Chapter 3

Set-Semigraceful Graphs

This chapter identifies certain set-semigraceful graphs and obtains some properties of them. Certain bounds for the order and size of such set-semigraceful graphs are derived. More set-semigraceful graphs from given ones are also obtained through various graph theoretic methods.

3.1 Certain Set-Semigraceful Graphs

In 1986, B. D. Acharya introduced the concept of set-semigraceful graphs:

Definition 3.1.1. [2] A graph $G$ is said to be set-semigraceful if $\gamma(G) = \lfloor \log_2( |E| + 1) \rfloor$.

The following are some observations recorded for the convenience of future references.

Remark 3.1.2. (i). The Heawood graph is set-semigraceful by theorem 1.0.10.
(ii). The Petersen graph is not set-semigraceful by theorem 1.0.11.

(iii). The stars are set-semigraceful by theorem 2.1.4.

(iv). The paths $P_n$ is set-semigraceful if and only if $n \neq 2^m; m > 1$ by theorem 2.4.2.

(v). $C_3$ and $C_4$ are set-semigraceful by theorem 2.3.1 and theorem 2.3.11 respectively.

(vi). For $1 \leq n \leq 15$, $K_n$ is set-semigraceful if and only if $n \in \{1, \ldots, 7, 9, 12\}$ by theorem 1.0.8 and theorem 2.1.16.

(vii). The helm graph $H_{2^n-1}; n \geq 2$ is set-semigraceful by theorem 2.3.7.

(viii). All set-graceful graphs are set-semigraceful.

(ix). The path $P_7$ is set-semigraceful but it is not set-graceful.

(x). A graph $G$ of size $2^m - 1$ is set-semigraceful if and only if it is set-graceful.

(xi). Not all graphs of size $2^m - 1$ is set-semigraceful. For example $K_{3,5}$ is not set-semigraceful by theorem 1.0.15.

(xii). The complete bi-partite graphs $K_{2^m-1,n}; m > \lceil \log_2 n \rceil$ is set-semigraceful by (iii) and theorem 2.2.1.

(xiii). $K_{2^m,n}$ is set-semigraceful by theorem 2.2.2.

(xiv). The complete tri-partite graphs $K_{1,2^m-1,n}$ is set-semigraceful by theorem 2.2.5.

(xv). The complete 4-partite graphs $K_{1,1,2^n-1,2^n-1}$ is set-semigraceful by theorem 2.2.6.
(xvi). The complete 5-partite graphs $K_{1,1,1,1,2^{n-1}}$ is set-semigraceful by theorem 2.2.7.

(xvii). $N_n$ is set-semigraceful if and only if $n = 1$.

In order to identify the set-semigraceful double stars, we derive the set-indexing numbers of them in the following two theorems.

**Theorem 3.1.3.** For a double star graph $ST(m, n)$ with $|V| = 2^l; l \geq 2$

$$\gamma(ST(m, n)) = \begin{cases} l & \text{if } m \text{ is even}, \\ l + 1 & \text{if } m \text{ is odd}. \end{cases}$$

**Proof.** Let $V(K_{1,m}) = \{u_0, u_1, \ldots, u_m\}$ and $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$ where $u_0$ and $v_0$ are the central vertices. By theorem 1.0.5, $\gamma(ST(m, n)) \geq l$. Consider the set $X = \{x_1, \ldots, x_l\}$. Since $|V| = m + n + 2 = 2^l$, either both $m$ and $n$ are even or odd.

Case 1: $n$ is even.

Consider the collection $A_1, A_2, \ldots, A_{\frac{n}{2}}$ of distinct subsets of $X$ chosen by the following rule: $A_1$ is any nonempty proper subset of $X$ and for $1 < i \leq \frac{n}{2}$, $A_i$ is a nonempty proper subset of $X$ distinct from $A_1, A_2, \ldots, A_{i-1}$ such that $A_i \neq A_k$ for all $k; 1 \leq k \leq i - 1$. Since $\frac{n}{2} < 2^{l-1} - 1$, the above choice is possible. Now, for $\frac{n}{2} + 1 \leq j \leq n$, let $B_j = A_{\frac{n}{2}} - A_{\frac{n}{2}}$. Define a set-indexer $f$ on $ST(m, n)$ as follows:

$$f(u_0) = 0, \quad f(v_0) = X$$

$$f(v_i) = A_i; \ 1 \leq i \leq \frac{n}{2} \text{ and}$$

$$f(v_j) = B_j; \ \frac{n}{2} + 1 \leq j \leq n.$$ 

Now assign the remaining $(m + n + 2) - (2 + \frac{n}{2} + \frac{n}{2}) = m$ subsets of $X$ to the vertices $u_1, u_2, \ldots, u_m$. Clearly, the edge labels

$$f(u_0, u_k) = f(u_k) \text{ for } 1 \leq k \leq m, \ f(u_0, v_0) = X$$
\[ f(v_0, v_i) = X \setminus f(v_i) = X \setminus A_i = B_{i+1} \] for \( 1 \leq i \leq \frac{n}{2} \) and
\[ f(v_0, v_j) = X \setminus f(v_j) = X \setminus B_j = A_{j-1} \] for \( \frac{n}{2} + 1 \leq j \leq n \)
are distinct. Consequently, in this case \( \gamma(ST(m, n)) = l \).

**Case 2: \( n \) is odd.**

If possible let \( \gamma(ST(m, n)) = l \) and let \( g \) be the corresponding set-indexer with indexing set \( X = \{x_1, \ldots, x_l\} \). Since \( |V| = 2^l \), all subsets of \( X \) are the labels of distinct vertices of \( ST(m, n) \). Then
\[ 2^X = \{g(u_i) = C_i ; 0 \leq i \leq m\} \cup \{g(v_j) = D_j ; 0 \leq j \leq n\}. \]

Let \( m = 1 \). Then the subsets \( D_0 \oplus D_1, D_0 \oplus D_2, \ldots, D_0 \oplus D_n, \)
\( D_0 \oplus C_0 \) and \( D_0 \oplus C_1 \) are the \( 2^l - 1 \) distinct nonempty subsets of \( X \). But by the definition of \( g \), \( C_0 \oplus C_1 \) is also a nonempty subset
\( X \) distinct from the above nonempty subsets of \( X \). This is not possible as \( |X| = l \). Consequently, if \( \gamma(ST(m, n)) = l \) then \( m > 1 \). Similarly, we also get \( n \neq 1 \). So we may assume that \( m, n \geq 3 \).

Suppose that \( C_0 = \emptyset \). Then \( g(v_0) = g(u_0, v_0) \neq g(u_0, v_j) \) for all \( j = 1, \ldots, n \). Since \( n \) is odd, we can find a vertex \( v_k; 1 \leq k \leq n \)
such that \( g(u_0, v_k) = D_0 \oplus D_k \) is distinct from all \( D_i \)'s; \( 1 \leq j \leq n \).

