Chapter 6

Strongly T-Set Graceful Graphs

This chapter examines the topological set-gracefulness of subgraphs, particularly spanning subgraphs of a topologically set-graceful graph. Certain families of graphs for which every spanning subgraph is topologically set graceful are identified and termed strongly t-set graceful. Further, a subclass of strongly t-set graceful graphs called discrete t-set graceful are introduced and studied.

6.1 T-Set Graceful Subgraphs

The topological set gracefulness of subgraphs with a special emphasis on spanning subgraphs is investigated here. There is every reason to begin with the following simple result.

Theorem 6.1.1. $2K_2$ is t-set graceful with t-number 3.

Proof. By theorem 1.0.5, $\gamma(2K_2) \geq \lceil \log_2(|V| + 1) \rceil = 2$. Suppose
\( \gamma(2K_2) = 2 \) and let \( f \) be the corresponding set-indexer with indexing set \( X = \{a, b\} \). Clearly, \( \emptyset, \{a\}, \{b\}, \{a, b\} \) are the vertex labels of the vertices \( v_1, v_2, v_3, v_4 \) of \( 2K_2 \) under \( f \). Without loss of generality, let \( 2K_2 = (v_1, v_2) \cup (v_3, v_4) \) where \( f(v_1) = \emptyset \). Since, 
\[
\emptyset \oplus \{a\} = \{a\} = \{b\} \oplus \{a, b\} \\
\emptyset \oplus \{b\} = \{b\} = \{a\} \oplus \{a, b\} \\
\emptyset \oplus \{a, b\} = \{a, b\} = \{a\} \oplus \{b\},
\]
it follows that both the edges \((v_1, v_2)\) and \((v_3, v_4)\) have the same edge label which is not possible. Thus, \( \gamma(2K_2) \geq 3 \). By theorem 4.1.4, theorem 4.1.8 and theorem 4.2.2, \( 3 \leq \gamma(2K_2) \leq \tau(2K_2) \leq \tau(K_4) = 3 \). 

Theorem 6.1.2. A graph is t-set graceful if and only if it has a t-set graceful subgraph with the same t-number.

Proof. As every graph is a subgraph of itself, the necessity part is obvious.

Conversely, let \( G' \subset G \) with \( \tau(G') = \tau(G) \). Suppose \( G' \) be t-set graceful with t-number \( m \). Then, \( m = \gamma(G') \)
\[
\leq \gamma(G), \text{ by theorem 1.0.6} \\
\leq \tau(G), \text{ by theorem 4.1.4} \\
= \tau(G') \\
= m.
\]
Consequently, \( \gamma(G) = \tau(G) = m \) and \( G \) is t-set graceful.

Remark 6.1.3. All subgraphs of a t-set graceful graph with the same t-number may not be t-set graceful. Consider the fan graph \( F_7 = P_6 \lor \{u\}; P_6 = (u_1, \ldots, u_6) \). By theorem 2.4.11 and theorem 4.1.4, \( 4 = \gamma(F_7) \leq \tau(F_7) \). Now define a t-set indexer \( f \) on \( F_7 \) with t-indexing set \( X = \{x_1, x_2, x_3, x_4\} \) as follows: Assign \( \emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_2\}, \{x_1, x_3\} \) and \( X \) to the vertices \( u_1, \ldots, u_6 \) and
6.1 T-Set Graceful Subgraphs

Thus, $F_7$ is t-set graceful with t-number 4. Since, $P_7$ is a spanning subgraph of $F_7$, by theorem 4.3.4 and theorem 4.1.8, $4 = \tau(P_7) \leq \tau(F_7) \leq 4$. But, by theorem 2.4.2, $\gamma(P_7) = 3$ so that $P_7$ is not t-set graceful.

Theorem 6.1.4. [2] If $G$ is a $(p,q)$-graph with $p \geq 6$, then $\gamma(G) \leq p - 2$.

Corollary 6.1.5. Let $G$ be any graph such that $K_n \setminus E(C_n) \subseteq G \subseteq K_n; n \geq 6$. Then $G$ is not t-set graceful.

Proof. By theorem 4.2.2, $\tau(G) = n - 1$
\[ > n - 2 \]
\[ \geq \gamma(G), \text{ by theorem 6.1.4.} \]

Lemma 6.1.6. For $n \geq 9$, $\gamma(K_n) \leq n - 3$.

Proof. Let $X = \{x_1, \ldots, x_{n-3}\}; n \geq 9$. Assigning $\emptyset, \{x_1\}, \{x_2\}, \ldots, \{x_{n-3}\}, \{x_1, \ldots, x_4\}, \{x_3, \ldots, x_6\}$ to the vertices of $K_n$ we get a set-indexer of $K_n$.

Theorem 6.1.7. $\gamma(K_n \setminus E(K_{1,3})) \leq n - 3; n \geq 9$.

Proof. The proof follows from theorem 1.0.6 and lemma 6.1.6.

Theorem 6.1.8. Let $G$ be any graph such that $K_{n-1} \subseteq G \subseteq K_n \setminus E(K_{1,3}); n \geq 9$. Then $G$ is not t-set graceful if $\tau(G) = n - 2$.

Proof. The proof follows from theorem 1.0.6 and theorem 6.1.7.

Theorem 6.1.9. If $\tau(G) = \lfloor \log_2 |V| \rfloor$, then $G$ is t-set graceful.

Proof. The proof follows from theorem 4.1.4 and theorem 2.1.11.
Remark 6.1.10. The converse of corollary 6.1.9 is not true in general. By theorem 2.3.15 and theorem 4.3.11 it follows that the cycle $C_5$ is t-set graceful. However, $\lfloor \log_2 |V| \rfloor = 3 < 4 = \tau(C_5)$. Thus, there are t-set graceful graphs $G$ with $\tau(G) > \lfloor \log_2 |V| \rfloor$.

Theorem 6.1.11. For any t-set graceful graph $G$ of order $n \geq 2$, $\tau(G) \leq \lfloor \log_2 \left( \frac{n^3 - 3n^2 + 8n + 6}{6} \right) \rfloor$.

