Chapter 7

Topoline Graphs

This chapter studies the concept of a topoline set-indexer which induces a topology on the edge set of a graph. Unlike topological set-indexers, not all graphs have topoline set-indexers and this caused the origin of topoline graphs. Every topoline graph admits a topoline number which in turn produces topoline set-graceful graphs. It is derived that all set-graceful graphs are topoline and topoline set-graceful stars are topogenic. Many families of topoline and topoline set-graceful graphs have been identified.

7.1 Topoline Set-Indexer

We introduce the concept of a topoline set-indexer by requiring that the edge labels together with the empty set form a topology on the indexing set. The minimum of the cardinalities of such topoline indexing sets is called the topoline number of the graph.

Definition 7.1.1. A set-indexer $f : V \cup E \rightarrow 2^X$ of a nonempty
graph $G$ is said to be a topoline set-indexer if $f(E) \cup \emptyset$ is a topology on $X$ and $X$ is called the topoline indexing set of $G$. The minimum among the cardinalities of such topoline indexing sets is called the topoline number of $G$ and is denoted by $\tau_e(G)$. A nonempty graph $G$ is said to be topoline if it has a topoline set-indexer.

**Theorem 7.1.2.** $\gamma(G) \leq \tau_e(G)$.

**Proof.** Since every topoline set-indexer is a set-indexer, the result follows trivially. \qed

The following are two easy consequences from theorem 2.1.11 and theorem 1.0.5.

**Corollary 7.1.3.** $\lceil \log_2 |V| \rceil \leq \tau_e(G)$.

**Corollary 7.1.4.** $\lceil \log_2 (|E| + 1) \rceil \leq \tau_e(G)$.

**Remark 7.1.5.** By theorem 2.3.1, $C_7$ is set-graceful and let $f$ be a set-graceful labeling of $C_7$. Then $f(E) \cup \emptyset$ is the discrete topology on a set $X$ with $|X| = 3$. Thus, the set-graceful labeling on $C_7$ is a topoline set-indexer and hence $\lceil \log_2 |V| \rceil = \lceil \log_2 7 \rceil = 3 = \tau_e(C_7)$ so that equality holds in corollary 7.1.3. But, by assigning $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}$ and $X = \{a, b, c, d\}$ to the vertices of the complete graph $K_6$ we get a topoline set-indexer of $K_6$ and by theorem 1.0.5 and theorem 7.1.2 it follows that $4 = \lceil \log_2 (|E| + 1) \rceil \leq \gamma(K_6) \leq \tau_e(K_6) = 4$. Clearly, $\lceil \log_2 |V| \rceil = \lceil \log_2 6 \rceil = 3 < \tau_e(K_6)$ so that strict inequality holds in corollary 7.1.3.

**Remark 7.1.6.** By theorem 2.4.1, $P_4 = (u_1, \ldots, u_4)$ is not set-graceful and hence $\tau_e(P_4) \geq \gamma(P_4) \geq 3$. Now, by assigning the subsets $\emptyset, \{a\}, \{b\}, \{a, c\}$ of the set $X = \{a, b, c\}$ to the vertices $u_1, \ldots, u_4$ in that order we get a topoline set-indexer of $P_4$. 
Clearly, \([\log_2(|E| + 1)| = \log_2 4| = 2 < \tau_e(P_4) = 3\) so that the strict inequality holds in corollary 7.1.4. Now, by assigning the subsets \(\emptyset, \{c\}, \{a, b\}\) and \(\{a\}\) of the set \(X = \{a, b, c\}\) to the vertices \(v_1, \ldots, v_4\) of \(K_4 \setminus (v_1, v_2)\) we get a topoline set-indexer. Clearly, \([\log_2(|E| + 1)| = \log_2 6| = 3 = \tau_e(K_4 \setminus (v_1, v_2))\) so that the equality holds in corollary 7.1.4.

**Remark 7.1.7.** By theorem 4.1.8 we know that, if \(G'\) is a spanning subgraph of \(G\), then \(\tau(G') \leq \tau(G)\). But this is not true in the case of a topoline set-indexer. For example, it can be easily verified that \(\tau_e(K_{1, 7}) = 3\) while \(\tau_e(K_{1, 7} \setminus e) = 4\) by theorem 1.0.36.

**Theorem 7.1.8.** Any set-graceful labeling is a topoline set-indexer. Moreover, \(\gamma(G) = \tau_e(G)\) for a set-graceful graph \(G\).

**Proof.** Let \(f\) be a set-graceful labeling of a graph \(G\) with indexing set \(X\). Then \(\gamma(G) = \log_2(|E| + 1)| = |X|\) so that \(|E| = 2|X| - 1\). Therefore, \(f(E) \cup \emptyset = 2^X\) so that \(f\) is a topoline set-indexer of \(G\). This implies that \(\tau_e(G) \leq \gamma(G)\). Now by theorem 7.1.2, \(\gamma(G) = \tau_e(G)\).