Since \( g(u_0, u_i) = g(u_i); 1 \leq i \leq m \), we cannot assign \( D_0 \oplus D_k \)
to any vertex \( u_i; 1 \leq i \leq m \). Consequently, it must be \( C_0 \neq \emptyset \).

Similarly we can prove that \( D_0 \neq D_k \).

Now suppose that one of the \( C_i \)'s \((1 \leq i \leq m)\), say \( C_m = \emptyset \). Then \( g(u_0, u_m) = C_0 \). So we cannot assign \( C_0 \oplus D_0 \) to any of the \( v_j \)'s; \( 1 \leq j \leq n \). Without loss of generality, suppose that
\( g(u_{m-1}) = C_0 \oplus D_0 \). If \( g(u_0, u_i) = g(u_p) \) for some \( p; 1 \leq p \leq m \), then \( g(u_i) = g(u_0, u_p) \). Consequently, since \( m - 2 \) is odd,
there exists at least one vertex \( u_k \) such that \( g(u_k) \neq g(u_0, u_j) \) and \( g(u_0, u_k) \neq g(u_j); 1 \leq j \leq m \). Thus, \( C_k \neq g(u_0, u_j); 1 \leq j \leq m \).

Then \( C_k = g(v_0, v_q) \) for some \( 1 \leq q \leq n \). Then \( D_q = g(v_q) = D_0 \oplus C_k \neq g(v_0, v_j) \) for all \( 1 \leq j \leq n \). Also we must have \( g(u_0, u_k) \)
\[ C_0 \oplus C_k = g(v_w) = D_w \text{ for some } 1 \leq w \leq n \text{ such that } v_a \neq v_w. \]

Consequently, \( D_w = g(v_w) \neq g(v_0, v_j) \) for all \( 1 \leq j \leq n \). Thus, corresponding to every \( u_k \) such that \( g(u_k) \notin \{ C_0 \oplus C_i/1 \leq i \leq m \} \), there exists distinct vertices \( v_q \) and \( v_w \) such that \( g(v_q), g(v_w) \in \{ C_0 \oplus C_i/1 \leq i \leq m \} \). Since \( m \) and \( n \) are finite, this is not possible. Consequently, \( (ST(m, n)) = l + 1 \).

Consider the set \( Y = X \cup \{ x_{i+1} \} \). Now by assigning \( \emptyset \) to \( u_0 \), \( Y \) to \( v_0 \) and the \( 2^l - 2 \) distinct nonempty proper subsets of \( X \) to the remaining vertices we get a set-indexer \( h \) of \( ST(m, n) \) with the edge labels:

\[
\begin{align*}
    h(u_0, v_0) &= Y, \\
    h(u_0, u_i) &= h(u_i) \quad \text{for } 1 \leq i \leq m \text{ and} \\
    h(v_0, v_j) &= Y \setminus h(v_j) \quad \text{for } 1 \leq j \leq n.
\end{align*}
\]

Thus, \( \gamma(ST(m, n)) = l + 1. \) \( \square \)

**Theorem 3.1.4.** \( \gamma(ST(m, n)) = l + 1 \) if \( 2^l < |V| < 2^{l+1}. \)

**Proof.** Because of theorem 1.0.5 and theorem 1.0.6 we need only to consider the double star \( ST(m, n) \) with \( |V| = 2^{l+1} - 1 \). Let \( V(K_{1,m}) = \{u_0, u_1, \ldots, u_m\} \) and \( V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\} \) where \( u_0 \) and \( v_0 \) are the central vertices. By theorem 1.0.5, \( \gamma(ST(m, n)) \geq l + 1. \) Since \( m + n + 2 = 2^{l+1} - 1 \), then exactly one of \( m \) and \( n \), say \( n \) is even. Let \( X = \{x_1, \ldots, x_{l+1}\} \) and \( A_1 \) be any nonempty proper subset of \( X \). For \( 1 < i \leq \frac{n}{2} \), define \( A_i \) to be any nonempty proper subset of \( X \) distinct from \( A_j \) and \( A_j^c \) for all \( j \) such that \( 1 \leq j \leq i - 1 \). Now, for \( \frac{n}{2} + 1 \leq j \leq n \), let \( B_j = A_{j-\frac{n}{2}}^c \). Now consider the set-indexer \( f \) of \( ST(m, n) \) with indexing set \( X \) defined as follows:

\[
\begin{align*}
    f(u_0) &= \emptyset, \\
    f(v_0) &= X, \\
    f(v_i) &= A_i; \ 1 \leq i \leq \frac{n}{2} \quad \text{and} \\
    f(v_j) &= B_j; \ \frac{n}{2} + 1 \leq j \leq n.
\end{align*}
\]
Now assign the remaining \((m + n + 2) - (2 + \frac{n}{2} + \frac{n}{2}) = m\) subsets of \(X\) to the vertices \(u_1, u_2, \ldots, u_m\) in any order. Clearly, the edge labels
\[
f(u_0, v_0) = X, \quad f(u_0, u_k) = f(u_k), \quad 1 \leq k \leq m
\]
\[
f(u_0, v_i) = X \setminus f(v_i) = B_{i + \frac{n}{2}}, \quad 1 \leq i \leq \frac{n}{2}
\]
\[
f(u_0, v_j) = X \setminus f(v_j) = A_{j - \frac{n}{2}} \text{ for } \frac{n}{2} + 1 \leq j \leq n
\]
are distinct. Thus, \(\gamma(ST(m, n)) = l + 1\) if \(|V| = 2^{l+1} - 1\).

**Theorem 3.1.5.** A double star is set-semigraceful if its order is not a power of 2.

**Proof.** Let \(ST(m, n)\) be a double star whose order is not a power of 2. Then there exists a positive integer \(l \geq 2\) such that \(2^l < |V| < 2^{l+1}\). By theorem 3.1.4, \(\gamma(ST(m, n)) = \lceil \log_2 |V| \rceil = \lceil \log_2(|E| + 1) \rceil\), since \(ST(m, n)\) is a tree. Therefore, \(ST(m, n)\) is set-semigraceful.

**Remark 3.1.6.** The converse of theorem 3.1.5 is not true. For example consider the double star \(ST(m, n); m = 4\) and \(n = 2\). By theorem 3.1.3, \(\gamma(ST(m, n)) = 3 = \lceil \log_2(|E| + 1) \rceil\) so that \(ST(m, n)\) is set-semigraceful.