Proof. When $n = 2, 3, n - 1 \leq \gamma(G)$, by theorem 2.1.11
$$\leq \tau(G), \text{ by theorem 4.1.4}$$
$$\leq \tau(K_n), \text{ by theorem 4.1.8}$$
$$= n - 1, \text{ by theorem 4.2.2.}$$

Now assume that $n \geq 4$ and let $G$ be a t-set graceful graph of order $n$. Then, $\tau(G) = \gamma(G)$
$$\leq \gamma(K_n), \text{ by theorem 1.0.6}$$
$$\leq \left\lfloor \log_2 \left( 1 + \sum_{j=0}^{3} \binom{n-1}{j} \right) \right\rfloor, \text{ by theorem 1.0.9.}$$

Clearly,
$$1 + \sum_{j=0}^{3} \binom{n-1}{j} = 1 + \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3}$$
$$= 1 + 1 + (n - 1) + \frac{(n-1)(n-2)}{2} + \frac{(n-1)(n-2)(n-3)}{6}$$
$$= \frac{n^3 - 3n^2 + 8n + 6}{6}.$$

Remark 6.1.12. $C_6$, $F_6$ and $W_5$ are certain t-set graceful graphs that attain the above bound for $\tau(G)$.

Theorem 6.1.13. $\gamma(K_n) \geq \frac{4}{3} \lfloor \log_2 n \rfloor + 2; n \geq 7$.

Proof. By theorem 1.0.8, $\gamma(K_7) = 5$. For $8 \leq n \leq 15$, by theorem 2.1.16, $\gamma(K_n) \geq 6$. For $16 \leq n \leq 31$, by theorem 2.1.16 and theorem 2.1.11, $\gamma(K_n) \geq 8$. Now for $m \geq 5$, choose an integer $n$ such that $2^m \leq n \leq 2^{m+1} - 1$. By theorem 1.0.5,
$$\gamma(K_n) \geq \left\lceil \log_2 \left( \frac{n(n-1)}{2} + 1 \right) \right\rceil$$
6.1 T-Set Graceful Subgraphs

\[ m = \frac{4}{3} \left\lfloor \log_2 n \right\rfloor + 2 > m + \left( \frac{m}{3} + 2 \right) = \frac{4m}{3} + 2 = \frac{4}{3} \left\lfloor \log_2 n \right\rfloor + 2. \]

Now by combining theorem 1.0.9 and theorem 6.1.13 we get,

**Corollary 6.1.14.** \( \frac{4}{3} \left\lfloor \log_2 n \right\rfloor + 2 \leq \gamma(K_n) \leq \left\lfloor \log_2 \left( \frac{n^{3 - 3n^2 + 8n + 6}}{6} \right) \right\rfloor; \)
\( n \geq 7. \)

**Theorem 6.1.15.** \( m \leq \tau(K_{1,n}) \leq \frac{4m}{3} + 2; \) \( 2^m - 1 \leq n \leq 2^{m+1} - 2. \)

**Proof.** Let \( X \) be a set of minimum cardinality so that there is a topology \( \tau \) on \( X \) with \( n + 1 \) open sets. Then by theorem 1.0.38, \( m \leq |X| \leq \frac{4m}{3} + 2. \) Now by assigning the open sets of \( \tau \) to the vertices of \( N_{n+1} \) we get a t-set indexer of \( N_{n+1}. \) But, by theorem 4.1.6, \( \tau(N_{n+1}) = \tau(K_{1,n}). \) Consequently, \( m \leq \tau(K_{1,n}) \leq \frac{4m}{3} + 2. \)

**Corollary 6.1.16.** \( \tau(K_{1,n}) \leq \gamma(K_{n+1}). \)

**Proof.** For \( n \geq 1, \) there exists an \( m \) such that \( 2^m \leq n + 1 \leq 2^{m+1} - 1. \) For \( 1 \leq n \leq 5, \) by theorem 2.1.4, by theorem 5.1.8 and theorem 1.0.8, \( \tau(K_{1,n}) \leq \gamma(K_{n+1}). \) If \( n \geq 6, \) from theorem 6.1.13 and theorem 6.1.15, \( \tau(K_{1,n}) \leq \frac{4m}{3} + 2 \leq \gamma(K_{n+1}). \)

Thus, the t-number of a star graph does not exceed the set-indexing number of the complete graph of the same of order.

**Conjecture 6.1.17.** For every \( n > 1, \) there exists a t-set graceful nonempty graph of order \( n. \)
Lemma 6.1.18. The wheel graph $W_6$ is t-set graceful with t-number 4.

Proof. Let $W_6 = C_6 \cup \{u\}; \; C_6 = (u_1, \ldots, u_6, u_1)$. By theorem 4.1.4, $\tau(W_6) \geq \gamma(W_6) = 4$, by theorem 2.3.24.

But, we can define a t-set indexer $f$ on $W_6$ with t-indexing set $X = \{a, b, c, d\}$ as follows: Assign $\emptyset, \{a\}, \{a, b, d\}, \{b\}, \{a, b, c\}, \{a, b\}$ and $X$ to the vertices $u, u_1, \ldots, u_6$ in that order. Clearly, $f(V)$ is a topology on $X$.

Theorem 6.1.19. Let $G$ be a graph of order 7 such that $C_7 \subseteq G \subseteq W_6$. Then $G$ is t-set graceful.

Proof. By theorem 1.0.5, $4 \leq \gamma(G)$

\[ \leq \tau(G), \text{ by theorem 4.1.4} \]

\[ \leq \tau(W_6), \text{ by theorem 4.1.8} \]

\[ = 4, \text{ by lemma 6.1.18} \]

Thus, $G$ is t-set graceful. \qed

By theorem 4.2.2, it is known that every complete graph $G$ satisfies $\tau(G) = o(G) - 1$ and our quest for t-set graceful complete graphs results in the following:

Theorem 6.1.20. A graph $G$ with t-number $|V| - 1$ is t-set graceful if only if $2 \leq |V| \leq 5$.

Proof. Let $G$ be a t-set graceful graph with $\tau(G) = |V| - 1$. Then, by theorem 4.1.2 and theorem 1.0.13, $2 \leq |V| \leq 5$.

Conversely, suppose $\tau(G) = |V| - 1; \; 2 \leq |V| \leq 5$. Then, the following four cases may occur.
Case 1: $|V| = 2$.

Then $G = K_2$ or $N_2$. Now by applying theorem 2.1.11,

\[ 1 = \lfloor \log_2 2 \rfloor \]

\[ \leq \gamma(G) \]

\[ \leq \tau(G), \text{ by theorem 4.1.4} \]

\[ \leq \tau(K_2), \text{ by theorem 4.1.8} \]

\[ = 1, \text{ by theorem 4.2.2.} \]

Case 2: $|V| = 3$.

Then $G$ is a spanning subgraph of $K_3$ and by applying theorem 2.1.11,

\[ 2 = \lfloor \log_2 3 \rfloor \]

\[ \leq \gamma(G) \]

\[ \leq \tau(G), \text{ by theorem 4.1.4} \]

\[ \leq \tau(K_3), \text{ by theorem 4.1.8} \]

\[ = 2, \text{ by theorem 4.2.2.} \]

Case 3: $|V| = 4$.