**Remark 7.1.9.** The converse of theorem 7.1.8 is not true. Consider the path \(P_5 = (v_1, \ldots, v_5)\). By corollary 7.1.3, \(\tau_e(P_5) \geq 3\). Now by assigning the subsets \(\{a\}, \emptyset, \{a, b\}, \{c\}\) and \(\{b, c\}\) of \(X = \{a, b, c\}\) to the vertices \(v_1, \ldots, v_5\), in that order we get a topoline set-indexer of \(P_5\). Then by theorem 2.4.2 and theorem 7.1.2, \(3 = \gamma(P_5) \leq \tau_e(P_5) \leq 3\), but \(P_5\) is not set-graceful.

**Remark 7.1.10.** In the light of theorem 2.1.1, it follows that \(f^d\) is a topoline set-indexer if and only if \(f\) is so.

**Theorem 7.1.11.** For a graph \(G\), if \(\tau_e(G) = \log_2(|E| + 1)\) then \(G\) is set-graceful.
Proof. By theorem 1.0.5,
\[ \lceil \log_2(|E| + 1) \rceil \leq \gamma(G) \]
\[ \leq \tau_r(G), \text{ by theorem 7.1.2} \]
\[ = \log_2(|E| + 1). \]
\[ \Rightarrow \gamma(G) = \log_2(|E| + 1) \text{ so that } G \text{ is set-graceful.} \]

In contrast with t-set indexers, not all graphs have topoline set-indexers.

Theorem 7.1.12. \( K_4 \) is not topoline.

Proof. If possible, let \( f \) be a topoline set-indexer of \( K_4 \) with topoline indexing set \( X \). Let \( A_1, A_2, A_3 \) and \( A_4 \) be the vertex labels of \( K_4 \) under \( f \). Since \( f(E) \cup \emptyset \) is a topology say \( \tau \) on \( X \), one of the edge labels is \( X \). Without loss of generality we may assume that \( A_1 \oplus A_2 = X \) so that \( A_1 \cap A_2 = \emptyset \). Now the remaining edge labels are \( A_1 \oplus A_3, A_1 \oplus A_4, A_2 \oplus A_3, A_2 \oplus A_4 \) and \( A_3 \oplus A_4 \) and by the definition of topology,

\[
(A_1 \oplus A_3) \cap (A_2 \oplus A_3) = (A_1 \cap A_2) \oplus A_3 = A_3 \in \tau
\]
\[
(A_1 \oplus A_4) \cap (A_2 \oplus A_4) = (A_1 \cap A_2) \oplus A_4 = A_4 \in \tau
\]
\[
(A_1 \oplus A_3) \cup (A_2 \oplus A_3) = (A_1 \cup A_2) \oplus A_3 = X \setminus A_3 \in \tau
\]
\[
(A_1 \oplus A_4) \cup (A_2 \oplus A_4) = (A_1 \cup A_2) \oplus A_4 = X \setminus A_4 \in \tau.
\]

Now we claim that \( A_3 \neq \emptyset \).

For if \( A_3 = \emptyset \), then the edge labels of \( K_4 \) are \( X, A_1, A_2, A_4, A_1 \oplus A_4 \) and \( A_2 \oplus A_4 \). Since \( f \) is an injection on both \( V \) and \( E \), \( A_4 \neq \emptyset \) and \( A_4 \neq X \). Consequently, one of the edge labels \( A_1, A_2, A_1 \oplus A_4 \) and \( A_2 \oplus A_4 \) is \( X \setminus A_4 \). Now, \( X \setminus A_4 = A_1 \) or \( A_2 \)

\[ \Rightarrow A_4 = A_2 \text{ or } A_1, \text{ a contradiction.} \]

\( X \setminus A_4 = A_1 \oplus A_4 \) or \( A_2 \oplus A_4 \Rightarrow X = A_1 \) or \( A_2 \) - a contradiction.

So we must have \( A_3 \neq \emptyset \) and hence the claim.
Further, by theorem 2.1.1, we must have $A_3 \neq X$ also.

By similar arguments, $A_4 \neq \emptyset, X$.

Thus, $X, A_3, A_4, X \setminus A_3, X \setminus A_4$ and $\emptyset$ are 6 distinct elements of $\tau$. Now, the following cases arise.

Case 1: $A_3 \cap A_4 = \emptyset$.

Then, $A_3 \cup A_4$ and $X \setminus (A_3 \cup A_4)$ are two distinct elements of $\tau$ other than the above six elements, which is a contradiction.

Case 2: $A_3 \cap A_4 \neq \emptyset$.

Subcase-2.1: If $A_3 \subseteq A_4$, then $A_3 \cup (X \setminus A_4)$ and $A_4 \cap (X \setminus A_3)$ are two distinct elements of $\tau$ other than the above six elements, which is a contradiction. Similarly, the case $A_4 \subseteq A_3$ is also not possible.

Subcase-2.2: Let $A_3 \cup A_4 \neq A_3, A_4$, then $A_3 \cap A_4$ and $X \setminus (A_3 \cap A_4)$ are two distinct elements of $\tau$ other than the above six elements, which is a contradiction.