**Theorem 3.1.7.** The double star \(ST(m, n)\) is set-semigraceful if \(m\) is even.

**Proof.** Since \(o(ST(m, n)) \geq 4\), there exists a positive integer \(l \geq 2\) such that \(2^l \leq |V| \leq 2^{l+1} - 1\). If \(|V| = 2^l\), then by theorem 3.1.3, \(\gamma(ST(m, n)) = l = \lceil \log_2 |V| \rceil = \lceil \log_2(|E| + 1) \rceil\) so that \(ST(m, n)\) is set-semigraceful. Otherwise, the required result follows from theorem 3.1.5.

**Remark 3.1.8.** The converse of the above theorem is not true as the double star \(ST(5, 7)\) is set-semigraceful by theorem 3.1.5.
The following theorem characterizes the set-semigraceful double stars.

**Theorem 3.1.9.** The double star $ST(m,n)$ is set-semigraceful if and only if either the order is not a power of 2 or $m$ is even.

**Proof.** For a double star $ST(m,n)$, suppose the order is a power of 2 and $m$ is odd. Then by theorem 3.1.3, $\gamma(ST(m,n)) = \lceil \log_2 |V| \rceil + 1 = \lceil \log_2 (|E|+1) \rceil + 1 > \lceil \log_2 (|E|+1) \rceil$.

$\Rightarrow$ $ST(m,n)$ is not set-semigraceful. Consequently, if $ST(m,n)$ is set-semigraceful, then either the order is not a power of 2 or $m$ is even.

The converse part follows from theorem 3.1.5 and theorem 3.1.7.

**Remark 3.1.10.** In 1986, B. D. Acharya [3] conjectured that the cycle $C_{2^n-1}; n \geq 2$ is set-graceful and in 1989, Mollard and Payan [31] settled this in the affirmative. The idea of their proof is the following:

Consider the field $GF(2^n)$ constructed by a binary primitive polynomial say, $p(x)$ of degree $n$. Let $\alpha$ be a root of $p(x)$ in $GF(2^n)$. Then $GF(2^n) = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^n-2}\}$. Now by assigning $\alpha^i \mod p(\alpha), 1 \leq i \leq 2^n - 1$, to the vertices $v_i$ of the cycle $C_{2^n-1} = (v_1, \ldots, v_{2^n-1}, v_1)$ we get a set-graceful labeling of $C_{2^n-1}$ with indexing set $X = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Note that for each $\alpha^i \in GF(2^n) \setminus \{0\}$ we have $\alpha^i = a_0\alpha^0 + a_1\alpha^1 + \ldots + a_{n-1}\alpha^{n-1}; a_i = 0$ or 1 for $0 \leq i \leq n - 1$ with $\alpha^0 = 1$ and we identify it as $\{\alpha^i : a_i = 1; 0 \leq i \leq n - 1\}$ which is a subset of $X$.

Using the above set-valuation of $C_{2^n-1}$, we derive the set-indexing numbers of many families of cycles below and establish
their set-semigracefulness. Note that this includes some of the cycles that we have already dealt with in the previous chapter.

**Theorem 3.1.11.** The cycles $C_k; 2^n - 1 \leq k \leq 2^n + 2^{n-1} - 2$, $n \geq 3$ are set-semigraceful.

**Proof.** The cycle $C_2^{2n-1} = (v_1, \ldots, v_{2n-1}, v_1)$ has a set-graceful labeling $f$ as described in the above remark 3.1.10. Take $l = 2^n - 1 - 1$ new vertices $u_1, \ldots, u_l$ and form the cycle $C_2^{2n+l-1} = (v_1, u_1, v_2, v_3, u_2, v_4, v_5, u_3, v_6, v_7, \ldots, v_{2^n-3}, u_l, v_{2^n-2}, v_{2^n-1}, v_1)$. Now define a set-indexer $g$ of $C_2^{2n+l-1}$ with indexing set $Y = X \cup \{x\}$ as follows:

$$g(v_i) = f(v_i); 1 \leq i \leq 2^n - 1$$
$$g(u_j) = f(v_{2j-1}, v_{2j}) \cup \{x\}; 1 \leq j \leq l.$$

Then by theorem 1.0.5, $\gamma(C_2^{2n+l-1}) = n + 1$. Now by removing the vertices $u_j; 2 \leq j \leq l$ and joining $(v_{2j-1}, v_{2j})$ in succession we get the cycles $C_2^{2n+l-2}, C_2^{2n+l-3}, \ldots, C_2^{2n}$. Clearly $g$ induces optimal set-indexers for these cycles by theorem 1.0.5 and $\gamma(C_k) = n + 1$; $2^n \leq k \leq 2^n + l - 2$ so that these cycles are set-semigraceful. \qed

A class of set-semigraceful wheels are identified below:

**Theorem 3.1.12.** The wheels $W_k; 2^n - 1 \leq k \leq 2^n + 2^{n-1} - 2$, $n \geq 2$ are set-semigraceful.

**Proof.** With the help of theorem 2.3.25, it can be easily verified that $W_k$ is set-semigraceful for $k = 2^n - 1, 2^n, 2^n + 1, 2^n + 2; n \geq 2$. Now assume that $2^n + 3 \leq k \leq 2^n + 2^{n-1} - 2$. By theorem 1.0.5,

$$\lceil \log_2(|E| + 1) \rceil \leq \gamma(W_k)$$
$$\leq \gamma(C_k) + 1, \text{ by theorem 2.3.28}$$
$$= \lceil \log_2(k + 1) \rceil + 1, \text{ since } C_k \text{ is set-semigraceful by theorem 3.1.11}$$
$= n + 2$
$= \lfloor \log_2(2k + 1) \rfloor$,
\hspace{1cm} since $2^{n+1} + 6 \leq 2k \leq 2^{n+1} + 2^n - 4$
$= \lfloor \log_2(|E| + 1) \rfloor$.

Thus, the wheels $W_k$ are set-semigraceful for the above values of $k$. \hfill \square

The following theorem provides a class of $n$-suns that are set-semigraceful.

**Theorem 3.1.13.** The $k$-suns; $2^n - 1 \leq k \leq 2^n + 2^n - 2$, $n \geq 3$ are set-semigraceful.

**Proof.** By theorem 2.3.23 it follows that $k$-suns where $k = 2^n - 1, 2^n, 2^n + 1, 2^n + 2$; $n \geq 2$ are set-semigraceful. Now assume that $2^n + 3 \leq k \leq 2^n + 2^n - 2$. By theorem 1.0.5,

$$\left\lfloor \log_2(|E| + 1) \right\rfloor \leq \gamma(k - \text{sun})$$

\hspace{1cm} $\leq \gamma(C_k) + 1$, by theorem 2.3.29
\hspace{1cm} $= \lfloor \log_2(k + 1) \rfloor + 1$, since $C_k$ is set-semigraceful
\hspace{1cm} by theorem 3.1.11
\hspace{1cm} $= n + 2$
\hspace{1cm} $= \lfloor \log_2(2k + 1) \rfloor$,
\hspace{1cm} since $2^{n+1} + 6 \leq 2k \leq 2^{n+1} + 2^n - 4$
\hspace{1cm} $= \lfloor \log_2(|E| + 1) \rfloor$.