Then $G$ is a spanning subgraph of $K_4$ and also a spanning supergraph of $2K_2$. Now by applying theorem 6.1.1,

\[ 3 = \gamma(2K_2) \]

\[ \leq \gamma(G), \text{ by theorem 1.0.6} \]

\[ \leq \tau(G), \text{ by theorem 4.1.4} \]

\[ \leq \tau(K_4), \text{ by theorem 4.1.8} \]

\[ = 3, \text{ by theorem 4.2.2.} \]

Case 4: $|V| = 5$.

Then $G$ is a spanning subgraph of $K_5$ and also a spanning supergraph of $C_5$. Now by applying theorem 2.3.15,

\[ 4 = \gamma(C_5) \]

\[ \leq \gamma(G), \text{ by theorem 1.0.6} \]
\[ \leq \tau(G), \text{by theorem } 4.1.4 \]
\[ \leq \tau(K_5), \text{by theorem } 4.1.8 \]
\[ = 4, \text{by theorem } 4.2.2. \]

**Corollary 6.1.21.** $K_n$ is $t$-set graceful if and only if $2 \leq n \leq 5$.

**Proof.** The proof follows from theorem 4.2.2 and theorem 6.1.20.

**Remark 6.1.22.** There are $t$-set graceful graphs $G$ for which $\tau(G) \neq |V| - 1; 2 \leq |V| \leq 5$. For example, $P_5$ is $t$-set graceful with $t$-number 3.

The following theorem characterizes graphs for which all the spanning supergraphs are $t$-set graceful.

**Theorem 6.1.23.** Every spanning supergraph of a $t$-set graceful graph is $t$-set graceful if and only if $2 \leq |V| \leq 5$.

**Proof.** The necessary part follows from corollary 6.1.21 as every graph with $n \geq 6$ vertices has $K_n$, as a spanning super graph.

Conversely, let $G$ be a $t$-set graceful graph with $2 \leq |V| \leq 5$. To prove that any spanning supergraph $G'$ of $G$ is $t$-set graceful.

Suppose $|V| = 2$ or 3. Then, by theorem 2.1.11,
\[ |V| - 1 = \left\lceil \log_2 |V| \right\rceil \]
\[ \leq \gamma(G') \]
\[ \leq \tau(G'), \text{by theorem } 4.1.4 \]
\[ \leq \tau(K_n), \text{by theorem } 4.1.8 \]
\[ = |V| - 1, \text{by theorem } 4.2.2. \]

Suppose $|V| = 4$ or 5. Then by theorem 2.1.11 and theorem 1.0.5,
\[ |V| - 2 = \left\lceil \log_2 |V| \right\rceil \leq \gamma(G') \leq |V| - 1 \text{ so that } \gamma(G') = |V| - 1 \text{ or } |V| - 2. \]
Case 1: $\gamma(G') = |V| - 1$.

Then by theorem 4.1.4,

$|V| - 1 = \gamma(G')$

$\leq \tau(G')$

$\leq \tau(K_n)$, by theorem 4.1.8

$= |V| - 1$, by theorem 4.2.2.

Case 2: $\gamma(G') = |V| - 2$.


Then $\gamma(G') = 2$ and by theorem 1.0.5, $|E(G')| \leq 3$. Also, by theorem 1.0.6 and theorem 2.4.2, $P_4$ is not subgraph of $G'$. Consequently, $G'$ is a spanning subgraph of $H = K_4 \setminus E(K_{1,3})$ or $H = K_4 \setminus E(K_3)$. Then, by theorem 4.1.4,

$2 = \gamma(G')$

$\leq \tau(G')$

$\leq \tau(H)$, by theorem 4.1.8

$\leq 2$, by theorem 4.2.5 and theorem 4.2.6.

Subcase 2.2. $|V| = 5$.

Consider the complete graph $K_5$. Let $H_1 = K_5 \setminus E(K_{1,3})$, $H_2 = K_5 \setminus E(K_3)$ and $H_3$ be any proper spanning subgraph of $K_5 \setminus E(C_4)$.

By theorem 4.2.5 and theorem 4.2.6, $\tau(H_1) = 3 = \tau(H_2)$.

It can be easily verified that $\tau(H_3) = 3$. Since $\gamma(G') = 3$, by theorem 1.0.5, $|E(G')| \leq 7$. By theorem 1.0.6 and theorem 2.3.15, $C_5 \notin G'$. Consequently, $G'$ is a spanning subgraph of $H_1$ or $H_2$ or $H_3$. Then by theorem 4.1.4 and theorem 4.1.8, $3 = \gamma(G') \leq \tau(G') \leq \tau(H_i) = 3$; $i = 1, 2, 3$.

Corollary 6.1.24. Every graph $G$ with $2 \leq |V| \leq 5$ is t-set graceful.

Proof. By theorem 5.1.12 and theorem 5.1.8 it follows that $N_n$
is t-set graceful for $2 \leq n \leq 5$. Now, the corollary follows from theorem 6.1.23.

Remark 6.1.25. Every proper spanning subgraph of a t-set graceful graph need not be t-set graceful. Consider the graph $G = C_6 \cup \{u_7\}; C_6 = (u_1, \ldots, u_6, u_1)$. By theorem 2.3.19,

$$4 = \gamma(C_6) \leq \gamma(G), \text{ by theorem 1.0.6}$$

$$\leq \tau(G), \text{ by theorem 4.1.4.}$$

We can define a t-set indexer $f$ of $G$ with t-indexing set $X = \{x_1, x_2, x_3, x_4\}$ as follows: Assign $X$, $\emptyset$, $\{x_1\}$, $\{x_2\}$, $\{x_1, x_2, x_3\}$, $\{x_1, x_2\}$, $\{x_1, x_3\}$ to the vertices $u_1, \ldots, u_7$ in that order. Let $G'$ be any proper spanning subgraph of $G$ and by theorem 2.1.11,

$$3 \leq \gamma(G') \leq \gamma(P_6 \cup K_1), \text{ by theorem 1.0.6}$$

$$\leq \gamma(P_7)$$

$$= 3, \text{ by theorem 2.4.2}$$

$$< \tau(G'), \text{ by theorem 4.1.15}$$

$$\leq \tau(G), \text{ by theorem 4.1.8}$$

$$\leq 4$$

so that $G'$ is not t-set graceful.

6.2 Strongly T-Set Graceful Graphs

Unlike the case with spanning supergraphs, there are a variety of graphs for which every spanning subgraph is t-set graceful and this motivates the following:

Definition 6.2.1. A graph $G$ is said to be strongly t-set graceful, if every spanning subgraph of $G$ is t-set graceful.
Obviously, strongly t-set graceful graphs are t-set graceful. Also, $G$ is strongly t-set graceful if and only if every spanning subgraph of $G$ is strongly t-set graceful.

