertices \( w_1, \ldots, w_{2^n - 1} \) in any order. Clearly,

\[
f(u_1, u_2) = f(u_1) \oplus f(u_2) = f(u_2) = C
\]
f(u_1, v_i) = f(u_1) \oplus f(v_i) = f(v_i) \\
\quad f(u_1, w_j) = f(u_1) \oplus f(w_j) = f(w_j) \\
\quad f(u_2, v_i) = f(u_2) \oplus f(v_i) = C \cup f(v_i) \\
\quad f(u_2, w_j) = f(u_2) \oplus f(w_j) = C \cup f(w_j) \text{ and} \\
\quad f(v_i, w_j) = f(v_i) \oplus f(w_j) \\
\quad \quad = f(v_i) \cup f(w_j) \text{ for } 1 \leq i \leq 2^n - 1 \text{ and } 1 \leq j \leq 2^n - 1.

Note that, the vertex labels and the edge labels are distinct so that \( f \) is a set-indexer of \( K_{1,1,2^{n-1},2^n-1} \) with \( A \cup B \cup C \) as the indexing set.

\[ \square \]

**Theorem 2.2.7.** \( \gamma(K_{1,1,1,1,2^n-1}) = n + 3. \)

**Proof.** Let \( V = \{v_1, \ldots, v_n, w_1, \ldots, w_{2^n-1}\} \); \( d(v_i) = 2^n + 2 \) for \( 1 \leq i \leq 4 \) and \( d(w_j) = 4 \) for \( 1 \leq j \leq 2^n - 1 \). Now consider the sets \( X = \{x_1, \ldots, x_{n+3}\}, Y = X \setminus \{x_{n+3}\}, Z = Y \setminus \{x_{n+2}\} \) and \( W = Z \setminus \{x_{n+1}\} \). Define a set-indexer \( f \) of \( K_{1,1,1,1,2^n-1} \) with \( X \) as the indexing set as follows: Assign \( 0 \) to \( v_1, X \) to \( v_2, Y \) to \( v_3 \) and \( Z \) to \( v_4 \). Again assign the \( 2^n - 1 \) distinct nonempty subsets of \( W \) to the remaining vertices \( w_1, \ldots, w_{2^n-1} \) in any order. Then the edge labels are given by,

\[
\begin{align*}
\quad f(v_1, w_i) &= f(v_1) \oplus f(w_i) = f(w_i) \\
\quad f(v_2, w_i) &= f(v_2) \oplus f(w_i) = X \setminus f(w_i) \\
\quad f(v_3, w_i) &= f(v_3) \oplus f(w_i) = Y \setminus f(w_i) \\
\quad f(v_4, w_i) &= f(v_4) \oplus f(w_i) = Z \setminus f(w_i); \quad 1 \leq i \leq 2^n - 1 \\
\quad f(v_1, v_2) &= f(v_1) \oplus f(v_2) = X \\
\quad f(v_1, v_3) &= f(v_1) \oplus f(v_3) = Y \\
\quad f(v_1, v_4) &= f(v_1) \oplus f(v_4) = Z \\
\quad f(v_2, v_3) &= f(v_2) \oplus f(v_3) = X \setminus Y = \{x_{n+3}\} \\
\quad f(v_2, v_4) &= f(v_2) \oplus f(v_4) = X \setminus Z = \{x_{n+2}, x_{n+3}\} \\
\quad f(v_3, v_4) &= f(v_3) \oplus f(v_4) = Y \setminus Z = \{x_{n+2}\}.
\end{align*}
\]
Clearly, the edge labels are all distinct. Hence, \( f \) is a set-indexer on \( K_{1,1,1,1,2^n-1} \) with \( X \) as the indexing set so that \( \gamma(K_{1,1,1,1,2^n-1}) \leq n + 3 \). But by theorem 1.0.5, \( \gamma(K_{1,1,1,1,2^n-1}) \geq \lceil \log_2(2^{n+2} + 3) \rceil = n + 3 \) and hence the result.

The following is an upper bound for the set-indexing number of any complete \( k \)-partite graph.

**Theorem 2.2.8.** \( \gamma(K_{n_1,\ldots,n_k}) \leq \sum_{i=1}^{k} \lceil \log_2 n_i \rceil + k - 1 \).

**Proof.** Let \( V = V_1 \cup \ldots \cup V_k \) be a partition of the vertex set of \( K_{n_1,\ldots,n_k} \). Let \( p_1 = \lceil \log_2 n_1 \rceil \) and \( p_i = \lceil \log_2 n_i \rceil + 1 \) for \( 2 \leq i \leq k \). Let \( X_1, \ldots, X_k \) be any \( k \) disjoint sets with \( |X_i| = p_i \) for \( 1 \leq i \leq k \). Now, assign the vertices in \( V_i \) with distinct subsets of \( X_i \) and the vertices in \( V_i \); \( 2 \leq i \leq k \) with distinct nonempty subsets of \( X_i \), in any order. This gives a set-indexer for \( K_{n_1,\ldots,n_k} \) so that \( \gamma(K_{n_1,\ldots,n_k}) \leq \sum_{i=1}^{k} \lceil \log_2 n_i \rceil + k - 1 \). \( \qed \)

### 2.3 Cycles and Related Graphs

This section investigates mainly the set-indexing numbers of certain cycles and related graphs such as helms, wheels and suns. It is shown that for \( n \geq 3 \), the cycles \( C_{2^n}, C_{2^n+1} \) and \( C_{2^n+2} \) have the same set-indexing number that is equal to \( n + 1 \). It is also proved that for \( n \geq 2 \), the set-indexing numbers of the wheels \( W_{2^n}, W_{2^n+1} \) and \( W_{2^n+2} \) are equal to \( n + 2 \) and that of \( W_{2^n-1} \) is \( n + 1 \). Further, the set-indexing numbers of four consecutive families of \( k \)-suns where \( k = 2^n - 1, 2^n, 2^n + 1 \) and \( 2^n + 2; \ n \geq 2 \) are obtained.
In 1989, Mollard and Payan [31] proved that,

**Theorem 2.3.1.** [31] For any integer \( n \geq 2 \), the cycle \( C_{2^n-1} \) is set-graceful.

Consequently, \( \gamma(C_{2^n-1}) = n \). As \( C_{2^n-1} \) has only \( 2^n - 1 \) vertices, the unique unassigned subset of the indexing set can be used to label the isolated vertex in \( C_{2^n-1} \cup K_1 \). Thus, we have the following:

**Corollary 2.3.2.** For any integer \( n \geq 2 \), \( \gamma(C_{2^n-1} \cup K_1) = n \).

**Corollary 2.3.3.** For any integer \( n \geq 2 \), \( \gamma(P_{2^n-1}) = n \).

**Proof.** The proof follows from theorem 1.0.5, theorem 1.0.6 and theorem 2.3.1. \( \square \)

**Corollary 2.3.4.** For any integer \( n \geq 2 \), \( \gamma(P_{2^n-1} \cup K_1) = n \).

**Proof.** The proof follows from theorem 1.0.6 and corollary 2.3.2 and corollary 2.3.3. \( \square \)

**Theorem 2.3.5.** If \( G \) is the \((2^n - 1)\)-sun; \( n \geq 2 \), then \( \gamma(G) = n + 1 \).

**Proof.** As \( G \) is a sun of order \( 2(2^n - 1) \), it contains the cycle \( C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1) \) and the pendant edges \( \{(u_i, v_i) : i = 1, \ldots, 2^n - 1\} \). By theorem 2.3.1, \( \gamma(C_{2^n-1}) = n \) with \( X = \{x_1, \ldots, x_n\} \) as the indexing set and \( f \) be the corresponding set-indexer. By theorem 1.0.5, \( \gamma(G) \geq n + 1 \). Consider the set-indexer \( g \) of \( G \) with indexing set \( Y = X \cup \{x_n+1\} \) defined by \( g(v_i) = f(v_i) \), \( g(u_i) = f(v_{i-1}) \cup \{x_{n+1}\} \) for \( i \in \{1, \ldots, 2^n - 1\} \) and \( v_0 = v_{2^n-1} \). Clearly, \( g(u_i) \) and \( g(v_j) \) for distinct indices \( i, j \in \{1, \ldots, 2^n - 1\} \)
are all distinct. Note,
\[ g(v_{i-1}, u_i) = f(v_{i-1}, u_i) \]
and
\[ g(u_i, v_i) = g(u_i) \oplus g(v_i) = f(v_{i-1}) \cup \{x_{n+1}\} \oplus f(v_i) = f(v_{i-1}, u_i) \cup \{x_{n+1}\}; i \in \{1, \ldots, 2^n - 1\}, u_0 = u_{2^n - 1}, \]
are also distinct. Consequently, \( \gamma(G) = n + 1 \).

Theorem 2.3.6. The set-indexing number of the wheel \( W_{2^n-1} \); \( n \geq 2 \) is \( n + 1 \).

Proof. \( W_{2^n-1} = C_{2^n-1} \lor \{u\}; C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1) \) and is of size \( 2(2^n - 1) \). By theorem 2.3.1, \( \gamma(C_{2^n-1}) = n \). Let \( f \) and \( X = \{x_1, \ldots, x_n\} \) be the corresponding set-indexer and the indexing set respectively. By theorem 1.0.5, \( \gamma(W_{2^n-1}) \geq n + 1 \). Define a set-indexer \( g \) extending from \( f \) with indexing set \( Y = X \cup \{x_{n+1}\} \) as follows:
\[ g(v_i) = f(v_i) \text{ for } i \in \{1, \ldots, 2^n - 1\} \text{ and } g(u) = \{x_{n+1}\}. \]
Clearly, \( g(u) \) and \( g(v_i); i \in \{1, \ldots, 2^n - 1\} \) are all distinct. Note,
\[ g(v_{i-1}, v_i) = f(v_{i-1}, v_i) \]
and
\[ g(u, v_i) = g(u) \oplus g(v_i) = g(v_i) \cup \{x_{n+1}\}; i \in \{1, \ldots, 2^n - 1\}, u_0 = u_{2^n - 1}, \]
are all distinct. \( \square \)

Theorem 2.3.7. For every integer \( n \geq 2 \), \( \gamma(H_{2^n-1}) = n + 2 \), where \( H_{2^n-1} \) is the helm.

Proof. The helm \( H_{2^n-1} \) contains the wheel \( W_{2^n-1} = C_{2^n-1} \lor \{w\}; C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1) \) and \( \{(u_i, v_i); i = 1, \ldots, 2^n - 1\} \) are the pendant edges of \( H_{2^n-1} \). By theorem 2.3.6, \( \gamma(W_{2^n-1}) = n + 1 \) with \( X = \{x_1, \ldots, x_{n+1}\} \) as the indexing set and let \( f \) be the corresponding set-indexer. By theorem 1.0.5, \( \gamma(H_{2^n-1}) \geq n + 2 \).
Define a set-indexer $g$ of $H_{2^n-1}$ with indexing set $Y = X \cup \{x_{n+2}\}$ as follows:

\[
g(v_i) = f(v_i); \quad i \in \{1, \ldots, 2^n-1\}, \quad g(w) = f(w) \quad \text{and} \quad g(u_i) = f(v_{i-1}) \cup \{x_{n+2}\}; \quad i \in \{1, \ldots, 2^n-1\}, \quad v_0 = v_{2^n-1}.
\]

Clearly, the vertex labels, $g(w), g(u_i)$ and $g(v_j)$ for distinct indices $i, j \in \{1, \ldots, 2^n-1\}$ are all distinct. Further, the edge labels,

\[
g(v_{i-1}, v_i) = f(v_{i-1}, v_i), \quad g(w, v_i) = f(w, v_i) \quad \text{and} \quad g(u_i, v_i) = g(u_i) \oplus g(v_i) = f(v_{i-1}, v_i) \cup \{x_{n+2}\}; \quad i \in \{1, \ldots, 2^n-1\}, \quad v_0 = v_{2^n-1}
\]

are also all distinct. Consequently, $\gamma(C_{2^n-1} \cup P_2) = n + 2$. \hfill \Box

**Theorem 2.3.8.** For every integer $n \geq 2$, the set-indexing number of the graph $C_{2^n-1} \cup P_2$ is $n + 2$.

**Proof.** By theorem 1.0.5, $\gamma(C_{2^n-1} \cup P_2) \geq n + 2$. Let $C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1)$ and $P_2 = (u_1, u_2)$. By theorem 2.3.1, $C_{2^n-1}$ has a set-indexer $f$ with indexing set $X = \{x_1, \ldots, x_n\}$ and let $f(v_i) = A_i$ where $A_i$'s are distinct subsets of $X$. Then we can extend $f$ to a set-indexer $g$ of $C_{2^n-1} \cup P_2$ with indexing set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ as follows:

\[
g(v_i) = f(v_i) = A_i; \quad 1 \leq i \leq 2^n - 1
\]

\[
g(u_1) = \{x_{n+1}\} \quad \text{and} \quad g(u_2) = \{x_{n+2}\}.
\]

Clearly,

\[
g(v_i, v_{i+1}) = f(v_i, v_{i+1}), \quad g(u_1, v_i) = A_i \oplus \{x_{n+1}\} = A_i \cup \{x_{n+1}\}
\]

\[
g(u_1, u_2) = \{x_{n+1}\} \oplus \{x_{n+2}\} = \{x_{n+1}, x_{n+2}\} \quad \text{and}
\]

\[
g(u_2, v_i) = A_i \oplus \{x_{n+2}\} = A_i \cup \{x_{n+2}\}; \quad 1 \leq i \leq 2^n - 1, \quad v_1 = v_{2^n}
\]

are all distinct. Consequently, $\gamma(C_{2^n-1} \cup P_2) = n + 2$. \hfill \Box

**Theorem 2.3.9.** The set-indexing number of the graph $C_{2^n-1} \cup N_2; \ n \geq 2$ is $n + 2$. 

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Proof. This follows from theorem 1.0.5, theorem 1.0.6 and theorem 2.3.8. □

Theorem 2.3.10. The set-indexing number of the graph $C_{2m-1} \vee C_{2n}$ is $m + n + 1$ for integers $m \geq 2$ and $n \geq 3$.