Thus, we arrive at contradictions in all the possibilities. Consequently, $K_4$ has no topoline set-indexer.

\[ \square \]

Remark 7.1.13. Since $K_2, K_3$ and $K_6$ are set-graceful; by theorem 7.1.8, they are topoline. We strongly feel that no other complete graphs are topoline and put forward the following:

Conjecture 7.1.14. The complete graph $K_n; n > 1$ is topoline only if it is set-graceful.

Theorem 7.1.15. For $n \geq 3$, $\tau_e(K_2 \cup N_{n-2}) = \lceil \log_2 n \rceil$.

\textit{Proof.} Consider the graph $G = K_2 \cup N_{n-2}; V = \{v_1, \ldots, v_n\}$ and $K_2 = (v_1, v_2)$. Now we can find a topoline set-indexer $f$ of $G$ with topoline indexing set $X = \{x_1, \ldots, x_{\lceil \log_2 n \rceil}\}$ as follows: Assign $\emptyset$ and $X$ to $v_1$ and $v_2$. Now assign the $n - 2$ distinct unassigned
subsets of \( X \) to the remaining vertices of \( G \). Clearly, \( f \) is a set-indexer and \( f(E) \cup \emptyset \) is the indiscrete topology on \( X \).

Corollary 7.1.16. Let \( G \) be a topline graph of order \( n \geq 3 \). Then \( \tau_c(G) \geq \tau_c(K_2 \cup N_{n-2}) \).

**Proof.** The proof follows from theorem 7.1.15 and corollary 7.1.3.

Theorem 7.1.17. Every nonempty graph has a topline embedding.

**Proof.** The proof follows from theorem 7.1.15.

Remark 7.1.18. A graph even with a connected topline embedding need not be topline. For example, consider the graph \( K_4 \setminus e; V = \{v_1, \ldots, v_4\} \) and \( e = (v_1, v_2) \). Assigning \( \emptyset, \{c\}, \{a\} \) and \( \{a, b\} \) to the vertices \( v_1, \ldots, v_4 \) in that order we get a topline set indexer of \( K_4 \setminus e \). But by theorem 7.1.12, \( K_4 \) is not topline.

Theorem 7.1.19. \( \tau_c(K_{2^n+2^m,p}) \leq n + p + 1; 0 \leq m \leq n \).

**Proof.** Let \( V = \{u_1, \ldots, u_{2^n+2^m}, v_1, \ldots, v_p\} \); \( d(u_i) = p \) and \( d(v_j) = 2^n + 2^m, 1 \leq i \leq 2^n + 2^m, 1 \leq j \leq p \). Let \( X = \{x_1, \ldots, x_{n+1}\} \) and \( Y = \{y_1, \ldots, y_p\} \). By lemma 5.1.2, there exists a topology \( \tau \) on \( X \) with \( 2^n + 2^m \) open sets. Now define a topline set-indexer \( f \) of \( K_{2^n+2^m,p} \) with topline indexing set \( X \cup Y \) as follows: Assign the \( 2^n + 2^m \) distinct elements of \( \tau \) to the vertices \( u_1, \ldots, u_{2^n+2^m} \) and the subsets \( \{y_1\}, \{y_1, y_2\}, \ldots, \{y_1, \ldots, y_p\} \) of \( Y \) to the vertices \( v_1, \ldots, v_p \) in that order. Clearly, the edge labels together with \( \emptyset \) form a topology on \( X \cup Y \) and hence the result.
7.2 Certain Classes of Topoline Graphs

Many families of topoline graphs including trees, fans, complete bipartite graphs and forests are identified.

Theorem 7.2.1. Every star is topoline. Moreover $\tau_e(K_{1,n}) = \tau(K_{1,n})$.

Proof. Let $V = \{v_0, \ldots, v_n\}$; $v_0$ be the central vertex. Let $f$ be a t-set indexer of $K_{1,n}$ with t-indexing set say $X$. Let $A_0, A_1, \ldots, A_n$ be the vertex labels of $K_{1,n}$ under $f$. Without loss of generality we may assume that $A_0 = \emptyset$. Consider a set-indexer $g : V \rightarrow 2^X$ by $g(v_i) = A_i$; $0 \leq i \leq n$. Clearly, then $\emptyset \cup g(E) = g(V) = f(V)$ so that $g$ is a topoline set-indexer. Also, this implies that $\tau_e(K_{1,n}) \leq \tau(K_{1,n})$.

Further, let $A_1, A_2, \ldots, A_n$ be the edge labels of $K_{1,n}$ under any topoline set-indexer. Then by assigning $\emptyset, A_1, \ldots, A_n$ to the vertices $v_0, \ldots, v_n$ in that order, we get a topoline set-indexer of $K_{1,n}$ so that $\tau(K_{1,n}) \leq \tau_e(K_{1,n})$. Thus for a star graph, $\tau_e = \tau$. \qed

Corollary 7.2.2. All topoline set-graceful stars are topogenic.