**Theorem 6.2.2.** $K_{1,n}$ is t-set graceful if and only if it is strongly t-set graceful.

**Proof.** The proof follows from theorem 5.1.12. \hfill \Box

**Theorem 6.2.3.** If $K_{1,n}$ is not t-set graceful, then no graph of order $n + 1$ is strongly t-set graceful.

**Proof.** Since $K_{1,n}$ is not t-set graceful, by theorem 5.1.13, no spanning subgraph of it is t-set graceful. In particular, $N_{n+1}$ is not t-set graceful. \hfill \Box

**Remark 6.2.4.** The above theorem states that the t-set graceful-ness of the star graph of the same order is a necessary condition for a given graph to be strongly t-set graceful. However, the converse is not true. By corollary 6.1.21, $K_8$ is not strongly t-set graceful, but $K_{1,7}$ is t-set graceful by theorem theorem 5.1.8.

**Corollary 6.2.5.** No graph of order $n; 3 \cdot 2^{m-2} < n < 2^m$, $m \geq 3$ is strongly t-set graceful.

**Proof.** By theorem 4.1.15 and theorem 2.1.4, $K_{1,n-1}$ is not t-set graceful. Now, the corollary follows from theorem 6.2.3. \hfill \Box

**Theorem 6.2.6.** Every graph of order $m; 2 \leq m \leq 5$ is strongly t-set graceful.

**Proof.** The proof follows from corollary 6.1.24. \hfill \Box

**Theorem 6.2.7.** The graph $H = (K_8 \setminus E(K_{1,3})) \cup K_1$ is strongly t-set graceful.
Proof. Let \( u_1, \ldots, u_6 \) be the vertices of \( H \) such that \( d(u_1) = d(u_2) = d(u_3) = 3, d(u_4) = 4, d(u_5) = 1 \) and \( d(u_6) = 0 \). By theorem 2.1.11 and theorem 4.1.4, \( 3 \leq \gamma(H) \leq \tau(H) \). Let \( X = \{a, b, c\} \). Now by assigning \( \{a\}, \{a, b\}, X, \emptyset, \{a, c\}, \{b\} \) to the vertices \( u_1, \ldots, u_6 \) in that order we get a t-set indexer of \( H \) with \( X \) as the indexing set. Now the result follows from theorem 2.1.11, theorem 4.1.4 and theorem 4.1.8.

Theorem 6.2.8. Every graph \( G \) with \( \tau(G) = \lfloor \log_2 |V| \rfloor \) is strongly t-set graceful.

Proof. Let \( G' \) be a spanning subgraph of \( G \). Then by applying theorem 2.1.11, \[ \lceil \log_2 |V| \rceil \leq \gamma(G') \]
\[ \leq \gamma(G) \], by theorem 1.0.6
\[ \leq \tau(G) \], by theorem 4.1.4
\[ = \lfloor \log_2 |V| \rfloor \].

Also, from theorem 2.1.11, \( \lfloor \log_2 |V| \rfloor \leq \gamma(G') \)
\[ \leq \tau(G') \], by theorem 4.1.4
\[ \leq \tau(G) \], by theorem 4.1.8
\[ = \lceil \log_2 |V| \rceil \].

Thus, \( \gamma(G') = \tau(G') = \lfloor \log_2 |V| \rfloor \), so that \( G' \) is t-set graceful.

Remark 6.2.9. Every spanning subgraph of \( K_5 \) is t-set graceful, by corollary 6.1.24 and hence \( K_5 \) is strongly t-set graceful. But, \( \tau(K_5) = 4 > 3 = \lfloor \log_2 |V(K_5)| \rfloor \). Thus, the converse of theorem 6.2.8 is not true.

Theorem 6.2.10. Every t-set graceful path \( P_n; n \neq 2^m \) is strongly t-set graceful.

Proof. The proof follows from theorem 2.4.2 and theorem 6.2.8.
Theorem 6.2.11. Double star of order $2^n + 2^m$; $m, n \geq 0$ is strongly t-set graceful.

Proof. Let $G$ be a double star of order $2^n + 2^m$.

Case 1: $m = n$. Then $|V| = 2^{n+1}$.

Subcase 1.1. Both $m$ and $n$ are even.

Then, by theorem 5.1.29, $G$ is t-set graceful and by theorem 3.1.3, $\gamma(G) = \tau(G) = n + 1 = \lceil \log_2 |V| \rceil$. Now, by theorem 6.2.8, every spanning subgraph of $G$ are t-set graceful.

Subcase 1.2. Both $m$ and $n$ are odd.

By theorem 5.1.29, $G$ is t-set graceful and by theorem 3.1.3, $\gamma(G) = \tau(G) = n + 2$. Let $G'$ be any spanning subgraph of $G$. Then, by theorem 2.1.11 and theorem 1.0.6, $\lceil \log_2 |V| \rceil = \lceil \log_2 2^{n+1} \rceil = n + 1 \leq \gamma(G') \leq \gamma(G) = n + 2$. Thus, $\gamma(G') = n + 1$ or $n + 2$.

If $\gamma(G') = n + 1$, then the optimal set-indexer $f$ of $G'$ becomes a t-set indexer forming the discrete topology $f(V)$ on the corresponding indexing set.

If $\gamma(G') = n + 2$, then by theorem 4.1.4 and theorem 4.1.8 it follows that $n + 2 = \gamma(G') \leq \tau(G') \leq \tau(G) = n + 2$. Thus, if $m = n$, then every spanning subgraph of $G$ is t-set graceful with t-number either $n + 1$ or $n + 2$.

Case 2: $m \neq n$.

By theorem 5.1.31, $\gamma(G) = \tau(G) = n + 1$. Let $G'$ be any spanning subgraph of $G$. Then, $n + 1 = \lceil \log_2 |V| \rceil$

$\leq \gamma(G')$, by theorem 2.1.11

$\leq \tau(G')$, by theorem 4.1.4

$\leq \tau(G)$, by theorem 4.1.8

$= n + 1$. 