Proof. By theorem 2.3.1, $C_{2m-1} = (u_1, \ldots, u_{2m-1}, u_1)$ and $C_{2n-1} = (v_1, \ldots, v_{2n-1}, v_1)$ have optimal set-indexers $f$ and $g$ with $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ as the indexing sets respectively. By theorem 1.0.12, $g(v_1) = 0$, without the loss of generality. Now consider the sets $Y \setminus g(v_2)$ and $Y \setminus g(v_{2n-1})$. Clearly, there exists a vertex $v_k; 3 \leq k \leq 2^n - 1$ in $C_{2n-1}$ such that $g(v_{k-1})$ and $g(v_k)$ both distinct from the above two sets. Consider the cycle $C_{2n}$ which is obtained by inserting a vertex $w$ in the edge $(v_{k-1}, v_k)$ of $C_{2n-1}$. By theorem 1.0.5, $\gamma(C_{2m-1} \vee C_{2n}) \geq m + n + 1$.

Now consider the set $Z = Y \cup \{y_{n+1}\}$. We can define a set-indexer $h$ of $C_{2m-1} \vee C_{2n}$ with indexing set $X \cup Z$ as follows:

- $h(u_i) = f(u_i)$ for $1 \leq i \leq 2^m - 1$, $h(w) = \{y_{n+1}\}$
- $h(v_1) = Z$ and $h(v_j) = g(v_j)$ for $2 \leq j \leq 2^n - 1$.

Clearly, for $1 \leq i \leq 2^m - 1$,

- $h(u_i, u_{i+1}) = f(u_i, u_{i+1}); u_1 = u_{2m}$
- $h(v_1, u_i) = Z \cup f(u_i)$, $h(w, u_i) = f(u_i) \cup \{y_{n+1}\}$
- $h(u_i, v_j) = f(u_i) \cup g(v_j)$ for $2 \leq j \leq 2^n - 1$
- $h(v_j, v_{j+1}) = g(v_j, v_{j+1})$ for $2 \leq j \leq 2^n - 2$ ($j \neq k - 1$)
- $h(v_1, v_2) = Z \setminus g(v_2)$, $h(v_{k-1}, w) = g(v_{k-1}) \cup \{y_{n+1}\}$
- $h(w, v_k) = g(v_k) \cup \{y_{n+1}\}$ and $h(v_{2n-1}, v_1) = Z \setminus g(v_{2n-1})$.

Consequently, $h$ is a set-indexer of $C_{2m-1} \vee C_{2n}$ and the result follows from theorem 1.0.5. □

Now, we can find an optimal set-indexer of the cycle $C_{2n}$ by using the set-gracefulness of $C_{2n-1}$.
Theorem 2.3.11. For any integer \( n \geq 2 \), the set-indexing number of the cycle \( C_{2^n} \) is \( n+1 \).

Proof. By theorem 2.3.1, \( \gamma(C_{2^n-1}) = n; C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1) \) and let \( f \) be the corresponding optimal set-indexer with \( X = \{x_1, \ldots, x_n\} \) as the indexing set. By theorem 1.0.5, \( \gamma(C_{2^n}) \geq n+1 \). The set-indexer \( f \) can be extended to \( g \) on \( C_{2^n} = (v_1, \ldots, v_{2^n}, v_1) \) with indexing set \( Y = X \cup \{x_{n+1}\} \) by defining \( g(v_i) = f(v_i) \) for \( 1 \leq i \leq 2^n - 1 \) and \( g(v_{2^n}) = \{x_{n+1}\} \). Clearly, for \( 1 \leq i \leq 2^n - 2 \),

\[
g(v_i, v_{i+1}) = f(v_i, v_{i+1}),
\]

\[
g(v_{2^n}, v_1) = g(v_{2^n}) \oplus g(v_1) = f(v_1) \cup \{x_{n+1}\} \quad \text{and} \quad g(v_{2^n-1}, v_{2^n}) = g(v_{2^n-1}) \oplus g(v_{2^n}) = f(v_{2^n-1}) \cup \{x_{n+1}\}.
\]

Thus, \( g \) is a set-indexer of \( C_{2^n} \) so that \( \gamma(C_{2^n}) = n+1 \). \( \square \)

Theorem 2.3.12. For \( n \geq 2 \), \( \gamma(2^n - \text{sun}) = n + 2 \).

Proof. Let \( G \) consist the cycle \( C_{2^n} = (v_1, \ldots, v_{2^n}, v_1) \) and the pendant edges \( \{(u_i, v_i); i = 1, \ldots, 2^n\} \). By theorem 2.3.11, \( \gamma(C_{2^n}) = n + 1 \) with \( X = \{x_1, \ldots, x_{n+1}\} \) as the indexing set and let \( f \) be the corresponding optimal set-indexer. By theorem 1.0.5, \( \gamma(G) \geq n + 2 \). Consider the set-indexer \( g \) of \( G \) with indexing set \( Y = X \cup \{x_{n+2}\} \) defined as follows:

\[
g(v_i) = f(v_i) \quad \text{for} \quad i \in \{1, \ldots, 2^n\}
\]

\[
g(u_i) = f(v_{i-1}) \cup \{x_{n+2}\}; \quad i \in \{1, \ldots, 2^n\}, \quad v_0 = v_{2^n}.
\]

Clearly, \( g(u_i) \) and \( g(v_j) \) for distinct indices \( i, j \in \{1, \ldots, 2^n\} \) are all distinct. Note, \( g(u_{i-1}, v_i) = f(v_{i-1}, v_i) \) and \( g(u_i, v_i) = g(u_i) \oplus g(v_i) = f(v_{i-1}, v_i) \cup \{x_{n+2}\} \) for \( i \in \{1, \ldots, 2^n\}, v_0 = v_{2^n} \) are all distinct. Consequently, \( \gamma(G) = n + 2 \). \( \square \)

Theorem 2.3.13. For every integer \( n \geq 2 \), the set-indexing number of the wheel \( W_{2^n} \) is \( n + 2 \).
Proof. Let $W_{2^n} = C_{2^n} \lor \{u\}; C_{2^n} = (v_1, \ldots, v_{2^n}, v_1)$. By theorem 2.3.11, $\gamma(C_{2^n}) = n + 1$ with $X = \{x_1, \ldots, x_{n+1}\}$ as the indexing set and let $f$ be the corresponding optimal set-indexer. By theorem 1.0.5, $\gamma(W_{2^n}) \geq n + 2$. Define a set-indexer $g$ of $W_{2^n}$ with indexing set $Y = X \cup \{x_{n+2}\}$ as follows:

$$g(u_i) = f(u_i) \text{ for } i \in \{1, \ldots, 2^n\} \text{ and } g(u) = \{x_{n+2}\}.$$ 

Clearly, $g(u)$ and $g(v_i); i \in \{1, \ldots, 2^n\}$, are all distinct. Also, $g(u_{i-1}, u_i) = f(u_{i-1}, u_i)$ and $g(u, u_i) = g(u) \lor g(u_i) = f(u_i) \lor \{x_{n+2}\}$ for $i \in \{1, \ldots, 2^n\}$, $v_0 = v_{2^n}$, are all distinct. Hence, the result follows.