Proof. The proof follows from theorem 5.1.28 and theorem 7.2.1. \qed

Remark 7.2.3. Not all topogenic stars are t-set graceful. For example consider the star $K_{1,6}$, it is topogenic but it is not t-set graceful.

Theorem 7.2.4. The complete bipartite graph $K_{m,n}$ is topoline.

Proof. Let $V = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$; $d(u_i) = n$ and $d(v_j) = m$, $1 \leq i \leq m$, $1 \leq j \leq n$. Let $X$ be a set in which there is
a topology with $m$ open sets say $A_1, \ldots, A_m$. Consider the set $Y = \{y_1, \ldots, y_n\}$ which is disjoint from $X$. Now we can define a topoline set-indexer $f$ of $K_{m,n}$ with topoline indexing set $X \cup Y$ as follows: Assign $A_1, \ldots, A_m$ to the vertices $u_1, \ldots, u_m$ respectively and assign $\{y_1, \ldots, y_i\}$ to the vertex $u_i$; $1 \leq i \leq n$. Clearly, the edge labels of $K_{m,n}$ together with $\emptyset$ will form a topology on $X \cup Y$.

Theorem 7.2.5. All paths are topoline.

Proof. Assigning the subsets $\emptyset$, $\{x_1\}$, $\{x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_1, x_3, x_8\}$, $\ldots$ of the set $X = \{x_1, \ldots, x_{n-1}\}$ to the vertices $v_1, \ldots, v_n$ of the path $P_n = (v_1, \ldots, v_n)$ in that order, we get a topoline set-indexer of $P_n$.

Theorem 7.2.6. All trees are topoline.

Proof. The proof is by induction on the order $n$ of trees. Clearly, by theorem 7.2.5, $K_2$ and $P_3$ are topoline and hence the result is true for $n = 2$ and 3.

Suppose that any tree of order $m$ is topoline.

Let $T$ be a tree of order $m + 1$ and $u$ be a pendant vertex of $T$. By assumption $T \setminus u$ is topoline, let $f$ be a topoline set-indexer of $T \setminus u$ with $X = \{x_1, \ldots, x_k\}$ as the topoline indexing set. Now we can define a topoline set-indexer say $g$ of $T$ with topoline indexing set $Y = X \cup \{x_{k+1}\}$ as follows:

$g(v) = f(v)$ for all $v \in V(T \setminus u)$ and $g(u) = Y \setminus f(w); w \in N(u)$. Thus $T$, the tree of order $m + 1$ is topoline and hence the result is true for $n = m + 1$ also.

Theorem 7.2.7. Nonempty forests are topoline.
7.2 Certain Classes of Topoline Graphs

Proof. The proof is by induction on the order \( n \) of the forest. Clearly, \( K_2, K_2 \cup K_1 \) and \( P_3 \) are topoline, follows from theorem 7.1.15 and theorem 7.2.5 and hence the result is true for \( n = 2, 3 \).

Assume that all forests of order \( m \) are topoline.

Let \( G \) be a forest of order say \( m+1 \) and let \( u \) be a pendant vertex of \( G \). Then \( G \setminus u \) is a forest of order \( m \) and by assumption let \( f \) be a topoline set-indexer of \( G \setminus u \) with topoline indexing set say \( X = \{x_1, \ldots, x_k\} \). Now assigning

\[
g(v) = f(v) \quad \text{for all } v \in V(G) \quad \text{and} \quad g(u) = Y \setminus f(w); w \in N(u),
\]

we get a topoline set-indexer say \( g \) of \( G \) with indexing set \( Y = X \cup \{x_{k+1}\} \). Hence the result is true for \( n = m+1 \) also. ☐

Corollary 7.2.8. Every nonempty spanning subgraph of a tree is topoline.

Proof. The proof follows from theorem 7.2.7. ☐

Theorem 7.2.9. All fans are topoline.

Proof. Let \( F_n = P_{n-1} \cup \{u\}; P_{n-1} = (v_1, \ldots, v_{n-1}) \). Now assign the subsets \( \emptyset, \{x_1\}, \{x_2\}, \{x_1, x_3\}, \{x_2, x_4\}, \{x_1, x_3, x_5\}, \ldots \) of the set \( X = \{x_1, \ldots, x_{n-1}\} \) to the vertices \( u, v_1, \ldots, v_{n-1} \) in that order. Clearly,

\[
f(v_{n-1}) = \begin{cases} 
\{x_1, x_3, \ldots, x_{n-3}, x_{n-1}\} & \text{if } n-1 \text{ is odd,} \\
\{x_2, x_4, \ldots, x_{n-3}, x_{n-1}\} & \text{if } n-1 \text{ is even}
\end{cases}
\]

we get a topoline set-indexer of \( F_n \) with \( X \) as the topoline indexing set. ☐

Theorem 7.2.10. The complete tripartite graph \( K_{1,m,n} \) is topoline.