Thus, in this case every spanning subgraph of \( G \) is t-set graceful with t-number \( n + 1 \).

\[ \square \]

**Theorem 6.2.12.** *Every tree of order 6 is strongly t-set graceful.*

**Proof.** By theorem 5.1.4 and theorem 6.2.2, \( K_{1,5} \) is strongly t-set graceful. Again by theorem 5.2.7 and theorem 6.2.10, \( P_6 \) is strongly t-set graceful. Further, by theorem 6.2.11, the two double stars of order 6 are strongly t-set graceful. Let \( u_1, \ldots, u_6 \) be the vertices of the remaining two trees (viz. the caterpillars of diameter 4) say, \( T_1 \) and \( T_2 \). Let \( (u_1, u_2, u_3, u_4, u_5) \) be the longest path in both \( T_1 \) and \( T_2 \) and let \( (u_2, u_6) \in T_1 \) and \( (u_3, u_6) \in T_2 \). Let \( X = \{a, b, c\} \). Then by assigning \( \{a, c\}, \emptyset, \{a, b\}, X, \{a\} \) and \( \{b\} \) to the vertices \( u_1, \ldots, u_6 \) of \( T_1 \) in that order and by assigning \( \{a, c\}, \{b\}, \emptyset, \{a, b\}, X \) and \( \{a\} \) to the vertices \( u_1, \ldots, u_6 \) of \( T_2 \) in that order we get t-set indexers of \( T_1 \) and \( T_2 \) with \( X \) as the t-indexing set. Now by theorem 2.1.11 and theorem 4.1.4, \( 3 = \lceil \log_2 |V_i| \rceil \leq \gamma(T_i) \leq \tau(T_i) \leq 3 \) for \( i = 1, 2 \). Consequently, both \( T_1 \) and \( T_2 \) are strongly t-set graceful by theorem 6.2.8. \[ \square \]

**Corollary 6.2.13.** *Every tree of order at most 6 is strongly t-set graceful.*

**Proof.** The proof follows from theorem 6.2.12 and theorem 6.1.24. \[ \square \]

**Example 6.2.14.** *\( C_6 \) is strongly t-set graceful.*

Let \( C_6 = (v_1, \ldots, v_6, v_1) \). By theorem 4.1.4 and theorem 2.3.19, \( \tau(C_6) \geq \gamma(C_6) = 4 \). Let \( X = \{a, b, c, d\} \). Now by assigning \( \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X, \{a, c\} \) to the vertices \( v_1, \ldots, v_6 \), in that order, we get a t-set indexer of \( C_6 \) with t-indexing set \( X \). Thus, \( C_6 \) is t-set graceful and the result follows from theorem 6.2.12.
Example 6.2.15. \( C_5 \cup K_1 \) is strongly t-set graceful.

Let \( C_5 = (u_1, \ldots, u_5, u_1) \) and \( K_1 = \{u\} \). By theorem 2.3.15, theorem 1.0.6 and theorem 4.1.4, \( 4 = \gamma(C_5) \leq \gamma(C_5 \cup K_1) \leq \tau(C_5 \cup K_1) \). Now define a t-set indexer \( f \) on \( C_5 \cup K_1 \) with t-indexing set \( X = \{a, b, c, d\} \) as follows: Assign \( \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X \) and \( \{b\} \) to the vertices \( u_1, \ldots, u_5 \) and \( u \), in that order. Thus, \( C_5 \cup K_1 \) is t-set graceful and by theorem 6.2.12, \( C_5 \cup K_1 \) is strongly t-set graceful.

Example 6.2.16. \( K_4 \cup K_2 \) is strongly t-set graceful.

Let \( V = \{u_1, \ldots, u_6\} \); \( K_2 = (u_5, u_6) \). By theorem 2.1.11, \( \gamma(K_4 \cup K_2) \geq 3 \). Let \( X = \{a, b, c\} \). Then,

\[
\{a\} = \emptyset \oplus \{a\} = \{a, b\} \oplus \{b\} = \{a, c\} \oplus \{c\} = X \oplus \{b, c\} \quad \text{and} \\
\{b\} = \emptyset \oplus \{b\} = \{a, b\} \oplus \{a\} = \{b, c\} \oplus \{c\} = X \oplus \{a, c\}.
\]

Consequently, there does not exist a set-indexer of \( K_4 \cup K_2 \) with \( X \) as the indexing set. But, by assigning \( \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, Y = \{a, b, c, d\} \) and \( \{b\} \) to the vertices \( u_1, \ldots, u_6 \) in that order we get a t-set indexer \( f \) of \( K_4 \cup K_2 \) with \( Y \) as the t-indexing set. Thus, \( K_4 \cup K_2 \) is t-set graceful with t-number 4.

Now consider the spanning subgraphs \( G_1 \) and \( G_2 \) of \( K_4 \cup K_2 \) given by \( G_1 = K_4 \cup K_2 \setminus (u_1, u_3) \) and \( G_2 = K_4 \cup K_2 \setminus (u_5, u_6) \). Clearly, every spanning subgraph of \( K_4 \cup K_2 \) is either a spanning subgraph of \( G_1 \) or \( G_2 \). Assigning \( \emptyset, \{a, b\}, X, \{a\}, \{b\} \) and \( \{a, c\} \) to \( u_1, \ldots, u_6 \), in that order, we get t-set indexers for \( G_1 \) and \( G_2 \) with \( X \) as the indexing set. Then, by theorem 2.1.11 and theorem 4.1.4, \( G_1 \) and \( G_2 \) are t-set graceful with t-number \( 3 = \lceil \log_2 |V| \rceil \).

Now, the result follows from theorem 6.2.8.

Example 6.2.17. \( 2K_3 \) is strongly t-set graceful.

Let \( G = 2K_3 = (u_1, u_2, u_3, u_1) \cup (v_1, v_2, v_3, v_1) \). By theorem 2.1.11,
Consider the set \( X = \{x_1, x_2, x_3\} \). Since \( \{A_1 \oplus A_2, A_1 \oplus A_3, A_2 \oplus A_3\} \cap \{B_1 \oplus B_2, B_1 \oplus B_3, B_2 \oplus B_3\} \neq \emptyset \), we cannot find a set-indexer of \( G \) with \( X \) as the indexing set. So \( \gamma(G) \geq 4 \). Now, by assigning \( \emptyset, \{a\}, \{a, b\}, \{b\}, \{a, b, c\} \) and \( Y = \{a, b, c, d\} \) to the vertices \( u_1, u_2, u_3, v_1, v_2 \) and \( v_3 \) in that order we get a \( t \)-set indexer of \( G \) with \( Y \) as the \( t \)-indexing set. Thus \( G \) is \( t \)-set graceful with \( t \)-number 4.

Let \( H = G \setminus (u_1, u_2) \). Assign \( \emptyset, \{b\}, \{a, c\}, \{a\}, \{a, b\} \) and \( X \) to the vertices \( u_1, u_2, u_3, v_1, v_2 \) and \( v_3 \) in that order, we get a \( t \)-set indexer of \( H \). Then, by theorem 2.1.11 and theorem 4.1.4, \( 3 = \lceil \log_2 |V| \rceil \leq \gamma(H) \leq \tau(H) = 3 \) and hence, \( H \) is \( t \)-set graceful with \( t \)-number \( \lceil \log_2 |V| \rceil = 3 \). Consequently, \( G \) is strongly \( t \)-set graceful by theorem 6.2.8.