Thus, the $k$-suns are set-semigraceful for the specified values of $k$. \hfill \square
3.2 Some Properties

This section sheds more light on set-semigraceful graphs. Apart from many properties, certain bounds for the order and size of set-semigraceful graphs are also derived.

The following is an immediate consequence of the definition of set-semigraceful graphs.

Theorem 3.2.1. Let $G$ be a $(p, q)$-graph with $\gamma(G) = m$. Then $G$ is set-semigraceful if and only if $2^{m-1} \leq q \leq 2^m - 1$.

It has an obvious result.

Corollary 3.2.2. If $T$ is a set-semigraceful tree of order $p \geq 2$, then $2^{\gamma(T)-1} + 1 \leq p \leq 2^{\gamma(T)}$.

Corollary 3.2.3. Let $G'$ be a $(p, q')$ subgraph of a set-semigraceful $(p, q)$-graph $G$ with $q' \geq 2^{\gamma(G)-1}$. Then $G'$ is also set-semigraceful. Moreover, $\gamma(G') = \gamma(G)$.

Proof. Since $G' \subseteq G$, by theorem 3.2.1,

$2^{\gamma(G)} - 1 \geq q \geq q' \geq 2^{\gamma(G)-1}$

$\Rightarrow 2^{\gamma(G)} \geq q' + 1 \geq 2^{\gamma(G)-1} + 1$

$\Rightarrow \lceil \log_2(q' + 1) \rceil = \gamma(G)$

$\geq \gamma(G')$, by theorem 1.0.6

$\geq \lceil \log_2(q' + 1) \rceil$, by theorem 1.0.5

$\Rightarrow G'$ is set-semigraceful and $\gamma(G') = \gamma(G)$. □

While theorem 3.2.1 gives bounds for the size of a set-semigraceful graph in terms of the set-indexing number, the following one gives the same in terms of its order.
3.2 Some Properties

Theorem 3.2.4. Let $G$ be a set-semigraceful $(p, q)$-graph; $p \geq 2$. Then $2^{|\log_2 p| - 1} \leq q \leq 2^{|\log_2 (\frac{\pi(p-1)}{2})| + 1} - 1$.

Proof. By theorem 2.1.4 and corollary 2.1.7, $|\log_2 p| \leq \gamma(K_{p, p-1}) \leq \gamma(G) = |\log_2 (q + 1)|$, since $G$ is set-semigraceful. Then by theorem 3.2.1, $2^{|\log_2 p| - 1} \leq q \leq 2^{|\log_2 (\frac{\pi(p-1)}{2})| + 1} - 1$, since $q \leq \frac{p(p-1)}{2}$.

Remark 3.2.5. The converse of theorem 3.2.4 is not true. By theorem 2.3.15, $|\log_2 (|E(C_5)| + 1)| = 2 < \gamma(C_5) = 4$. But $C_5$ is not set-semigraceful even if $2^2 \leq |E(C_5)| \leq 2^4 - 1$ holds. Further, as a consequence of the above theorem, the graphs $C_6 \cup N_3$, $C_6 \cup N_4$, $C_6 \cup 2K_2$, $C_{14} \cup N_6$ and $C_{14} \cup 2K_2$ are not set-semigraceful.

Remark 3.2.6. By theorem 1.0.6, for any subgraph $G'$ of $G$, $\gamma(G') \leq \gamma(G)$. But subgraphs of a set-semigraceful graph need not be set-semigraceful. For example $K_6$ is set-semigraceful, but the spanning subgraph $C_6$ of $K_6$ is not set-semigraceful, by theorem 1.0.8 and theorem 2.3.19.

Theorem 3.2.7. If the complete graph $K_n$; $n \geq 2$ is set-semigraceful, then $2m - 1 \leq \gamma(K_n) \leq 2m + 1$; $m = |\log_2 n|$.

Proof. If $K_n$ is set-semigraceful then, 

\[ |\log_2 \frac{n(n-1)}{2} + 1| = \gamma(K_n) \geq |\log_2 \frac{n(n-1)}{2}| = |\log_2 n + \log_2 (n-1) - \log_2 2| = |\log_2 n + \log_2 (n-1) - 1|. \]

For any $n$, there exists $m$ such that $2^m \leq n \leq 2^{m+1} - 1$ so that from above $\gamma(K_n) \geq 2m - 1$. But, 

\[ \gamma(K_n) = |\log_2 \frac{n(n-1)}{2} + 1| = |\log_2 \frac{n(n-1)+2}{2}| \leq 2m + 1. \]
Thus, $2m - 1 \leq \gamma(K_n) \leq 2m + 1; m = \lfloor \log_2 n \rfloor$. \qed

Remark 3.2.8. The converse of theorem 3.2.7 is not always true. For example, by theorem 2.1.16

$$2\lfloor \log_2 n \rfloor - 1 \leq \gamma(K_8) \leq 2\lfloor \log_2 n \rfloor + 1.$$ 

But the complete graph $K_8$ is not set-semigraceful. Also by theorem 2.1.16 and theorem 3.2.7, $K_{13}$, $K_{14}$ and $K_{15}$ are not set-semigraceful.

Theorem 3.2.9. If a $(p, q)$-graph $G$ has a set-semigraceful labeling with respect to a set $X$ of cardinality $m \geq 2$, there exists a partition of the vertex set $V$ into two nonempty sets $V_1$ and $V_2$ such that the number of edges joining the vertices of $V_1$ with those of $V_2$ is at most $2^{m-1}$.

Proof. Let $f: V \cup E \to 2^X$ be a set-semigraceful labeling of $G$ with indexing set $X$ of cardinality $m$. Let $V_1 = \{u \in V : |f(u)| \text{ is odd}\}$ and $V_2 = V \setminus V_1$. We have $|A \oplus B| = |A| + |B| - 2|A \cap B|$ for any two subsets $A$, $B$ of $X$ and hence $|A \oplus B| = 1 \pmod{2} \iff A$ and $B$ do not belong to the same set $V_i; i = 1, 2$. Therefore, all odd cardinality subsets of $X$ in $f(E)$ must appear on edges joining $V_1$ and $V_2$. Consequently, there exists at most $2^{m-1}$ edges between $V_1$ and $V_2$. \qed

Theorem 3.2.10. If $G$ is a set-semigraceful graph with $\gamma(G) = m$, then every subgraph $H$ of $G$ with $2^{m-1} \leq |E(H)| \leq 2^m - 1$ is also set-semigraceful.