Theorem 2.3.14. For every integer $n \geq 3$, $\gamma(C_{2^n+1}) = n + 1$.

Proof. Consider the cycle $C_{2^n+1} = (v_1, \ldots, v_{2^n+1}, v_1)$. Choose $k$ such that $3 \leq k \leq 2^n - 2$. Now, consider the cycle $C_{2^n+1} = (u_1, \ldots, u_{2^n+1}, u_1); u_i = v_i$ for $1 \leq i \leq k$ and $u_j = v_{j+1}$ for $k + 1 \leq j \leq 2^n - 1$. By theorem 2.3.1, $\gamma(C_{2^n-1}) = n$ and let $f$ be the corresponding optimal set-indexer with the indexing set $X = \{x_1, \ldots, x_n\}$. By theorem 1.0.12, without loss of generality we can take $f(u_1) = \emptyset$. Let $f(u_{2^n-1}) = A$, $f(u_k) = B$ and $f(u_{k+1}) = C$ where $A, B, C \subseteq X$. By theorem 1.0.5, $\gamma(C_{2^n+1}) \geq n + 1$. Now, we can extend $f$ to a set-indexer $g$ of $C_{2^n+1}$ with indexing set $Y = X \cup \{x_{n+1}\}$ as follows:

$$g(v_i) = f(u_i); 1 \leq i \leq k, g(v_{j+1}) = f(u_j); k + 1 \leq j \leq 2^n - 1,$$

$$g(v_{2^n+1}) = A \cup \{x_{n+1}\} \text{ and } g(v_{k+1}) = \{x_{n+1}\}.$$ 

Clearly,

$$g(u_i, u_{i+1}) = f(u_i, u_{i+1}) \text{ for } 1 \leq i \leq k - 1,$$

$$g(u_k, u_{k+1}) = g(v_k) \lor g(v_{k+1}) = B \cup \{x_{n+1}\},$$

$$g(v_{k+1}, v_{k+2}) = g(u_{k+1}) \lor g(v_{k+2}) = C \cup \{x_{n+1}\},$$

$$g(u_j, u_{j+1}) = f(u_{j-1}, u_j) \text{ for } k + 2 \leq j \leq 2^n - 1,$$
\[ g(v_{2n}, v_{2n+1}) = g(v_{2n}) \oplus g(v_{2n+1}) = \{x_{n+1}\} \text{ and} \\
g(v_{2n+1}, v_1) = g(v_{2n+1}) \oplus g(v_1) = g(v_{2n+1}) = A \cup \{x_{n+1}\}. \]

Hence, the set-indexing number of the cycle \( C_{2n+1} \) is \( n+1 \). \hfill \Box

The above result is not true when \( n = 2 \) because Acharya has already proved that,

**Theorem 2.3.15.** \([4]\) \( \gamma(C_5) = 4 \).

However, it holds when \( n = 1 \) as \( \gamma(C_3) = 2 \).

As a consequence of theorem 2.3.14, the set-indexing numbers of the \((2n+1)\)-sun and the wheel \( W_{2n+1} \) are derived in the following theorems.

**Theorem 2.3.16.** If \( G \) is the \((2n+1)\)-sun, then \( \gamma(G) = n+2 \).

**Proof.** By theorem 2.3.5, the result is true for \( n = 1 \). When \( n = 2 \), \( G \) is the 5-sun of order 10 and by theorem 1.0.5, \( \gamma(G) \geq 4 \). Then the set-valuation given in figure 2.1 shows that \( \gamma(5\text{-sun}) = 4 \) so that the result is true for \( n = 2 \).

Now assume that \( n \geq 3 \). Let \( G \) contain the cycle \( C_{2n+1} = (v_1, \ldots, v_{2n+1}, v_1) \) and \( \{(u_i, v_i) ; i = 1, \ldots, 2n+1\} \) be the pendant edges. By theorem 2.3.14, \( \gamma(C_{2n+1}) = n+1 \). Let \( X = \{x_1, \ldots, x_{n+1}\} \) be the indexing set and \( f \) be the corresponding set-indexer. By theorem 1.0.5, \( \gamma(G) \geq n+2 \). Define a set-indexer \( g \) of \( G \) with indexing set \( Y = X \cup \{x_{n+2}\} \) as follows:

\[ g(v_i) = f(v_i) \text{ and} \]
\[ g(u_i) = f(v_{i-1}) \cup \{x_{n+2}\}; i \in \{1, \ldots, 2n+1\}, v_0 = v_{2n+1}. \]

Clearly, \( g(u_i) \) and \( g(v_j) \) for distinct indices \( i, j \in \{1, \ldots, 2n+1\} \) are all distinct. Note, \( g(v_{i-1}, v_i) = f(v_{i-1}, v_i) \) and \( g(u_i, v_i) = g(u_i) \oplus g(v_i) = f(v_{i-1}, v_i) \cup \{x_{n+2}\} \) for \( i \in \{1, \ldots, 2n+1\} \), \( v_0 = v_{2n+1} \) are all distinct. Hence, \( \gamma(G) = n+2 \). \hfill \Box
Theorem 2.3.17. The set-indexing number of the wheel \( W_{2^n+1} \) is \( n + 2 \).

Proof. By theorem 2.3.6, the result is true for \( n = 1 \). The setvaluation of \( W_5 \) given in figure 2.2 together with theorem 1.0.5 shows that \( \gamma(W_5) = 4 \).

Now assume that \( n \geq 3 \). Suppose \( W_{2^n+1} \) contains the cycle \( C_{2^n+1} = (v_1, \ldots, v_{2^n+1}, v_1) \) and \( K_1 = \{u\} \). By theorem 2.3.14, \( \gamma(C_{2^n+1}) = n + 1 \). Let \( X = \{x_1, \ldots, x_{n+1}\} \) be the indexing set and \( f \) be the corresponding set-indexer. By theorem 1.0.5, \( \gamma(W_{2^n+1}) \geq n + 2 \).