Remark 6.2.18. For the strongly \( t \)-set graceful graphs given by the above examples, \( \tau(G) > \lceil \log_2 |V| \rceil \). Thus, the converse of theorem 6.2.8 is wrong.

Corollary 6.2.19. If \( P_n \cup K_1; n \geq 4 \) is \( t \)-set graceful, then it is strongly \( t \)-set graceful.

Proof. The proof follows from theorem 6.2.8 and theorem 2.4.3.

Theorem 6.2.20. \( C_{2^n-1} \cup K_1 \) is strongly \( t \)-set graceful.

Proof. Let \( G' \) be any spanning subgraph of \( G = C_{2^n-1} \cup K_1 \). Then, \( n = \lceil \log_2 |V| \rceil \)

\[ \leq \gamma(G'), \text{ by theorem 2.1.11} \]

\[ \leq \gamma(G), \text{ by theorem 1.0.6} \]

\[ = n, \text{ by theorem 2.3.2.} \]

Thus, \( \gamma(G') = n \) and the optimal set-indexer \( f \) defines the discrete
topology \( f(V) \) on the corresponding indexing set. Consequently, 
\( G' \) is t-set graceful. \( \square \)

**Remark 6.2.21.** Set-graceful and t-set graceful are two independent notions. For example, \( C_{2n-1} \) is set-graceful but not t-set graceful; whereas \( P_5 \) is t-set graceful but not set-graceful. Furthermore, \( C_{2n-1} \cup K_1 \) is both set-graceful and t-set graceful.

The following theorem provides a class of graphs for which neither the graph nor the spanning subgraphs are t-set graceful.

**Theorem 6.2.22.** Spanning subgraphs of \( C_{2n-1} \); \( n \geq 3 \) are not t-set graceful.

**Proof.** Let \( G \) be any spanning subgraph of \( C_{2n-1} \). Then,
\[
 n = \lceil \log_2 |V| \rceil 
\leq \gamma(G), \text{ by theorem 2.1.11} 
\leq \gamma(C_{2n-1}), \text{ by theorem 1.0.6} 
= n, \text{ by theorem 2.3.1.} 
\]
But, by theorem 4.1.15, \( \tau(G) \geq n + 1 \) so that \( G \) is not t-set graceful. \( \square \)

Thus, \( C_{2n-1}; n \geq 3 \) is not strongly t-set graceful. However, based on the investigations carried out we strongly believe that the same is not true in the case of \( C_{2n} \) and even put forward the following:

**Conjecture 6.2.23.** \( C_{2n} \) is strongly t-set graceful.

**Remark 6.2.24.** There are t-set graceful graphs for which no proper spanning subgraph is t-set graceful. Consider the graph \( G = C_6 \cup K_1 \). By theorem 2.3.19,
4 = γ(C_6)
≤ γ(G), by theorem 1.0.6
≤ τ(G), by theorem 4.1.4
≤ τ(F_7), by theorem 4.1.8
= 4, by remark 6.1.3
and hence, G is t-set graceful with t-number 4. Consider P_6 ∪ K_1
and by theorem 2.1.11,
3 ≤ γ(P_6 ∪ K_1)
≤ γ(P_7), by theorem 1.0.6
= 3, by theorem 2.4.2
< τ(P_6 ∪ K_1), by theorem 4.1.15.
Clearly, P_6 ∪ K_1 is not t-set graceful. Let G' be any proper
spanning subgraph of G. Clearly, G' ⊆ G and by theorem 2.1.11,
3 ≤ γ(G')
≤ γ(P_6 ∪ K_1), by theorem 1.0.6
= 3, by theorem 2.4.3 and theorem 2.4.2
< τ(G'), by theorem 4.1.15.
Thus, G' is not t-set graceful. This motivates the following defi-
dition.

Definition 6.2.25. A t-set graceful graph is said to be weakly t-
set graceful if no proper spanning subgraph of it is t-set graceful.

Any nonempty strongly t-set graceful graph is not weakly t-set
graceful. There are empty graphs N_{10}, N_9, N_8, N_6 and N_5 which
are both strongly and weakly t-set graceful.

Example 6.2.26. C_5 ∪ N_2 is weakly t-set graceful.
By theorem 2.3.15, 4 = γ(C_5)
≤ γ(C_5 ∪ N_2), by theorem 1.0.6
6.2 Strongly T-Set Graceful Graphs

\[ \leq \tau(C_5 \cup N_2), \text{ by theorem 4.1.4} \]
\[ \leq \tau(F_7), \text{ by theorem 4.1.8} \]
\[ = 4, \text{ by remark 6.1.3}. \]

Let \( G' \) be any proper spanning subgraph of \( C_5 \cup N_2 \). Clearly, \( G' \subseteq P_7 \) and \( 3 = \lceil \log_2 |V| \rceil \)
\[ \leq \gamma(G'), \text{ by theorem 2.1.11} \]
\[ \leq \gamma(P_7), \text{ by theorem 1.0.6} \]
\[ = 3, \text{ by theorem 2.4.2} \]
\[ < \tau(G'), \text{ by theorem 4.1.15}. \]

Thus, \( G' \) is not t-set graceful. Hence, \( C_5 \cup N_2 \) is weakly t-set graceful.

Remark 6.2.27. There are t-set graceful graphs which are neither strongly nor weakly t-set graceful. By remark 6.1.3, \( F_7 \) is a t-set graceful graph which is not strongly t-set graceful. But, by theorem 6.1.19 it follows that \( F_7 \) is not weakly t-set graceful.

Theorem 6.2.28. Every graph \( G \) of order \( 2^n + 2^m; m, n \geq 0 \) have at least one t-set graceful spanning subgraph.

Proof. By theorem 5.1.7, \( K_{1,2^n+2^m-1} \) is t-set graceful and \( N_{2^n+2^m} \subseteq K_{1,2^n+2^m-1} \). But, by theorem 5.1.12, \( N_{2^n+2^m} \) is t-set graceful.

Remark 6.2.29. Every graph with atleast two vertices contains the t-set graceful subgraph \( N_2 \). But, not all graph with at least two vertices contain a t-set graceful spanning subgraph. Consider the path \( P_7 \), by theorem 2.4.2, \( \gamma(P_7) = 3 \). Let \( G \) be any spanning subgraph of \( P_7 \). Then, \( 3 = \lceil \log_2 7 \rceil \)
\[ \leq \gamma(G), \text{ by theorem 2.1.11} \]
\[ \leq \gamma(P_7), \text{ by theorem 1.0.6} \]
\[ = 3, \text{ by theorem 2.4.2} \]
< \tau(G)$, by theorem 4.1.15.