Proof. Since every set-indexer of $G$ is a set-indexer of $H$, the result follows from theorem 1.0.5. \qed
Corollary 3.2.11. Let $G \subseteq K_{1,n}$ and $\gamma(K_{1,n}) = m$. Then $G$ is set-semigraceful with $\gamma(G) = m$ if and only if $2^{m-1} \leq |E(G)| \leq 2^m - 1$.

Proof. The proof follows from theorem 3.2.10 and theorem 3.2.1. 

Theorem 3.2.12. If $G$ is a $(p,q)$-graph such that $p > 2^n > q$ for some $n$, then $G$ is not set-semigraceful.

Proof. By theorem 2.1.11, $\gamma(G) \geq \lceil \log_2 p \rceil > \lceil \log_2 (q + 1) \rceil$, by hypothesis. Thus, $\gamma(G) > \lceil \log_2 (|E| + 1) \rceil$ so that $G$ is not set-semigraceful. 

Remark 3.2.13. The above theorem provides a class of disconnected graphs that are not set-semigraceful. However, there are disconnected graphs that are not set-semigraceful and do not belong to the above class. For instance, $K_8 \cup K_1$ and $K_{10} \cup K_1$. Further, the converse of theorem 3.2.12 is not true as there are infinitely many connected graphs that are not set-semigraceful by remark 3.1.2 (iv).

Theorem 3.2.14. Let $G$ be a set-semigraceful $(p,q)$-graph with $\gamma(G) = m$. Then for every $p'$ satisfying $p < p' < 2^m$, $G$ can be embedded in a set-semigraceful $(p',q)$-graph.

Proof. Let $f$ be a set-semigraceful labeling of $G$ with indexing set $X$ of cardinality $m = \gamma(G)$. Now add $p' - p$ isolated vertices to $G$ and assign the unassigned subsets of $X$ under $f$ to these vertices in a one to one manner. Clearly the resulting graph is set-semigraceful. 

Theorem 3.2.15. [5] Every graph can be embedded as an induced subgraph of a connected set-graceful graph.

Since every set-graceful graph is set-semigraceful, from the above theorem is follows that,

Theorem 3.2.16. Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph.

However, below we prove:

Theorem 3.2.17. Every graph can be embedded as an induced subgraph of a connected set-semigraceful graph which is not set-graceful.

Proof. Any graph $H$ with $o(H) \leq 5$ and $s(H) \leq 2$ and the graphs $P_4$, $P_4 \cup K_1$, $P_3 \cup K_2$ and $P_5$ are induced subgraphs of the set-semigraceful cycle $C_{10}$ which is not set-graceful. Again any graph $H'$ with $3 \leq o(H') \leq 5$ and $3 \leq s(H') \leq 9$ can be obtained as an induced subgraph of $H_1 \vee K_1$ for some graph $H_1$ with $o(H_1) = 5$ and $3 \leq s(H_1) \leq 9$. Then $3 < \log_2(|E(H_1 \vee K_1)| + 1) < 4$, since $8 \leq s(H_1 \vee K_1) < 15$ and hence $H_1 \vee K_1$ is not set-graceful. By theorem 1.0.5,

$4 = \lceil \log_2(|E(H_1 \vee K_1)| + 1) \rceil$

$\leq \gamma(H_1 \vee K_1)$

$\leq \gamma(K_6)$, by theorem 1.0.6

$= 4$, by theorem 1.0.8

so that $H_1 \vee K_1$ is set-semigraceful. Further, note that $K_6$ is set-semigraceful but not set-graceful.

Now let $G = (V,E); V = \{v_1, \ldots, v_n\}$ be a graph of order $n \geq 6$. Consider a set-indexer $g$ of $G$ with indexing set $X = \{1, 2, \ldots, 2^{\gamma(K_6)} - 1\}$.
\{x_1, \ldots, x_n\} defined by \(g(u_i) = \{x_i\}; 1 \leq i \leq n\). Let \(S = \{g(e): e \in E\} \cup \{g(v): v \in V\}\). Note that \(|S| = |E| + n\). Now take a new vertex \(u\) and join with all the vertices of \(G\). Let \(m\) be any integer such that \(2^{n-1} < m < 2^n - (|E| + n + 1)\). Since \(|E| \leq \frac{n(n-1)}{2}\) and \(n \geq 6\), such an integer always exists. Take \(m\) new vertices \(u_1, \ldots, u_m\) and join all of them with \(u\). A set-indexer \(f\) of the resulting graph \(G'\) can be defined as follows:

\[
f(u) = 0, \quad f(u_i) = g(u_i); 1 \leq i \leq n.
\]

Besides, \(f\) assigns the vertices \(u_1, \ldots, u_m\) with any \(m\) distinct elements of \(2^X \setminus (S \cup \emptyset)\). Thus, \(\gamma(G') \leq n\). But we have \(2^n > |E| + n + m + 1 > m > 2^{n-1}\) so that \(\gamma(G') \geq n\), by theorem 1.0.5. Hence, \(\log_2(|E(G')| + 1) < \log_2(|E(G)| + 1) = n = \gamma(G')\). This shows that \(G'\) is set-semigraceful, but not set-graceful.

Theorem 3.2.18. Let \(G = (V, E)\) be a spanning subgraph of \(K_{1,n}; n \geq 2\). Then \(G\) is set-semigraceful if and only if \(2^{\lceil \log_2 n \rceil - 1} \leq |E| \leq 2^{\lceil \log_2 n \rceil} - 1\).

Proof. Suppose \(G\) is set-semigraceful. By theorem 2.1.4 and theorem 2.1.6, \(\lceil \log_2 n \rceil = \gamma(N_{n+1})\)

\[
\leq \gamma(G), \text{ by theorem 1.0.6}
\]

\[
\leq \gamma(K_{1,n})
\]

\[
= \lceil \log_2 n \rceil, \text{ by theorem 2.1.4}
\]

\[
\Rightarrow \lceil \log_2(|E| + 1) \rceil = \lceil \log_2 n \rceil
\]

\[
\Rightarrow 2^{\lceil \log_2 n \rceil - 1} < |E| + 1 \leq 2^{\lceil \log_2 n \rceil}
\]

\[
\Rightarrow 2^{\lceil \log_2 n \rceil - 1} \leq |E| \leq 2^{\lceil \log_2 n \rceil} - 1.
\]

Conversely suppose \(2^{\lceil \log_2 n \rceil - 1} \leq |E| \leq 2^{\lceil \log_2 n \rceil} - 1\). By theorem 2.1.4, theorem 2.1.6, \(\lceil \log_2 n \rceil = \gamma(N_{n+1})\)

\[
\leq \gamma(G), \text{ by theorem 1.0.6}
\]

\[
\leq \gamma(K_{1,n})
\]
= \lceil \log_2 n \rceil$, by theorem 2.1.4.