Define a set-indexer \( g \) of \( W_{2^n+1} \) with indexing set \( Y = X \cup \{x_{n+2}\} \) as follows:

\[
g(v_i) = f(v_i) \text{ for } i \in \{1, \ldots, 2^n + 1\} \text{ and } g(u) = \{x_{n+2}\}.
\]

Clearly, \( g(u) \) and \( g(v_i), i \in \{1, \ldots, 2^n + 1\} \) are all distinct. Also, \( g(v_{i-1}, u_i) = f(v_{i-1}, v_i) \) and \( g(u, v_i) = g(u) \oplus g(v_i) = g(v_i) \cup \{x_{n+2}\} \).
for $i \in \{1, \ldots, 2^n + 1\}$, $v_0 = v_{2^n + 1}$ are all distinct. Thus, the set-indexing number of $W_{2^n + 1}$ is $n + 2$. 

\[ \text{Figure 2.2: An optimal set-indexer of the wheel } W_6 \]

Now we derive an upper bound for the set-indexing number of the cycle $C_{2^{n-2}}$ by using the set-graceful labeling of $C_{2^{n-1}}$.

**Theorem 2.3.18.** For every integer $n \geq 3$, $\gamma(C_{2^{n-2}}) \leq n + 1$.

**Proof.** Let $f$ be a set-indexer of the cycle $C_{2^{n-1}} = (v_1, \ldots, v_{2^{n-1}}, v_1)$ with indexing set $X = \{x_1, \ldots, x_n\}$. Consider the cycle $C_{2^{n-2}} = C_{2^{n-1}} \setminus v_{2^{n-1}} \cup (v_{2^{n-2}}, v_1)$. Define a set-indexer $g$ on $C_{2^{n-2}}$ with indexing set $Y = X \cup \{x_{n+1}\}$ as follows:

\[ g(v_i) = f(v_i) \text{ for } i \in \{1, \ldots, 2^n - 3\} \text{ and } g(v_{2^n-2}) = Y. \]

Clearly,

\[ g(v_{2^n-2}, v_1) = Y \setminus f(v_1), \quad g(v_{2^n-3}, v_{2^n-2}) = Y \setminus f(v_{2^n-3}) \text{ and } g(v_i, v_{i+1}) = f(v_i, v_{i+1}) \text{ for } i \in \{1, \ldots, 2^n - 4\} \]

are all distinct. Hence, $\gamma(C_{2^{n-2}}) \leq n + 1$. \qed
Theorem 2.3.19. $\gamma(C_{2^n-2}) = n + 1; n \geq 3.$

Proof. By theorem 1.0.5 and theorem 2.3.18, $n \leq \gamma(C_{2^n-2}) \leq n + 1$ where $C_{2^n-2} = (v_1, \ldots, v_{2^n-2}, v_1)$. Suppose $\gamma(C_{2^n-2}) = n$ and let $f$ be the corresponding set-indexer with indexing set $X$. Let $f(v_i) = A_i; 1 \leq i \leq 2^n - 2$. Then the edge labels given by $A_1 \oplus A_2, A_2 \oplus A_3, \ldots, A_{2^n-2} \oplus A_1$ are $2^n - 2$ distinct nonempty subsets of $X$. Hence, there exists exactly one nonempty subset say, $B$ of $X$ such that $A_i \neq B$ for $1 \leq i \leq 2^n - 2$. Consequently, $\gamma(C_{2^n-2}) = n + 1$, as required. □

The following theorem gives the set-indexing number of the cycle $C_{2^n+2}; n \geq 3$.

Theorem 2.3.20. For every integer $n \geq 3$, $\gamma(C_{2^n+2}) = n + 1$.

Proof. Consider the cycle $C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1)$. By theorem 2.3.1, $C_{2^n-1}$ has a set-indexer $f$ with indexing set $X = \{x_1, \ldots, x_n\}$. By theorem 1.0.12 we shall assume that $f(v_1) = 0$. Then there exists a unique nonempty set $A \subseteq X$ such that $f(v_i) \neq A$ for $i \in \{1, \ldots, 2^n - 1\}$. Consider the cycle $C_{2^n+2} = (w_1, \ldots, w_{2^n+2}, w_1)$ where $w_i = v_i$ for $1 \leq i \leq 2^n - 1$. Then a set-indexer $g$ on $C_{2^n+2}$ with indexing set $Y = X \cup \{x_{n+1}\}$ can be defined as follows: $g(w_i) = f(v_i)$ for $1 \leq i \leq 2^n - 1$, $g(w_{2^n}) = \{x_{n+1}\}$ and $g(w_{2^n+1}) = A$.

If $f(w_{2^n-1}) = X$, choose a nonempty proper subset $B$ of $X$ other than $A$ such that $A \cup B \neq X$ and assign $B \cup \{x_{n+1}\}$ to the vertex $w_{2^n+2}$. Clearly, $f(w_i)$ are all distinct for $1 \leq i \leq 2^n + 2$. Also,
\[ g(w_i, w_{i+1}) = f(v_i, v_{i+1}) \text{ for } 1 \leq i \leq 2^n - 2 \]
\[ g(w_{2^n-1}, w_{2^n}) = X \cup \{x_{n+1}\} = Y \]
\[ g(w_{2^n}, w_{2^{n+1}}) = A \cup \{x_{n+1}\} \]
\[ g(w_{2^{n+1}}, w_{2^{n+2}}) = A \oplus (B \cup \{x_{n+1}\}) = (A \oplus B) \cup \{x_{n+1}\} \]
\[ g(w_{2^{n+2}}, w_1) = B \cup \{x_{n+1}\} \]

so that the edge labels are all distinct. Hence, it follows from theorem 1.0.5 that the set-indexing number of \( C_{2^n+2} \) is \( n + 1 \).

On the other hand, if \( f(v_{2^n-1}) \neq X \) then there arises two cases:

Case 1: \( A = X \).

Let \( C \) be any nonempty proper subset of \( X \) such that \( g(w_{2^n-1}) \notin \{C, X \setminus C\} \). Now, assign \( C \cup \{x_{n+1}\} \) to the vertex \( w_{2^n+2} \). Clearly, \( f(w_i) \) are all distinct for \( 1 \leq i \leq 2^n + 2 \). Note,
\[ g(w_i, w_{i+1}) = f(v_i, v_{i+1}) \text{ for } 1 \leq i \leq 2^n - 2 \]
\[ g(w_{2^n-1}, w_{2^n}) = g(w_{2^n-1}) \cup \{x_{n+1}\} \]
\[ g(w_{2^n}, w_{2^{n+1}}) = A \cup \{x_{n+1}\} = Y \]
\[ g(w_{2^{n+1}}, w_{2^{n+2}}) = A \oplus (C \cup \{x_{n+1}\}) = (X \setminus C) \cup \{x_{n+1}\} \]
\[ g(w_{2^{n+2}}, w_1) = C \cup \{x_{n+1}\} \]

so that the edge labels are also all distinct. Thus, it follows from theorem 1.0.5 that the set-indexing number of \( C_{2^n+2} \) is \( n + 1 \).

Case 2: \( A \neq X \).