Proof. Let \( V(K_{1,m,n}) = \{u, v_1, \ldots, v_m, w_1, \ldots, w_n\}; d(u) = m+n, \\
d(v_i) = n + 1, d(w_j) = m + 1, 1 \leq i \leq m \) and \( 1 \leq j \leq n \).
Now assign 0 to $u$. Let $X$ be a nonempty set in which there is a topology say $\tau_1$ on $X$ with $m + 1$ open sets say $A_1, \ldots, A_{m+1}$. Similarly, let $Y$ be a nonempty set disjoint from $X$ such that there is a topology say $\tau_2$ on $Y$ with $n + 1$ open sets say $B_1, \ldots, B_{n+1}$. Without loss of generality we may assume that $A_{m+1} = B_{n+1} = 0$. Now define a topoline set-indexer $f$ of $K_{1,m,n}$ with topoline indexing set $X \cup Y$ as follows:

\[ f(u) = 0, \quad f(v_i) = A_i; \quad 1 \leq i \leq m \text{ and} \]
\[ f(w_j) = B_j; \quad 1 \leq j \leq n. \]

The following theorem will help us to construct a topoline graph from a graph which is topoline or not.

**Theorem 7.2.11.** Every graph can be embedded into a connected topoline graph as an induced subgraph.

**Proof.** Let $\{v_1, \ldots, v_n\}$ be the vertex set of the given graph $G$. Now take a new vertex say $u$ and join it with all the vertices of $G$. Consider the set $X = \{x_1, \ldots, x_n\}$. Let $m = 2^n - (|E| + n) - 1$. Take $m$ new vertices $u_1, \ldots, u_m$ and join them with $u$. A topoline set-indexer of the resulting graph $G'$ can be defined as follows:

Assign 0 to $u$, $\{x_i\}$ to $v_i$; $1 \leq i \leq n$. Let $S = \{f(e); \ e \in E\} \cup \{\{x_i\}; \ 1 \leq i \leq n\}$. Note that $|S| = |E| + n$. Now by assigning the $m$ elements of $2^X \setminus (S \cup \emptyset)$ to the vertices $u_1, \ldots, u_m$ in any order, we get a topoline set-indexer of $G'$ with $X$ as the topoline indexing set. \hfill $\square$

**Lemma 7.2.12.** Let $f$ be a topoline set-indexer of a graph $G$ with topoline indexing set $X$. If $f(V)$ is a topology on $X$ with $f(E) \subseteq f(V)$, then $G \cup N_m$ is topoline for each positive integer $m$. 
\textbf{7.3 Topoline Set-Graceful Graphs}

\textit{Proof.} We can find a topoline set-indexer say $g$ of $G \vee N_n; V(N_n) = \{v_1, \ldots, v_n\}$ with topoline indexing set $Y = X \cup \{x_1, \ldots, x_n\}; X \cap \{x_1, \ldots, x_n\} = \emptyset$ as follows:

\begin{itemize}
  \item $g(u) = f(u)$ for all $u \in V(G)$ and
  \item $g(v_i) = \{x_1, \ldots, x_i\}; 1 \leq i \leq n.$
\end{itemize}

\hfill $\square$

\textbf{Theorem 7.2.13.} Let $G$ be a set-graceful tree. Then $G \vee N_n$ is topoline.

\textit{Proof.} By theorem 1.0.18, $|E(G)| = 2^m - 1$ for some positive integer $m$. Since $G$ is a tree, then $|V(G)| = 2^m$. Let $f$ be a set-graceful labeling of $G$ with indexing set $X$ of cardinality $m$. By the definition of set-graceful labeling, $\gamma(G) = \log_2(|E(G)| + 1) = m = |X|$. Then we must have $f(V) = f(E) \cup \emptyset = 2^X$ so that by lemma 7.2.12, $G \vee N_n$ is topoline. \hfill $\square$

\textbf{Remark 7.2.14.} Let $f$ be a topoline set-indexer of $K_3$ such that $f(V)$ is also a topology on the topoline indexing set say $X$. Then $f$ is given by $f(v_1) = \emptyset$, $f(v_2) = A$ and $f(v_3) = X$ for some $A \subset X$; $V(K_3) = \{v_1, v_2, v_3\}$. Then clearly $f(E) \nsubseteq f(V)$. Note that here $K_3 \vee K_1 = K_4$ is not topoline by theorem 7.1.12.

\textbf{7.3 Topoline Set-Graceful Graphs}

This section defines topoline set-graceful graphs, graphs where set-indexing number and topoline number are equal. Clearly, all set-graceful graphs are topoline set-graceful but the converse is not true. Further, we note that topologically set-graceful and topoline set-graceful are two independent notions. However, a
star is topologically set-graceful if and only if it is topoline set-graceful.

Definition 7.3.1. A graph $G$ is said to be topoline set-graceful if it is topoline and $\gamma(G) = \tau_e(G)$.

By theorem 7.1.8, all set-graceful graphs are topoline set-graceful. But the converse is not true as is already seen in remark 7.1.9.

Theorem 7.3.2. $K_2 \cup N_{n+2}$ is topoline set-graceful.