But, \[ \log_2(|E| + 1) = \log_2 n = \gamma(G) \]
so that $G$ is set-semigraceful.

\[ \square \]

### 3.3 More Set-Semigraceful Graphs

In this section, we construct more set-semigraceful graphs from given ones through various graph theoretic methods. It is shown that the wheels and suns obtained from set-semigraceful cycles are also set-semigraceful.

**Theorem 3.3.1.** If a $(p, p-1)$-graph $G$ is set-semigraceful, then $G \lor N_{2^n-1}$ is set-semigraceful.

**Proof.** Let $G$ be set-semigraceful with set-indexing number $m$. By theorem 1.0.5, \[
\gamma(G \lor N_{2^n-1}) \geq \lceil \log_2(|E(G \lor N_{2^n-1})| + 1) \rceil \\
= \lceil \log_2(p - 1 + p(2^n - 1) + 1) \rceil \\
= \lceil \log_2(p2^n) \rceil \\
= \lceil \log_2(2^n) + \log_2(p) \rceil \\
\geq n + m.
\]

Let $f$ be a set-semigraceful labeling of $G$ with indexing set $X = \{x_1, \ldots, x_m\}$. Consider the set $Y = \{y_1, \ldots, y_n\}$ and let $V(N_{2^n-1}) = \{v_1, \ldots, v_{2^n-1}\}$. We can find a set-semigraceful labeling say $g$ of $G \lor N_{2^n-1}$ with indexing set $X \cup Y$ as follows: $g(u) = f(u)$ for all $u \in V(G)$ and assign the distinct nonempty subsets of $Y$ to the vertices $v_1, \ldots, v_{2^n-1}$ in any order. Thus $G \lor N_{2^n-1}$ is set-semigraceful.

\[ \square \]

**Remark 3.3.2.** The converse of theorem 3.3.1 is not true in general. For example consider the wheel graph $W_6 = C_5 \lor K_1$
3.3 More Set-Semigraceful Graphs

\[ (u_1, \ldots, u_5, u_1) \cup \{u\} \]. Now assigning the subsets \( \{a\}, \{a, b\}, \{a, b, c\}, \{a, d\}, \{a, b, c, d\} \) and \( \emptyset \) of the set \( X = \{a, b, c, d\} \) to the vertices \( u_1, \ldots, u_5 \) and \( u \) in that order we get \( W_5 \) as set-semigraceful whereas \( C_5 \) is not set-semigraceful by theorem 2.3.15.

Corollary 3.3.3. The triangular book \( K_2 \vee N_{2n-1} \) is set-semigraceful.

Proof. The proof follows from theorem 2.4.2 and theorem 3.3.1.

\[ \square \]

Theorem 3.3.4. The fan graph \( F_n; n \geq 2 \) is set-semigraceful if and only if \( n \neq 2^m + 1; m \geq 2 \).

Proof. Suppose \( n \neq 2^m + 1; n \geq 2 \) so that \( P_{n-1} \) is set-semigraceful (see remark 3.1.2 (iv)). Then by theorem 3.3.1, \( F_n = P_{n-1} \vee K_1 \) is set-semigraceful.

Conversely, if \( F_n \) is set-semigraceful, by theorem 3.2.1, \( \gamma(F_n) = \lceil \log_2(n - 1 + n - 2 + 1) \rceil = \lceil \log_2(2n - 2) \rceil = \lceil \log_2(n - 1) \rceil + 1. \) Then, by theorem 2.4.11, we must have \( n \neq 2^m + 1; m \geq 2. \) \[ \square \]

Theorem 3.3.5. The twing graph obtained from \( P_{2n-1}; n \geq 3 \) is set-semigraceful.

Proof. Let \( P_{2n-1} = (v_1, \ldots, v_{2n-1}) \). By theorem 2.4.2, \( \gamma(P_{2n-1}) = n \). Let \( f \) be an optimal set-indexer of \( P_{2n-1} \) with indexing set \( X \). Let \( T \) be a twing graph obtained from \( P_{2n-1} \) by joining each vertex \( v_i; i \in \{2, \ldots, 2n - 2\} \) of \( P_{2n-1} \) to two new vertices say \( u_i \) and \( w_i \) by pendant edges. Consider the set-indexer \( g \) of \( T \) with indexing set \( Y = X \cup \{x, y\} \) defined as follows:

\[ g(v) = f(v) \text{ for all } v \in V(P_{2n-1}) \]
\[ g(u_i) = f(u_{i-1}) \cup \{x\} \text{ and} \]
\[ g(w_i) = f(u_{i-1}) \cup \{y\}; \quad 2 \leq i \leq 2^n - 2. \]

Consequently, \( \gamma(T) \leq n + 2 \). But by theorem 1.0.5,
\[
\lceil \log_2(|E(T)| + 1) \rceil = \lceil \log_2(2^n - 2 + 2^n - 3 + 2^n - 3 + 1) \rceil \\
= n + 2 \\
\leq \gamma(T).
\]
Thus \( T \) is set-semigraceful. \( \Box \)

**Theorem 3.3.6.** The splitting graph \( S'(G) \) of a set-semigraceful bipartite \((p, q)\)-graph \( G \) with \( \gamma(G) = m \) and \( 3q \geq 2^{m+1} \), is set-semigraceful.

**Proof.** Let \( f \) be an optimal set-indexer of \( G \) with indexing set \( X \) of cardinality \( m \). Let \( V_1 = \{v_1, \ldots, v_n\} \) and \( V_2 = \{u_1, \ldots, u_l\} \) be the partition of \( V(G) \), where \( n = p - l \). Since \( G \) is set-semigraceful with \( \gamma(G) = m \), by theorem 3.2.1 we have \( 2^{m-1} \leq q \leq 2^m - 1 \). To form the splitting graph \( S'(G) \) of \( G \), for each \( v_i \) or \( u_j \) in \( G \), add a new vertex \( v'_i \) or \( u'_j \) and add edges joining \( v'_i \) or \( u'_j \) to all neighbors of \( v_i \) or \( u_j \) in \( G \) respectively. Since \( S'(G) \) has \( 3q \) edges, by theorem 1.0.5, \( \gamma(S'(G)) \geq \lceil \log_2(|E(S'(G))| + 1) \rceil = \lceil \log_2(3q + 1) \rceil \geq \lceil \log_2(2^{m+1} + 1) \rceil = m + 2 \). We can define a set-indexer \( g \) of \( S'(G) \) with indexing set \( Y = X \cup \{x, y\} \) as follows:
\[
g(v) = f(v) \quad \text{for all } v \in V(G), \\
g(v'_i) = f(v_i) \cup \{x\}; \quad 1 \leq i \leq n \quad \text{and} \\
g(u'_j) = f(u_j) \cup \{y\}; \quad 1 \leq j \leq l.
\]
Consequently, \( \gamma(S'(G)) = m + 2 = \lceil \log_2(|E(S'(G))| + 1) \rceil \) and hence \( S'(G) \) is set-semigraceful. \( \Box \)

**Remark 3.3.7.** (i). Even though \( C_3 \) is not bipartite, both \( C_3 \) and its splitting graph are set-semigraceful. If \( H \) is the splitting graph of \( C_3 \), by theorem 1.0.5, \( \gamma(H) \geq 4 \). Now the following set-valuation asserts that \( \gamma(H) = 4 \) so that
3.3 More Set-Semigraceful Graphs

$H$ is set-semigraceful. Note that $C_3$ is set-semigraceful by remark 3.1.2 (v).

