Chapter 3

Banach Fixed Point Theorem for Fuzzy Space

$E^n$

3.1. Introduction

Fixed point theorems find wide applications in different branches of analysis. The Banach fixed point theorem reveals the unifying power of functional analytic methods and also acts as a source of existence and uniqueness theorems in diverse branches of analysis. The Banach fixed point theorem relates to contraction mappings of a complete metric space and gives sufficient conditions for the existence and uniqueness of fixed point. Further an iterative process to obtain approximations to the fixed point is also given by the above theorem [26]. Application of the above theorem for fuzzy space $(E^n, D_1)$ is revealed in this chapter.
**Definition 3.1.1.** A fixed point of a mapping \( T : E^n \to E^n \) of a set \( E^n \) in to itself is an \( u \in E^n \) which is mapped on to itself that is \( Tu = u \).

**Definition 3.1.2 (Contraction).** As \((E^n, D_1)\) be a metric space, a mapping \( T : E^n \to E^n \) is called a contraction on \( E^n \) if there is a positive real number \( \alpha < 1 \) such that for all \( u, v \in E^n \)

\[
D_1(Tu, Tv) \leq \alpha D_1(u, v) \quad \alpha < 1. \quad (3.1.1)
\]

**Theorem 3.1.3 (Banach fixed point theorem for fuzzy sets. (Contraction theorem for fuzzy sets)).** Consider the complete metric space \( E^n = (E^n, D_1) \) where \( E^n \neq \emptyset \) and \( T : E^n \to E^n \) be a contraction on \( E^n \). Then \( T \) has exactly one fixed point.

**Proof.** In \( E^n \) each fuzzy number \( u \) is expressed in parameterized form \( u = [a(r), b(r)] \) \( r \in I \) and the metric \( D_1 \) is defined as

\[
D_1([a(r), b(r)], [c(r), d(r)]) = \left[ \min_{r \in I} \min_{i=1,2,\ldots,n} \{|a_i^1(r) - c_i^1(r)|, b_i^1(r) - d_i^1(r)|\}, \right.
\]

\[
\left. \min_{r \in I} \max_{i=1,2,\ldots,n} \{|a_i^1(r) - c_i^1(r)|, |b_i^1(r) - d_i^1(r)|\} \right]
\]

where

\[
a(r) = (a^1(r), a^2(r), \ldots, a^n(r)), \quad b(r) = (b^1(r), b^2(r), \ldots, b^n(r))
\]

\[
c(r) = (c^1(r), c^2(r), \ldots, c^n(r)), \quad d(r) = (d^1(r), d^2(r), \ldots, d^n(r))
\]

Constructing a Cauchy sequence \((u_n)\) by iteration so that it converges in the complete space \( E^n \), we prove that its limits is a fixed point of \( T \) and \( T \) has no further fixed points.
For any \( u_0 \in E^n \), let \( (u_n) \) be the image of \( u_0 \) under repeated application of \( T \). That is \( u_0, \ u_1 = T \ u_0, \ u_2 = T^2 \ u_0, \ldots \ u_n = T^n \ u_0, \ldots \) We show that \( (u_n) \) is a Cauchy sequence.

\[
D_1(u_{m+1}, u_m) = D_1(Tu_m, Tu_{m-1})
\]
\[
\leq \alpha D_1(u_m, u_{m-1})
\]
\[
= \alpha \cdot D_1(Tu_{m-1}, Tu_{m-2})
\]
\[
\leq \alpha^2 D_1(u_{m-1}, u_{m-2})
\]
\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldot
is a fixed point of the mapping \( T \). For that consider

\[
D_1(u, Tu) \leq D_1(u, u_m) + D_1(u_m, Tu) \quad \text{(using triangle inequality)}
\]

\[
\leq D_1(u, u_m) + \alpha D_1(u_{m-1}, u) \quad \text{(using contraction mapping)}
\]

\[
< \hat{\epsilon} > 0 \quad \text{because} \quad u_m \to u.
\]

We conclude that \( D_1(u, Tu) = 0 \). So that \( u = Tu \). This shows that \( u \) is a fixed point of \( T \). \( u \) is the only fixed point of \( T \) because from \( Tu = u \) and \( T\bar{u} = \bar{u} \) we obtain

\[
D_1(u, \bar{u}) = D_1(Tu, T\bar{u}) \leq \alpha D(u, \bar{u}) \quad \text{which implies} \quad D_1(u, \bar{u}) = 0 \quad \text{since} \quad \alpha < 1.
\]

Hence \( u = \bar{u} \) by the 2nd axiom of a metric. Thus the theorem is proved. \( \blacksquare \)

**Theorem 3.1.4** (Contraction of a ball). Consider the mapping \( T \) from the complete metric space, \( E^n = (E^n, D_1) \) into itself. If \( T \) is a contraction on a closed ball \( Y = \{ u / D_1(u, u_0) \leq [0, r] \} \), that is, \( T \) satisfies \( D_1(Tu, Tv) \leq \alpha D_1(u, v) \) for \( u, v \in Y \) and \( \alpha < 1 \). Also, assuming that

\[
D_1(u_0, Tu_0) < (1 - \alpha)[0, r]. \quad (3.1.3)
\]

The iterative sequence \( u_0, u_1 = Tu_0, u_2 = Tu_1 = T^2u_0, \ldots u_n = T^nu_0, \ldots \) converge to an \( u_0 \in Y \). This \( u \) is a fixed point of \( T \) and is the only fixed point of \( T \) in \( Y \).