Proof. By theorem 2.1.11,

\[
\log_2 |V| \leq \gamma(G) \leq \tau_e(G), \text{ by theorem 7.1.2} \leq \log_2 |V|, \text{ by theorem 7.1.15}.
\]

Remark 7.3.3. The notions of t-set graceful graphs and topoline set-graceful graphs are not related to each other. For the complete graph $K_6$, $\gamma(K_6) = 4 = \tau_e(K_6)$ and hence it is topoline set-graceful. But $K_6$ is not t-set graceful by theorem 6.1.21. On the other hand, the complete graph $K_4$ is t-set graceful, but by theorem 7.1.12, $K_4$ is not even topoline. But $K_3$ is both t-set graceful and topoline set-graceful.

Theorem 7.3.4. A star is t-set graceful if and only if it is topoline set-graceful.

Proof. The proof follows from theorem 7.2.1.

Theorem 7.3.5. Paths $P_{2^n+1}$ and $P_{2^n+2}$ are topoline set-graceful.

Proof. The cycle $C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1)$ has a set-graceful labeling $f$ as described in remark 3.1.10 with indexing set $X = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Consider the path $C_{2^n-1} \setminus (v_1, v_2)$. Take
two new vertices $u_1$ and $u_2$ and form the path $P_{2^{n+1}} = C_{2^{n-1}} \setminus (v_1, v_2) \cup \{(v_1, u_1), (u_1, u_2)\}$.

We can define a topoline set-indexer $g$ on $P_{2^{n+1}}$ with topoline indexing set $Y = X \cup \{y\}$ as follows:

\[
g(v_i) = f(v_i); \quad 1 \leq i \leq 2^n - 1
\]

\[
g(u_1) = Y \setminus \{1\}
\]

\[
g(u_2) = Y \setminus \{\alpha\}.
\]

By theorem 1.0.5 and theorem 7.1.2, $n + 1 = \lceil \log_2(|E(P_{2^{n+1}})| + 1) \rceil \\
\leq \gamma(P_{2^{n+1}}) \leq \tau_e(P_{2^{n+1}}) \leq n + 1$ and hence $P_{2^{n+1}}$ is topoline set-graceful.

Take a new vertex $u_3$ and form the path $P_{2^{n+2}} = P_{2^{n+1}} \cup (u_2, u_3)$. We can define a topoline set-indexer $h$ on $P_{2^{n+2}}$ with topoline indexing set $Y$ as follows:

\[
h(u) = g(u) \text{ for all } u \in V(P_{2^{n+1}}) \text{ and } h(u_3) = 0.
\]

Now by theorem 2.4.2 and theorem 7.1.2, $n + 1 = \gamma(P_{2^{n+2}}) \leq \tau_e(P_{2^{n+2}}) \leq n + 1$ and hence $P_{2^{n+2}}$ is topoline set-graceful. \qed

Lemma 7.3.6. [34] $T(n + 1, 2^n + 3) = 0; \quad 2 \leq n \leq 5$.

Theorem 7.3.7. The path $P_{2^{n+3}}; \quad 2 \leq n \geq 5$ is not topoline set-graceful.

Proof. By lemma 7.3.6 and theorem 7.2.6, $\tau_e(P_{2^{n+3}}) \geq n + 2$. Now the result follows from theorem 2.4.2. \qed

Theorem 7.3.8. $C_{2^{n+3}}$ is topoline set-graceful.

Proof. The cycle $C_{2^{n-1}} = (v_1, \ldots, v_{2^{n-1}}, v_1)$ has a set-graceful labeling $f$ as described in remark 3.1.10. Consider the path $C_{2^{n-1}} \setminus (v_1, v_2)$. Take four new vertices $u_1, u_2, u_3$ and $u_4$ and form the cycle $C_{2^{n+3}} = C_{2^{n-1}} \setminus (v_1, v_2) \cup \{(v_1, u_1), (u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, v_2)\}$. We can define a topoline set-indexer $g$ on $C_{2^{n+3}}$ with topoline indexing set $Y = X \cup \{y\}$ as follows:
\[ g(v_i) = f(v_i); \quad 1 \leq i \leq 2^n - 1 \]
\[ g(u_1) = Y \setminus \{1\}, \quad g(u_2) = Y \setminus \{\alpha\} \]
\[ g(u_3) = \emptyset \quad \text{and} \quad g(u_4) = Y \setminus \{1, \alpha\}. \]

Now by theorem 1.0.5 and theorem 7.1.2,
\[ n + 1 = \lceil \log_2(|E(C_{2^n+3})| + 1) \rceil \leq \gamma(C_{2^n+3}) \leq \tau_e(C_{2^n+3}) \leq n + 1 \]
and hence \( C_{2^n+3} \) is topoline set-graceful. \( \square \)