(ii). Splitting graph of the path $P_4$ is set-semigraceful. But $P_4$ is not set-semigraceful. Because of theorem 1.0.5, the following set-valuation of the splitting graph $G$ of $P_4$ shows that $\gamma(G) = 4 = \lceil \log_2(|E| + 1) \rceil$.

**Theorem 3.3.8.** For any set-graceful graph $G$, the graph $H; G \cup K_1 \subset H \subset G \cup K_1$ is set-semigraceful.

**Proof.** Let $m = \gamma(G) = \log_2(|E(G)| + 1)$. Then by theorem 1.0.5 we have $\gamma(H) \geq \lceil \log_2(|E(H)| + 1) \rceil \geq \lceil \log_2(2^m + 1) \rceil = m + 1.$
Let $f$ be a set-graceful labeling of $G$ with indexing set $X$. Now we can extend $f$ to a set-indexer $g$ of $G \lor K_1$ with indexing set $Y = X \cup \{x\}$ of cardinality $m + 1$ as follows:

$$g(u) = f(u) \text{ for all } u \in V(G) \text{ and } g(v) = \{x\}; \{v\} = V(G).$$

Clearly,

$$g(e) = f(e) \text{ for all } e \in E(G) \text{ and } g(u,v) = g(u) \cup \{x\}$$

are all distinct. Then by theorem 1.0.6 we have $\gamma(H) = m + 1 = [\log_2(|E(H) + 1|)].$ \qed

**Corollary 3.3.9.** The wheel $W_{2^{n}-1}; n \geq 2$ is set-semigraceful.

**Proof.** The proof follows from theorem 2.3.1 and theorem 3.3.8. \qed

**Theorem 3.3.10.** Let $G$ be a set-graceful $(p, p-1)$-graph, then $G \lor N_m$ is set-semigraceful.

**Proof.** Let $\gamma(G) = n$. For every $m$, there exists $l$ such that $2^l \leq m \leq 2^{l+1} - 1$. By theorem 1.0.5,

$$\gamma(G \lor N_m) \geq \left\lceil \log_2(|E(G \lor N_m)| + 1) \right\rceil$$

$$= \left\lceil \log_2(p - 1 + pm + 1) \right\rceil$$

$$= \left\lceil \log_2(p(m + 1)) \right\rceil$$

$$= \left\lceil \log_2(2^n)(m + 1) \right\rceil$$

$$= \left\lceil \log_2(2^n) + \log_2(m + 1) \right\rceil$$

$$\geq n + l + 1.$$

Let $f$ be a set-valuation of $G$ with $X = \{x_1, \ldots, x_n\}$ as the indexing set. Consider the set $Y = \{y_1, \ldots, y_{l+1}\}$ and $V(N_m) = \{v_1, \ldots, v_m\}$. Now by assigning the distinct nonempty subsets of $Y$ to the vertices $v_1, \ldots, v_m$ in any order, $f$ can be extended to get a set-indexer of $G \lor N_m$ with indexing set $X \cup Y$. \qed

**Corollary 3.3.11.** $K_{1, 2^{n}-1, m}$ is set-semigraceful.
Proof. The proof follows from theorem 1.0.16 and theorem 3.3.10.

\[\]
Proof. The proof follows from theorem 3.3.12 and theorem 1.0.16.

Theorem 3.3.14. Let $G$ be a set-semigraceful $(p, q)$-graph with $\gamma(G) = m$. If $p \geq 2^{m-1}$, then $G \vee K_1$ set-semigraceful.

Proof. By theorem 3.2.1, $2^{m-1} \leq |E(G)| \leq 2^m - 1$. Since $|V| \geq 2^{m-1}$, by theorem 1.0.5 we have $\gamma(G \vee K_1) \geq \lfloor \log_2(|E| + 1) \rfloor = m$.

Let $f$ be a set-indexer of $G$ with indexing set $X$ of cardinality $m = \gamma(G)$. Now we can define a set-indexer $g$ of $G \vee K_1$; $V(K_1) = \{v\}$ with indexing set $Y = X \cup \{x\}; x \notin X$ as follows:

$g(u) = f(u)$ for all $u \in V(G)$ and $g(v) = Y$.

This shows that $G \vee K_1$ is set-semigraceful.

Corollary 3.3.15. If $C_k$ is set-semigraceful, then $W_k$ is also set-semigraceful.

Proof. The proof follows from theorem 3.3.14.

Theorem 3.3.12 is now evident if we combine theorem 3.1.11 and corollary 3.3.15.

Corollary 3.3.16. $W_k$ where $2^n - 1 \leq k \leq 2^n + 2^{n-1} - 2$, $n \geq 3$ is set-semigraceful.

Remark 3.3.17. The converse of corollary 3.3.15 need not be true. By theorem 2.3.19, $4 = \gamma(C_6) > \lfloor \log_2(6 + 1) \rfloor = 3$ so that $C_6$ is not set-semigraceful. But by theorem 2.3.24, $\gamma(W_6) = 4 = \lfloor \log_2(12 + 1) \rfloor$ making $W_6$ set-semigraceful.

Theorem 3.3.18. If $W_{2k}$ where $\frac{2^n-1}{3} \leq k < 2^{n-2}$; $n \geq 4$ is set-semigraceful, then the gear graph of order $2k + 1$ is set-semigraceful.
3.3 More Set-Semigraceful Graphs

**Proof.** Let $G$ be the gear graph of order $2k + 1$. Then by theorem 1.0.5,

\[
\lceil \log_2(3k + 1) \rceil \leq \gamma(G)
\]

\[
\leq \gamma(W_{2k}), \text{ by theorem 1.0.6}
\]

\[
= \lceil \log_2(4k + 1) \rceil, \text{ since } W_{2k} \text{ is set-semigraceful}
\]

\[
= \lceil \log_2(3k + 1) \rceil, \text{ since } \frac{2^{n-1}}{3} \leq k < 2^{n-2}
\]

\[
\Rightarrow 2^n - 1 \leq 3k < 4k < 2^n
\]

\[
\Rightarrow 2^n - 1 + 1 \leq 3k + 1 < 4k + 1 \leq 2^n.
\]

Thus $\gamma(G) = \lceil \log_2(|E| + 1) \rceil$ so that $G$ is set-semigraceful. \qed

**Remark 3.3.19.** There are set-semigraceful wheel graphs not belonging to the above class but still producing set-semigraceful gear graphs. For example, both $W_8$ and the gear graph of order 9 obtained from $W_8$ are set-semigraceful.