Consequently, $G$ is not t-set graceful.

6.3 Discrete T-Set Graceful Graphs

This section characterizes graphs $G$ for which $o(G) = 2^{\gamma(G)}$. Surprisingly, they form a subclass of strongly t-set graceful graphs.

Definition 6.3.1. A graph $G$ with optimal set-indexer $f$ is said to be discrete topologically set-graceful (discrete t-set graceful) if $G$ is t-set graceful and $f(V)$ is the discrete topology.

Example 6.3.2. $K_{1,6} \cup K_1$ is discrete t-set graceful. Let $G = K_{1,6} \cup K_1$. By theorem 2.1.11 and theorem 4.1.4, $\tau(G) \geq \gamma(G) \geq 3$. But by assigning $0$ to the central vertex of $K_{1,6}$ and the distinct nonempty subsets of $X = \{a, b, c\}$ to the other vertices of $G$ in any order we get an optimal t-set indexer of $G$. Consequently, $\tau(G) = 3 = \gamma(G)$.

Remark 6.3.3. Discrete t-set graceful and set-graceful are two independent notions. For instance, $K_6$ is set-graceful (by theorem 1.0.17) but it is not discrete t-set graceful as it is not t-set graceful by corollary 6.1.21. On the other hand $K_{1,6} \cup K_1$ is discrete t-set graceful but it is not set graceful (by theorem 1.0.18).

Remark 6.3.4. Let $G$ be any graph. By theorem 2.1.11 and theorem 4.1.4, $\lceil \log_2 |V| \rceil \leq \gamma(G) \leq \tau(G)$

$\Rightarrow |V| \leq 2^{\gamma(G)} \leq 2^{\tau(G)}$.

$K_{1,6}$ is an example for which these inequalities become strict. Recall that $\gamma(K_{1,6}) = 3$ and $\tau(K_{1,6}) = 4$. Again, there are graphs that make only the first inequality into strict. Note that
\[ \gamma(P_6) = \tau(P_6) = 3. \] However, if \( |V| = 2^{\gamma(G)} \), then the optimal set-indexer \( f \) corresponding to \( \gamma(G) \) becomes a t-set indexer of \( G \) with discrete topology \( f(V) \). Consequently, \( \gamma(G) = \tau(G) \) so that \( |V| = 2^{\gamma(G)} = 2^{\tau(G)} \).

Thus we have,

**Theorem 6.3.5.** A graph \( G \) is discrete t-set graceful if and only if \( |V| = 2^{\gamma(G)} \).

**Remark 6.3.6.** From the above theorem it follows that a graph whose order is not a power of 2 is never discrete t-set graceful. For example, \( K_6 \) is not discrete t-set graceful even though it is t-set graceful by corollary 6.1.21.

**Corollary 6.3.7.** If \( G \) is discrete t-set graceful, then \( s(G) < o(G) \).

**Proof.** By theorem 1.0.5,
\[
\left\lfloor \log_2(|E| + 1) \right\rfloor \leq \gamma(G)
\]
\[
= \log_2 |V|, \text{ by theorem 6.3.5}
\]
\[
\Rightarrow |E| + 1 \leq |V|
\]
\[
\Rightarrow s(G) \leq o(G).
\]

**Remark 6.3.8.** Since \( K_{1,5} \) is not discrete t-set graceful, the converse of corollary 6.3.7 is not true.

**Corollary 6.3.9.** \( C_{2^n-1} \cup K_1 \) is discrete t-set graceful.

**Proof.** By corollary 2.3.2, \( \gamma(C_{2^n-1} \cup K_1) = n \). Now by theorem 6.3.5, \( C_{2^n-1} \cup K_1 \) is discrete t-set graceful.

The following theorem characterizes discrete t-set graceful trees.

\[ \gamma(P_6) = \tau(P_6) = 3. \] However, if \( |V| = 2^{\gamma(G)} \), then the optimal set-indexer \( f \) corresponding to \( \gamma(G) \) becomes a t-set indexer of \( G \) with discrete topology \( f(V) \). Consequently, \( \gamma(G) = \tau(G) \) so that \( |V| = 2^{\gamma(G)} = 2^{\tau(G)} \).
Theorem 6.3.10. A tree is discrete t-set graceful if and only if it is set-graceful.

Proof. Let \( T \) be a set-graceful tree. Then
\[
\gamma(T) = \log_2(|E| + 1) = \log_2 |V|
\]
\[
\Rightarrow o(T) = 2^{\gamma(T)}.
\]
Then \( T \) is discrete t-set graceful by theorem 6.3.5.

Conversely, let \( T \) be discrete t-set graceful. Then by theorem 6.3.5,
\[
\gamma(T) = \tau(T) = \log_2 |V| = \log_2(|E| + 1), \text{ since } T \text{ is a tree}
\]
\[
\Rightarrow T \text{ is set-graceful.} \qed
\]

Corollary 6.3.11. \( K_{1,2^r-1} \) is discrete t-set graceful.

Proof. By theorem 1.0.16, \( K_{1,2^r-1} \) is set-graceful. Now the corollary follows from theorem 6.3.10. \( \square \)

Corollary 6.3.12. The double star \( ST(m,n); m + n + 2 = 2^l \) and \( m \) is even is discrete t-set graceful.

Proof. By theorem 3.1.3, \( \gamma(ST(m,n)) = l \) so that it is set-graceful. Now, the corollary follows from theorem 6.3.10. \( \square \)

Theorem 6.3.13. Every spanning subgraph of a discrete t-set graceful graph is discrete t-set graceful.

Proof. Let \( H \) be any spanning subgraph of a discrete t-set graceful graph \( G \). By theorem 2.1.11,
\[
\lceil \log_2 |V| \rceil \leq \gamma(H)
\]
\[
\leq \tau(H), \text{ by theorem 4.1.4}
\]
\[
\leq \tau(G), \text{ by theorem 4.1.8}
\]
\[
= \log_2 |V|, \text{ by theorem 6.3.5.}
\]
\[
\Rightarrow \tau(H) = \log_2 |V|
\]
\[
\Rightarrow H \text{ is discrete t-set graceful, by theorem 6.3.5.} \qed
\]
Corollary 6.3.14. Every discrete t-set graceful graph is strongly t-set graceful.