**Theorem 7.3.9.** \( K_{2n,2} \) is topoline set-graceful.

**Proof.** Let \( V = \{u, v, w_1, \ldots, w_{2m}\} \); \( d(u) = d(v) = 2m \) and \( d(w_i) = 2, \quad 1 \leq i \leq 2m \). Define a topoline set-indexer \( f \) of \( K_{2n,2} \) with topoline indexing set \( X = \{x_1, x_2, \ldots, x_m\} \) as follows: Assign the distinct \( 2m \) subsets of \( X \setminus \{x_{m+1}, x_{m+2}\} \) to the vertices \( w_1, \ldots, w_{2m} \) in any order and finally assign \( \{x_{m+1}\} \) and \( \{x_{m+1}, x_{m+2}\} \) to the vertices \( u \) and \( v \) respectively. By theorem 7.1.2 and theorem 1.0.5, \( \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(2^{m+1} + 1) \rceil = m + 2 \leq \gamma(K_{2n,2}) \leq \tau_e(K_{2n,2}) \leq m + 2. \) \( \square \)

**Theorem 7.3.10.** A graph \( G \) of size \( m - 1 \) where \( 3 \cdot 2^{n-2} < m < 2^n; \quad n \geq 3 \) and \( \gamma(G) = n \) is not topoline set-graceful.

**Proof.** The proof follows from theorem 1.0.36. \( \square \)

**Remark 7.3.11.** From the above theorem it follows easily that the class of graphs given by \( P_{2^n-1}, K_{1,2^n-2}, \ ST(m_1, m_1); \ m_1 + m_2 = 2^n - 3 \) where \( n \geq 3 \) are not topoline set-graceful.

**Theorem 7.3.12.** \( K_{1,2^n-1,2m-1} \) is topoline set-graceful.

**Proof.** Let \( V(K_{1,2^n-1,2m-1}) = \{u, v_1, \ldots, v_{2^n-1}, w_1, \ldots, w_{2m-1}\} \);
\[ d(u) = 2^n + 2^m - 2, \quad d(v_i) = 2^n; \quad 1 \leq i \leq 2^n - 1 \quad \text{and} \quad d(w_j) = 2^n; \quad 1 \leq j \leq 2^m - 1. \]
Consider the sets \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_m\} \). Now we can define a topoline set-indexer \( f \) of
7.3 Topoline Set-Graceful Graphs

$K_{1,2^n-1,2^m-1}$ with topoline indexing set $X \cup Y$ as follows: Assign $\emptyset$ to $u$, the distinct nonempty sets of $X$ to $v_i; 1 \leq i \leq 2^n - 1$ and the distinct nonempty subsets of $Y$ to the vertices $w_j; 1 \leq j \leq 2^m - 1$. Clearly,

\[ \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(2^{m+n}) \rceil \]

\[ = m + n \]

\[ \leq \gamma(K_{1,2^n-1,2^m-1}), \text{ by theorem 1.0.5} \]

\[ \leq \tau_e(K_{1,2^n-1,2^m-1}), \text{ by theorem 7.1.2} \]

\[ \leq n + m. \]

\[ \square \]

**Theorem 7.3.13.** $K_{1,2^n,2^m-1}$ is topoline set-graceful.

**Proof.** Let $V(K_{1,2^n,2^m-1}) = \{u, v_1, \ldots, v_{2^n}, w_1, \ldots, w_{2^m-1}\}; d(u) = 2^n + 2^m - 1, d(v_i) = 2^m; 1 \leq i \leq 2^n$ and $d(w_j) = 2^n + 1; 1 \leq j \leq 2^m - 1$. Consider the sets $X = \{x_1, \ldots, x_{n+1}\}$ and $Y = \{y_1, \ldots, y_m\}$. Now we can define a topoline set-indexer $f$ of $K_{1,2^n,2^m-1}$ with topoline indexing set $X \cup Y$ as follows: Assign $\emptyset$ to $u$, the distinct nonempty sets of $X \setminus \{x_{n+1}\}$ to $v_i; 1 \leq i \leq 2^n - 1$, the distinct nonempty subsets of $Y$ to the vertices $w_j; 1 \leq j \leq 2^m - 1$ and finally assign $X$ to $v_{2^n}$. Clearly,

\[ \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(2^n + 2^m - 1 + 2^{n+m} - 2^n + 1) \rceil \]

\[ = m + n + 1 \]

\[ \leq \gamma(K_{1,2^n,2^m-1}), \text{ by theorem 1.0.5} \]

\[ \leq \tau_e(K_{1,2^n,2^m-1}), \text{ by theorem 7.1.2} \]

\[ \leq n + m + 1. \]

\[ \square \]

Since all set-graceful graphs are topoline set-graceful, by theorem 3.2.15 we get the following:

**Theorem 7.3.14.** Any graph $G$ can be embedded as an induced subgraph of a connected topoline set-graceful graph.
Note 7.3.15. (i). There are topoline set-graceful graphs that are neither set-graceful nor t-set graceful. $K_2 \cup N_{2^n-3}; n \geq 3$ is a family of such a graph. By theorem 7.3.2, $K_2 \cup N_{2^n-3}; n \geq 3$ is topoline set-graceful. But,

$$n = \lceil \log_2 |V| \rceil$$

$$\leq \gamma(K_2 \cup N_{2^n-3}), \text{ by theorem 2.1.11}$$

$$\leq \gamma(K_{1,2^n-2}), \text{ by theorem 1.0.6}$$

$$= n, \text{ by theorem 2.1.4}$$

$$< \tau(K_2 \cup N_{2^n-3}), \text{ by theorem 4.1.15.}$$

Thus, $K_2 \cup N_{2^n-3}$ is not t-set graceful and $\gamma(K_2 \cup N_{2^n-3}) = n \neq \log_2(|E|+1)$ so that it is not set-graceful.