Choose a nonempty proper subset \( E \) (other than \( A \) and \( g(w_{2^n-1}) \)) of \( X \) such that \( A \oplus E \neq g(w_{2^n-1}) \). Now, assign \( E \cup \{x_{n+1}\} \) to the vertex \( w_{2^n+2} \). Clearly, \( f(w_i) \) are all distinct for \( 1 \leq i \leq 2^n + 2 \). Also,
\[ g(w_i, w_{i+1}) = f(v_i, v_{i+1}) \text{ for } 1 \leq i \leq 2^n - 2 \]
\[ g(w_{2^n-1}, w_{2^n}) = g(w_{2^n-1}) \cup \{x_{n+1}\} \]
\[ g(w_{2^n}, w_{2^{n+1}}) = A \cup \{x_{n+1}\} \]
\[ g(w_{2^{n+1}}, w_{2^{n+2}}) = A \oplus (E \cup \{x_{n+1}\}) = (A \oplus E) \cup \{x_{n+1}\} \]
\[ g(w_{2^{n+2}}, w_1) = E \cup \{x_{n+1}\} \]
so that the edge labels are also all distinct. Thus, from theorem 1.0.5 it follows that the set-indexing number of $C_{2n+2}$ is $n+1$.

**Remark 2.3.21.** The above theorem does not hold good for $n = 1, 2$ as $\gamma(C_4) = 3$ by theorem 2.3.11 and $\gamma(C_6) = 4$ by theorem 2.3.19.

Now, the set-indexing numbers of the $(2^n + 2)$-sun and the wheel $W_{2n+2}$ are deduced from that of the cycle $C_{2n+2}$.

**Theorem 2.3.22.** If $G$ is the $(2^n + 2)$-sun; $n \geq 2$, then $\gamma(G) = n + 2$.

**Proof.** The following figure and theorem 1.0.5 account for the proof when $n = 2$.

![Figure 2.3: An optimal set-indexer of 6-sun](image)
Now suppose $n \geq 3$. Let $G$ contain the cycle $C_{2n+2} = (v_1, \ldots, v_{2n+2}, v_1)$ and $\{(u_i, v_i); i = 1, \ldots, 2^n + 2\}$ be the pendant edges. By theorem 2.3.20, $\gamma(C_{2n+2}) = n + 1$. Let $X = \{x_1, \ldots, x_{n+1}\}$ be the indexing set and $f$ be the corresponding set-indexer. By theorem 1.0.5, $\gamma(G) \geq n + 2$. Define a set-indexer $g$ of $G$ with indexing set $Y = X \cup \{x_{n+2}\}$ as follows:

$g(v_i) = f(v_i)$ and

$g(u_i) = f(v_{i-1}) \cup \{x_{n+2}\}$ for $i \in \{1, \ldots, 2^n + 2\}$, $u_0 = v_{2n+2}$.

Clearly, $g(u_i)$ and $g(v_j)$ for distinct indices $i, j \in \{1, \ldots, 2^n + 2\}$ are all distinct. Note, $g(v_{i-1}, v_i) = f(v_{i-1}, v_i)$ and $g(u_i, v_i) = g(u_i) \oplus g(v_i) = f(v_{i-1}, v_i) \cup \{x_{n+2}\}$ for $i \in \{1, \ldots, 2^n + 2\}$, $u_0 = v_{2n+2}$ are all distinct. Thus, $\gamma(G) = n + 2$. 

For the convenience of future references, the results of theorem 2.3.5, theorem 2.3.12, theorem 2.3.16 and theorem 2.3.22 are summarized as follows:

**Theorem 2.3.23.**

$$\gamma(k - \text{sun}) = \begin{cases} n + 1 & \text{if } k = 2^n - 1; \ n \geq 2 \\ n + 2 & \text{if } k = 2^n, 2^n + 1, 2^n + 2; \ n \geq 2 \end{cases}$$

**Theorem 2.3.24.** The set-indexing number of the wheel $W_{2n+2}$; $n \geq 2$ is $n + 2$.

**Proof.** In the light of theorem 1.0.5, the proof of the case $n = 2$ is evident from the set-indexer given in figure 2.4.

Now assume that $n \geq 3$. Suppose $W_{2n+2}$ contains the cycle $C_{2n+2} = (v_1, \ldots, v_{2n+2}, v_1)$ and $K_1 = \{u\}$. By theorem 2.3.20, $\gamma(C_{2n+2}) = n + 1$. Let $X = \{x_1, \ldots, x_{n+1}\}$ be the indexing set and $f$ be the corresponding set-indexer. By theorem 1.0.5, $\gamma(W_{2n+2}) \geq n + 2$. Define a set-indexer $g$ of $W_{2n+2}$ with indexing set $Y = X \cup \{x_{n+2}\}$
as follows:

\[ g(v_i) = f(v_i) \text{ for } i \in \{1, \ldots, 2^n + 2\} \text{ and } g(u) = \{x_{n+2}\}. \]

Clearly, \( g(u) \) and \( g(v_i) \), \( i \in \{1, \ldots, 2^n + 2\} \) are all distinct. Also, 
\[ g(v_{i-1}, v_i) = f(v_{i-1}, v_i) \text{ and } g(u, v_i) = g(u) \oplus g(v_i) = g(v_i) \cup \{x_{n+2}\} \]
for \( i \in \{1, \ldots, 2^n + 2\} \), \( v_0 = v_{2^n+2} \) are all distinct. Hence, the set-indexing number of the wheel \( W_{2^n+2} \) is \( n + 2 \).

\[ X = \{a, b, c, d\} \]

Figure 2.4: An optimal set-indexer of \( W_6 \)

Now we summarize the results of theorem 2.3.6, theorem 2.3.13, theorem 2.3.17 and theorem 2.3.24 for the convenience of future references.

**Theorem 2.3.25.**

\[ \gamma(W_k) = \begin{cases} 
\ n + 1 \text{ if } k = 2^n - 1; \ n \geq 2 \\
\ n + 2 \text{ if } k = 2^n, 2^n + 1, 2^n + 2; \ n \geq 2 
\end{cases} \]
Theorem 1.0.5 provides upper and lower bounds for the set-indexing number of a given graph $G$. But for many graphs these bounds are not so good. The next theorem obtains more optimal upper bounds for the set-indexing numbers of cycles.