Proof. Since every discrete t-set graceful graph is t-set graceful, the corollary follows from theorem 6.3.13. □

Remark 6.3.15. Obviously, all discrete t-set graceful graphs that are set-graceful will also be set-semi-graceful, t-set graceful and strongly t-set graceful. By theorem 1.0.16 and corollary 6.3.11, star graphs of order a power of 2 belong to the above category. However, not all graphs in this category are trees. For example, $C_{2^n-1} \cup K_1$ is both discrete t-set graceful and set-graceful by corollary 6.3.9 and corollary 2.3.2.

Note 6.3.16. (i). There are set-semi-graceful graphs that are neither set-graceful nor t-set graceful. For example, $P_{2^n-1}; n \geq 3$ is set-semi-graceful (see remark 3.1.2(iv)) but not set-graceful (by theorem 1.0.18). Again,

\[ \gamma(P_{2^n-1}) = n, \text{ by theorem 2.4.2} \]

\[ < \tau(P_{2^n-1}), \text{ by theorem 4.1.15} \]

so that $P_{2^n-1}; n \geq 3$ is not t-set graceful.

(ii). By theorem 2.3.1, the cycle $C_{2^n-1}; n \geq 3$ is set-graceful so that $\gamma(C_{2^n-1}) = n$

\[ < \tau(C_{2^n-1}), \text{ by theorem 4.1.15}. \]

Therefore, the cycles $C_{2^n-1}; n \geq 3$ constitute a class of set-graceful graphs which are not t-set graceful.

(iii). Recall from remark 4.2.10 that,

\[ \tau(K_6 \cup K_1) = 4 \]

\[ \geq \gamma(K_6 \cup K_1), \text{ by theorem 4.1.4} \]

\[ \geq \gamma(K_6), \text{ by theorem 1.0.6} \]
Thus, $K_6 \cup K_1$ is set-graceful as well as t-set graceful. However, it is not strongly t-set graceful as the spanning subgraph $N_7$ is not t-set graceful. Note that,
\[
\gamma(N_7) = \gamma(K_1, g), \text{ by theorem 2.1.6}
\]
\[
= 3, \text{ by theorem 2.1.4}
\]
\[
< \tau(N_7), \text{ by theorem 4.1.15}.
\]

(iv). It is known that, $K_3$ is set-graceful (by theorem 1.0.17) and strongly t-set graceful (by theorem 6.2.6). But $K_3$ is not discrete t-set graceful by theorem 1.0.8 and theorem 6.3.5.

(v). The family of stars $K_{1, 2^n - 1}$ is set-graceful as well as discrete t-set graceful by theorem 1.0.16 and theorem 6.3.10.

(vi). There are set-semigraceful graphs which are not set-graceful but discrete t-set graceful. By corollary 6.3.11 and theorem 6.3.13, $K_{1, 2^n - 2} \cup K_1$ is discrete t-set graceful. But by theorem 1.0.18, it is not set-graceful. Further,
\[
n = \lceil \log_2 |E(K_{1, 2^n - 2} \cup K_1)| + 1 \rceil
\]
\[
\leq \gamma(K_{1, 2^n - 2} \cup K_1), \text{ by theorem 1.0.5}
\]
\[
\leq \gamma(K_{1, 2^n - 1}), \text{ by theorem 1.0.6}
\]
\[
= n, \text{ by theorem 2.1.4}
\]
so that $K_{1, 2^n - 2} \cup K_1$ is set-semigraceful.

(vii). $K_{1, 2^n - 1} \cup N_{2^n}$ constitute a family of discrete t-set graceful graphs which are not set-semigraceful. Note that,
\[
\lceil \log_2 (|E| + 1) \rceil = n
\]
\[
< n + 1
\]
\[
= \lceil \log_2 |V| \rceil
\]
6.3 Discrete T-Set Graceful Graphs

\[ \leq \gamma(K_{1,2^{n-1}} \cup N_{2^n}), \text{ by theorem 2.1.11} \]
\[ \leq \gamma(K_{1,2^{n+1}-1}), \text{ by theorem 1.0.6} \]
\[ = n + 1, \text{ by theorem 2.1.4} \]

so that \( G \) is not set-semigraceful and \( \gamma(G) = n + 1 \). Then by theorem 6.3.5, \( G \) is discrete t-set graceful.

**(viii).** Now consider the family of graphs \( P_{2^n-1} \cup N_3; n \geq 3 \). Obviously, \( \lceil \log_2(|E| + 1) \rceil = n \)
\[ < n + 1 \]
\[ = \lceil \log_2 |V| \rceil \]
\[ \leq \gamma(P_{2^n-1} \cup N_3), \text{ by theorem 2.1.11} \]
\[ \leq \gamma(P_{2^n+2}), \text{ by theorem 1.0.6} \]
\[ = n + 1, \text{ by theorem 2.4.2.} \]

Thus, \( P_{2^n-1} \cup N_3 \) is not set-semigraceful and \( \gamma(P_{2^n-1} \cup N_3) = n + 1 \neq \log_2 |V| \) so that by theorem 6.3.5, \( P_{2^n-1} \cup N_3 \) is not discrete t-set graceful. Now by theorem 5.2.7, \( P_{2^n+2} \) is t-set graceful and hence strongly t-set graceful, by theorem 6.2.10. Being a spanning subgraph of a strongly t-set graceful graph \( P_{2^n-1} \cup N_3 \) is strongly t-set graceful. Thus, there are strongly t-set graceful graphs that are neither discrete t-set graceful nor set-semigraceful.

**(ix).** Further, there are t-set graceful graphs that are neither strongly t-set graceful nor set-semigraceful. For example, \( C_6 \cup K_1 \) is one of such graphs as shown in remark 6.2.24.

**(x).** We know that \( W_6 \) is set-semigraceful (by remark 3.3.17) and t-set graceful (by lemma 6.1.18). However, \( W_6 \) is not strongly t-set graceful as the spanning subgraph \( C_6 \cup K_1 \) is not strongly t-set graceful. Again, by theorem 1.0.18, \( W_6 \) is not set-graceful.
(xi). By theorem 6.2.6 and remark 3.1.2(vi), $K_4$ is strongly t-set graceful and set-semigraceful. But $K_4$ is not set-graceful by theorem 1.0.17. Finally, by theorem 1.0.8 and theorem 6.3.5, $K_4$ is not discrete t-set graceful.

The above findings are displayed in the following diagram.
6.3 Discrete T-Set Graceful Graphs

Fig 6.1: The Interrelations between Certain Categories of Graphs.

- **Set-Semigraceful Graphs**
- **Set-Graceful Graphs**
- **T-Set Graceful Graphs**
- **Strongly T-Set Graceful Graphs**
- **Discrete T-Set Graceful Graphs**