(ii). The family of cycles $C_{2^n-1}; n \geq 3$ are set-graceful but not t-set graceful (by note 6.3.16(ii)). Since all set-graceful graphs are topoline set-graceful, $C_{2^n-1}; n \geq 3$ is topoline set-graceful.

(iii). By note 6.3.16(iii), $K_6 \cup K_1$ is set-graceful as well as t-set graceful but not strongly t-set graceful. Being set-graceful, $K_6 \cup K_1$ is also topoline set-graceful.

(iv). $K_3$ is set-graceful and strongly t-set graceful, but not discrete t-set graceful (by note 6.3.16(iv)). Again $K_3$ is topoline set-graceful as it is set-graceful.

(v). By note 6.3.16(v), $K_{1,2^n-1}$ is discrete t-set graceful and set-graceful so that it is topoline set-graceful.

(vi). There are discrete t-set graceful graphs that are topoline set-graceful, but not set-graceful. By note 6.3.16(vii), $G = K_{1,2^n-1} \cup N_{2^n}$ is discrete t-set graceful, but not set-graceful.
Consider the set $X = \{x_1, \ldots, x_{n+1}\}$. Assign 0 to the central vertex of $K_{1,2^n-1}$ and the distinct nonempty subsets of the set $X \setminus \{x_n, x_{n+1}\}$ to any of $2^{n-1} - 1$ pendant vertices of $K_{1,2^n-1}$ in any order. Now assign the sets $\{A \cup \{x_n, x_{n+1}\} : A \in 2^X \setminus \{\emptyset\}\}$ to the remaining $2^{n-1}$ pendant vertices of $K_{1,2^n-1}$ in a one to one manner. Finally, use the remaining $2^n$ unassigned subsets of $X$ to label the isolated vertices of $G$ to get a topline set-indexer of $G$. Therefore, $n + 1 \geq \tau_s(G)$

$$\geq \gamma(G), \text{ by theorem 7.1.2}$$

$$= n + 1, \text{ by note 6.3.16(vii).}$$

Thus, $\tau_s(G) = \gamma(G)$ so that $G$ is topline set-graceful.

(vii). By note 6.3.16(vi), $K_{1,2^n-2} \cup K_1$; $n \geq 3$ is discrete t-set graceful and hence $\gamma(K_{1,2^n-2} \cup K_1) = n$ so that it is not topline set-graceful by theorem 7.3.10.

(viii). $K_4$ is strongly t-set graceful but not discrete t-set graceful (see note 6.3.16(ix)). Also $K_4$ is not topline by theorem 7.1.12.

(ix). By note 6.3.16(x), $W_6$ is t-set graceful but not strongly t-set graceful. Since $\gamma(W_6) = 4$, by theorem 7.3.10, $W_6$ is not topline set-graceful.

(x). The stars $K_{1,2^n+1}$ forms a class of strongly t-set graceful graphs that are not set-graceful. By theorem 5.1.7, $K_{1,2^n+1}$ is t-set graceful with t-number $n+1$ so that by theorem 6.2.2 they are strongly t-set graceful. Also by theorem 1.0.18, $K_{1,2^n+1}$ is not set-graceful. However, $K_{1,2^n+1}$ is topline set-graceful by theorem 7.3.4. Since $\gamma(K_{1,2^n+1}) = n + 1$, by
Theorem 6.3.5, $K_{1,2^n+1}$ is not discrete t-set graceful.

(xi). Let $G$ be the graph $((K_1 \cup K_2) \cup N_2) \cup N_2$.

By theorem 2.3.15, $4 = \gamma(G)$

$\leq \gamma(G)$, by theorem 1.0.6

$\leq \tau_e(G)$, by theorem 7.1.2.

Also by theorem 4.1.4, $\gamma(G) \leq \tau(G)$.

But $G$ admits a topoline set-indexer as well as a t-set indexer with indexing set $X = \{a, b, c, d\}$ as shown in figure 7.1.

Consequently, $\gamma(G) = \tau(G) = \tau_e(G) = 4$ so that $G$ is both topoline set-graceful and t-set graceful. However, $G$ is not set-graceful as $\gamma(G) \neq \log_2(|E| + 1)$. Further, by corollary 6.2.5, $G$ is not strongly t-set graceful.

The above facts are shown in the diagram below:
Fig 7.2: The Interrelations between Certain Categories of Graphs.

- Topoline Set-Graceful Graphs
- Set-Graceful Graphs
- T-Set Graceful Graphs
- Strongly T-Set Graceful Graphs
- Discrete T-Set Graceful Graphs