**Theorem 2.3.26.** For every integer $n \geq 4$, $\gamma(C_n) \leq m + 2$; $m = \lfloor \log_2 n \rfloor$.

**Proof.** The proof is by induction on $m (\geq 2)$. When $m = 2$, the result follows from theorem 2.3.1, theorem 2.3.11, theorem 2.3.14 and theorem 2.3.18. Now, suppose $m \geq 3$ and assume that the result is true up to $m - 1$. By theorem 2.3.1, theorem 2.3.11, theorem 2.3.14 and theorem 2.3.20 we have $\gamma(C_{2^m-1}) = m$, $\gamma(C_{2^m}) = m + 1$, $\gamma(C_{2^m+1}) = m + 1$ and $\gamma(C_{2^m+2}) = m + 1$. By theorem 2.3.18, $\gamma(C_{2^m+1-2}) \leq m + 2$ and from the proof, it follows that $C_{2^m+1-2} = (v_1, \ldots, v_{2^m+1-2}, v_1)$ has a set-indexer $f$ having indexing set $X = \{x_1, \ldots, x_{m+2}\}$ with $f(v_{2^m+1-2}) = X$. Now consider the cycle $C_{2^m+1-3} = C_{2^m+1-2} \setminus \{v_{2^m+1-2}\} \sqcup \{v_{2^m+1-3}, v_1\}$. A set-indexer $g$ of this cycle can be defined as follows:

$$g(v_{2^m+1-3}) = X \quad \text{and} \quad g(v_i) = f(v_i); i \in \{1, \ldots, 2^{m+1} - 4\}.$$ 

Consequently, $\gamma(C_{2^m+1-3}) \leq m + 2$. Repeating the above process successively, we get, $\gamma(C_{2^m+1-i}) \leq m + 2$ for $i \in \{4, 5, \ldots, 2^{m+1} - 2^m - 3\}$. Hence, the required result.

**Corollary 2.3.27.** For every integer $n \geq 3$, the cycle $C_n$ satisfies $m + 1 \leq \gamma(C_n) \leq m + 2$ where $m = \lfloor \log_2 n \rfloor$.

**Proof.** From theorem 1.0.5, theorem 2.3.1 and theorem 2.3.26 we have, $m + 1 \leq \gamma(C_n) \leq m + 2$ where $2^m \leq n < 2^{m+1}$ and hence the result.
The following theorem shows that the set-indexing number of $W_n$ is either $\gamma(C_n)$ or $\gamma(C_n) + 1$.

**Theorem 2.3.28.** $\gamma(C_n) \leq \gamma(W_n) \leq \gamma(C_n) + 1$.

**Proof.** Let $f$ be any set-indexer of $C_n$ with indexing set $X$. Then $f$ can be extended to a set-indexer $g$ of $W_n$ with indexing set $X \cup \{a\}; a \notin X$ defined as follows:

$g(v) = f(v)$ for all $v \in V(C_n)$ and $g(w) = \{a\}; V(K_1) = \{w\}$.

Hence, $\gamma(W_n) \leq \gamma(C_n) + 1$. Since $C_n \subset W_n$, the first inequality follows from theorem 1.0.6. \(\square\)

Similarly, the set-indexing number of $n$-sun is also either $\gamma(C_n)$ or $\gamma(C_n) + 1$.

**Theorem 2.3.29.** $\gamma(C_n) \leq \gamma(n - \text{sun}) \leq \gamma(C_n) + 1$.

**Proof.** Let the $n$-sun $G$ contain the cycle $C_n = (v_1, \ldots, v_n, v_1)$ and $\{(u_i, v_i); i = 1, \ldots, n\}$ be the pendant edges of $G$. Let $f$ be any set-indexer of $C_n$ with indexing set $X$. Now we can define a set-indexer $g$ of $G$ with indexing set $X \cup \{a\}; a \notin X$ as follows:

$g(v_i) = f(v_i); i \in \{1, \ldots, n\}$ and $g(u_i) = f(v_{i-1}) \cup \{a\}; i \in \{1, \ldots, n\}, v_0 = v_n$.

Now by theorem 1.0.6, $\gamma(C_n) \leq \gamma(n - \text{sun}) \leq \gamma(C_n) + 1$. \(\square\)

Thus, $\gamma(C_n) \leq \gamma(W_n), \gamma(n - \text{sun}) \leq \gamma(C_n) + 1$ and it has been already proved that, $\gamma(W_k) = \gamma(k - \text{sun}) = \gamma(C_k) + 1$ where $k = 2^n - 1, 2^n, 2^n + 1, 2^n + 2$ and $n \geq 3$. Also, $\gamma(W_k) = \gamma(k - \text{sun}) = \gamma(C_k) + 1$ where $k = 3, 4$ and $\gamma(W_k) = \gamma(k - \text{sun}) = \gamma(C_k)$ where $k = 5, 6$. In the light of the above observations we put forward the following:

**Conjecture 2.3.30.** $\gamma(W_n) = \gamma(n - \text{sun}); n \geq 3$. 

2.4 Paths and Related Graphs

A path $P_m$ is set-graceful if and only if $\gamma(P_m) = \log_2 m$. This can happen only if $m$ is a power of 2. But in 1986, Acharya proved that,

Theorem 2.4.1. [3] For every integer $n \geq 2$, the path $P_{2^n}$ is not set-graceful.

Consequently, no path of order at least 3 is set-graceful. Hence, by theorem 1.0.5, $\gamma(P_m) \geq \log_2 m + 1; m \geq 3$. Now in the following theorem, we compute the set-indexing numbers of all the paths.

Theorem 2.4.2. $\gamma(P_n) = \begin{cases} n-1 & \text{if } n \leq 2 \\ \lceil \log_2 n \rceil + 1 & \text{if } n \geq 3 \end{cases}$.

Proof. The proof is divided into the following cases.

Case 1: $m \leq 2$.
Theorem 1.0.8 accounts for this case.

Case 2: $m \geq 3$.

Subcase 2.1: $m$ is a power of 2.
By theorem 1.0.5 and theorem 2.4.1 it follows that $\gamma(P_{2^n}) \geq n + 1$. Now, the result follows from theorem 1.0.6 and theorem 2.3.11.

Subcase 2.2: $m$ is not a power of 2.
Then there exists a positive integer $n$ such that $2^{n-1} < m \leq 2^n - 1$.
Obviously, $P_{2^{n-1}} \subseteq P_m \subseteq P_{2^n - 1}$ and by theorem 1.0.6,

$\gamma(P_{2^{n-1}}) \leq \gamma(P_m) \leq \gamma(P_{2^n - 1}) \Rightarrow n \leq \gamma(P_m) \leq n$

by theorem 2.3.3

$\Rightarrow \gamma(P_m) = n = \lceil \log_2 m \rceil + 1$. □

Theorem 2.4.3. For every integer $n \geq 4$, $\gamma(P_n \cup K_1) = \gamma(P_n)$. 

2.4 Paths and Related Graphs

Proof. Let \( P_n = (v_1, \ldots, v_n) \). By theorem 2.4.2, \( \gamma(P_n) = m \) where \( m \) is the unique integer satisfying \( 2^{m-1} \leq n \leq 2^m - 1 \). Let \( f \) be a set-indexer of \( P_n \) having \( X = \{x_1, \ldots, x_m\} \) as the indexing set. Since \( |V(P_n)| \leq 2^m - 1 \), there is a subset \( A \) of \( X \) such that \( A \neq f(v_i) \) for all \( i \in \{1, \ldots, n\} \). Then by assigning \( A \) to the isolated vertex of \( P_n \cup K_1 \), we can extend \( f \) to get a set-indexer of \( P_n \cup K_1 \). Now, the required result follows from theorem 1.0.5.