**Theorem 3.3.20.** Let $C_k$ where $k = 2^n - m$ and $2^n + 1 > 3m$; $n \geq 2$ be set-semigraceful. Then the graph $C_k \vee K_2$ is set-semigraceful.

**Proof.** Let $G = C_k \vee K_2$; $K_2 = (u_1, u_2)$. By theorem 1.0.5,

\[
\gamma(G) \geq \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(3(2^n - m) + 2) \rceil = n + 2.
\]

But, $3m \leq 2^n + 1 \Rightarrow m < 2^{n-1}$. Therefore,

\[
2^n - (2^{n-1} - 1) \leq 2^n - m < 2^n
\]

\[
\Rightarrow 2^n - 1 \leq k < 2^n; \ k = 2^n - m
\]

\[
\Rightarrow 2^n - 1 + 1 \leq |E(C_k)| < 2^n
\]

\[
\Rightarrow \lceil \log_2(|E(C_k)| + 1) \rceil = n
\]

\[
\Rightarrow \gamma(C_k) = n, \text{ since } C_k \text{ is set-semigraceful.}
\]

Let $f$ be a set-indexer of $C_k$ with indexing set $X = \{x_1, \ldots, x_n\}$. Define a set-indexer $g$ of $G$ with indexing set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ as follows:
$g(v) = f(v)$ for every $v \in V(C_k)$
$g(u_1) = \{x_{n+1}\}$ and $g(u_2) = \{x_{n+2}\}$. □

Remark 3.3.21. The converse of the above theorem is not true.

By theorem 2.3.15, $\gamma(C_5) = 4 > 3 = \lceil \log_2(|E| + 1) \rceil$ and hence $C_5$ is not set-semigraceful. By assigning the subsets $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4\}$, $\{x_1, \ldots, x_4\}$, $\emptyset$ and $\{x_5\}$ of the set $X = \{x_1, \ldots, x_5\}$ to the vertices $v_1$, $v_5$, $u_1$ and $u_2$ of the graph $C_5 \vee K_2 = (v_1, \ldots, v_5, u_1) \vee (u_1, u_2)$ in that order we get $\gamma(C_5 \vee K_2) = 5$, by theorem 1.0.5. Thus, $C_5 \vee K_2$ is set-semigraceful. On the other hand there are set-semigraceful cycles $C_k$ not belonging to the above class but still give set-semigraceful graphs in joining with $K_2$. For example $C_4$ is one such cycle. Note that,

$4 = \lceil \log_2(|E(C_4 \vee K_2)| + 1) \rceil$
$\leq \gamma(C_4 \vee K_2)$, by theorem 1.0.5
$\leq \gamma(C_6)$, by theorem 1.0.6
$= 4$, by theorem 1.0.8

so that $C_4 \vee K_2$ is set-semigraceful.

**Theorem 3.3.22.** The double fan $P_k \vee K_2$ where $k = 2^n - m$ and $2^n \geq 3m; n \geq 3$ is set-semigraceful.

**Proof.** Let $G = P_k \vee K_2; K_2 = (u_1, u_2)$. By theorem 1.0.5,

$\gamma(G) \geq \lceil \log_2(|E| + 1) \rceil = \lceil \log_2(3(2^n - m) + 1) \rceil = n + 2.$

But, $3m \leq 2^n \Rightarrow m < 2^{n-1} - 1$. Therefore,

$2^n - (2^{n-1} - 2) \leq 2^n - m < 2^n - 1$
$\Rightarrow 2^{n-1} + 1 \leq 2^n - m - 1 < 2^n - 2$
$\Rightarrow 2^{n-1} + 1 \leq k - 1 < 2^n - 2; k = 2^n - m$
$\Rightarrow 2^{n-1} + 1 \leq |E(P_k)| < 2^n$
$\Rightarrow \lceil \log_2(|E(P_k)| + 1) \rceil = n$
$\Rightarrow \gamma(P_k) = n$, since $P_k$ is set-semigraceful by remark 3.1.2 (iv).
Let $f$ be a set-indexer of $P_k$ with indexing set $X = \{x_1, \ldots, x_n\}$. Define a set-indexer $g$ of $G$ with indexing set $Y = X \cup \{x_{n+1}, x_{n+2}\}$ as follows:

\[
g(v) = f(v) \text{ for every } v \in V(P_k) \\
g(u_1) = \{x_{n+1}\} \text{ and } g(u_2) = \{x_{n+2}\}.
\]

Remark 3.3.23. There are set-semigraceful double fans obtained from paths that are not set-semigraceful. $P_4 \vee K_2$ is one such double fan.

Theorem 3.3.24. Let $G$ be a set-semigraceful hamiltonian $(p,q)$-graph with $\gamma(G) = m$ and $p \geq 2^{m-1}$. If $G'$ is the graph obtained from $G$ by joining a pendant vertex to each vertex of $G$, then $G'$ is set-semigraceful.

Proof. Let $C = (v_1, \ldots, v_n, v_1)$ be a hamiltonian cycle in $G$. Let $f$ be a set-indexer of $G$ with $\gamma(G) = m$ and $X$ be the corresponding indexing set. Now, take $n$ new vertices $v'_i; 1 \leq i \leq n$ and let $G' = G \cup \{(v_i, v'_i) : 1 \leq i \leq n\}$. By theorem 1.0.5 we have $\gamma(G') \geq \lceil \log_2(|E(G')| + 1) \rceil = m + 1$. We can define a set-indexer $g$ of $G'$ with indexing set $Y = X \cup \{x\}$ as follows:

\[
g(u) = f(u) \text{ for all } u \in V(G) \\
g(v'_i) = f(v_i, v_{i+1}) \cup \{x\}; 1 \leq i \leq n \text{ with } v_{n+1} = v_1.
\]

Clearly $G'$ is set-semigraceful.

Corollary 3.3.25. If $C_k$ is set-semigraceful, then the sun-graph obtained from $C_k$ is set-semigraceful.

Proof. The proof follows from theorem 3.3.24.

The converse of the above corollary is not true. For instance, 5-sun is set-semigraceful whereas $C_5$ is not.
Along with theorem 3.1.11 the above corollary produces theorem 3.1.13 as,

Corollary 3.3.26. The sun-graph of order $2k$ where $2^n - 1 \leq k \leq 2^n + 2^{n-1} - 2; n \geq 3$ is set-semigraceful.