The set-indexing number of all the fans except that of the form \( F_{2n+1} \) are given in the following:

Theorem 2.4.4. For every integer \( n \geq 2 \), \( \gamma(F_{2^n+i}) = n + 2 \), \( i \in \{2, \ldots, 2^n\} \).

Proof. Let \( F_{2^n+i} \equiv \{u\} \cup P_{2^n+i-1} \) where \( P_{2^n+i-1} = (v_1, \ldots, v_{2^n+i-1}) \), \( i \in \{2, \ldots, 2^n\} \). By theorem 1.0.5, \( \gamma(F_{2^n+i}) \geq n + 2 \). By theorem 2.4.2, \( \gamma(P_{2^n+i-1}) = n + 1 \). Let \( g \) be the corresponding set-indexer with indexing set \( X = \{x_1, \ldots, x_{n+1}\} \) and let \( g(v_j) = A_j \) for \( j \in \{1, \ldots, 2^n + i - 1\} \). Now, the set-indexer \( f \) of \( F_{2^n+i} \), \( i \in \{2, \ldots, 2^n\} \) with the indexing set \( Y = X \cup \{x_{n+2}\} \) is defined as follows:

\[
f(v_j) = g(v_j) \text{ for } 1 \leq j \leq 2^n + i - 1 \text{ and } f(u) = \{x_{n+2}\}.
\]

Clearly, the edge labels,

\[
f(v_j, v_{j+1}) = g(v_j, v_{j+1}) \text{ for } j \in \{1, \ldots, 2^n + i - 2\} \text{ and } \]
\[
f(u, v_j) = f(u) \oplus f(v_j) = \{x_{n+2}\} \cup g(v_j) \text{ for } j \in \{1, \ldots, 2^n + i - 1\}
\]

are all distinct.

Definition 2.4.5. [28] A graph \( G \) is said to be set-sequential if there exists a nonempty set \( X \) and a bijective set-valued function \( f: V \cup E \to 2^X \setminus \emptyset \) such that \( f(u, v) = f(u) \oplus f(v) \) for every \( (u, v) \in E \).
Theorem 2.4.6. [33] A tree is set-graceful if and only if it is set-sequential.

Now, using theorem 2.4.1 it follows that

Theorem 2.4.7. For every integer \( n \geq 2 \), the path \( P_{2n} \) is not set-sequential.

Theorem 2.4.8. [4] A graph \( G \) is set-sequential if and only if \( G \vee K_1 \) is set-graceful.

Theorem 2.4.9. For every integer \( n \geq 2 \), \( F_{2n+1} \) is not set-graceful.

Proof. The proof follows from theorem 2.4.7 and theorem 2.4.8.

Now, it is not difficult to derive the set-indexing number of \( F_{2n+1} \).

Theorem 2.4.10. For every integer \( n \geq 2 \), the set-indexing number of the fan of order \( 2^n + 1 \) is \( n + 2 \).

Proof. Since \( F_{2n+1} \) is not set-graceful, \( \gamma(F_{2n+1}) > \log_2(|E| + 1) = n + 1 \). By theorem 2.4.4, \( \gamma(F_{2^n+2}) = n + 2 \). Then by theorem 1.0.6, it follows that \( \gamma(F_{2^n+1}) = n + 2 \).

The set-indexing numbers of all the fans can be computed from the following:

Theorem 2.4.11. \( \gamma(F_n) = \begin{cases} \lceil \log_2 n \rceil & \text{if } n = 2, 3 \\ \lceil \log_2 n \rceil + 1 & \text{if } n \geq 4 \end{cases} \).
Proof. Since $F_2$ and $F_3$ are complete graphs, the result for $n \leq 3$ follows from theorem 1.0.8. When $n = 4$,
\begin{align*}
3 &= \lfloor \log_2(n + 2) \rfloor \\
&\leq \gamma(F_n), \text{ by theorem 1.0.5} \\
&\leq \gamma(K_n), \text{ by theorem 1.0.6} \\
&= 3, \text{ by theorem 1.0.8}
\end{align*}
so that $\gamma(F_n) = \lfloor \log_2 n \rfloor + 1$ is true for $n = 4$.

When $n \geq 5$, the result follows from theorem 2.4.4 and theorem 2.4.10.

Theorem 2.4.12. For every integer $n > 4$, $\gamma(F_n) = \gamma(F_n \cup N_{2^{m+1}})$ where $2^m < n \leq 2^{m+1}$.

Proof. By theorem 2.4.11, $\gamma(F_n) = m + 2$ where $2^m < n \leq 2^{m+1}$. Let $f$ be the corresponding set-indexer with the indexing set $X = \{x_1, \ldots, x_{m+2}\}$. Since $2^m + 1 \leq |V(F_n)| \leq 2^{m+1}$, there exists at least $2^{m+1}$ subsets of $X$ which are not assigned to the $n$ vertices of $F_n$. Then by assigning these subsets of $X$ to the $2^{m+1}$ isolated vertices, we can extend $f$ to get a set-indexer of $F_n \cup N_{2^{m+1}}$. Now, the required result follows from theorem 1.0.6.

Theorem 2.4.13. For every integer $n \geq 2$, $\gamma(P_{2^n-1} \cup N_2) = n + 2$.

Proof. Let $P_{2^n-1} = (v_1, \ldots, v_{2^n-1})$ and $V(N_2) = \{u, v\}$. By corollary 2.3.3, $\gamma(P_{2^n-1}) = n$. Let $g$ be the corresponding set-indexer with indexing set $Y = \{x_1, \ldots, x_n\}$. Let $g(v_i) = A_i$ for $i \in \{1, \ldots, 2^n - 1\}$. Now, we can define a set-indexer $f$ of $P_{2^n-1} \cup N_2$ having indexing set $X = Y \cup \{x_{n+1}, x_{n+2}\}$ as follows:
\begin{align*}
  f(v_i) &= g(v_i) \text{ for } i \in \{1, \ldots, 2^n - 1\} \\
  f(u) &= \{x_{n+1}\} \text{ and } f(v) = \{x_{n+2}\}.
\end{align*}
Clearly, the edge labels
\[ f(u, v_i) = f(u) \oplus g(v_i) = A_i \cup \{x_{n+1}\}; \ i \in \{1, \ldots, 2^n - 1\} \]
\[ f(v, v_i) = f(v) \oplus g(v_i) = A_i \cup \{x_{n+2}\}; \ i \in \{1, \ldots, 2^n - 1\} \text{ and} \]
\[ f(v_i, v_{i+1}) = g(v_i) \oplus g(v_{i+1}) = A_i \oplus A_{i+1}; \ i \in \{1, \ldots, 2^n - 2\} \]
are all distinct. Thus, \( f \) is a set-indexer of \( P_{2^n-1} \lor N_2 \) and hence \( \gamma(P_{2^n-1} \lor N_2) = n + 2 \), by theorem 1.0.5. \qed