**Proof.** It is required to show that all \( u_m \)'s as well as \( u_0 \) lie in \( Y \). We put \( m = 0 \) in the equation

\[
D_1(u_m, u_n) \leq \frac{\alpha^m}{1 - \alpha} D_1(u_0, u_1) \quad (n > m).
\]
Change \( n \) to \( m \) and use (3.1.3) to get

\[
D_1(u_0, u_m) \leq \frac{1}{1 - \alpha} D(u_0, u_1) < [0, r].
\]

Hence all \( u_m \)'s are in \( Y \). Also \( u_0 \in Y \). Since \( u_m \to u_0 \) and \( Y \) is closed hence \( Y \) is complete. So by Banach fixed point theorem we get \( u_0 \) is the only fixed point of \( T \) in \( Y \). \( \square \)

**Result 3.1.5.** A contraction \( T \) on a metric space \( E^n \) is a continuous mapping.

### 3.2. Application of Banach Fixed point theorem for Fuzzy sets

The existence and uniqueness theorem for fuzzy differential and fuzzy integral equations is proved here using the Banach fixed point theorem for fuzzy sets. In connection with fuzzy function spaces, it is possible to find many important applications of Banach fixed point theorem for fuzzy sets. As in the classical theory, it is shown below that Picard’s theorem can be proved in \((E^n, D_1)\). Here an initial value problem is converted in to an integral equation, which defines a mapping and the conditions of the theorem shows that the mapping is a contraction. Hence its fixed point gives the solution of the problem. Here following the same approach in [26].
3.2.1. Picard’s existence and Uniqueness theorem for ordinary fuzzy
differential equations. Consider the fuzzy initial value problem

\[ u^1 = f(t, u) \tag{3.2.1} \]

with the initial condition \( u(t_0) = u_0 \), where \( f \) is a continuous fuzzy function on a closed region \( R = \{(t, u)/|t - t_0| \leq p \text{ and } D_2(u, u_0) \leq [\hat{0}, q]\} \). That is, \( f \in S' = \{f/f : R \to E', f \text{ continuous }\} \) and bounded on \( R \) say

\[ D_2(\hat{0}, f(t, u)) \leq [\hat{0}, m] \tag{3.2.2} \]

and \( u \in S' \). Suppose that \( f \) satisfies a Lipschitz condition on \( R \) with respect to its second argument, that is there is a constant \( K \) (Lipschitz constant) such that for \( (t, u), (t, v) \) in \( R \),

\[ D_2(f(t, u), f(t, v)) \leq KD_2(u, v). \tag{3.2.3} \]

The initial value problem (1) with \( t_0, u_0 \) in \( R \) has a unique solution. This solution exists on an interval \( J = [t_0 - \beta, t_0 + \beta] \) where

\[ \beta < \min \left\{ p, \frac{q}{m}, \frac{1}{K} \right\}. \tag{3.2.4} \]

**Proof.** Let \( S'(J) \) be the metric space of all real valued fuzzy continuous functions defined on the interval \( J = [t_0 - \beta, t_0 + \beta] \) with metric \( D_2 \) defined by
\[ D_2(u, v) = \left[ \inf_t \min_r \min \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\} \right. \]
\[ \left. \sup_t \max_r \max \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\} \right] \]

where \( u(t) = [a(r, t), b(r, t)] \), \( v(t) = [c(r, t), d(r, t)] \) be the parametric representation of \( u \) and \( v \) in \( S'(J) \), \( t \in J \). \( S'(J) \) is complete. Let \( \tilde{S} \) be the subspace of \( S'(j) \) consisting of all those functions \( u \in S'(J) \) that satisfy \( D_2(u, u_0) \leq \hat{0}, m] \beta \). \( \tilde{S} \) is closed in \( S'(J) \), so that \( S' \) is complete. By integration we can see that the initial value problem can be written as \( u = Tu \) where \( T : \tilde{S} \to \tilde{S} \) is defined by

\[ Tu(t) = u_0 + \int_{t_0}^t f(\tau, u(\tau))d\tau \] (3.2.5)

\( T \) is well defined for all \( u \in \tilde{S} \). Because \( m\beta < q \) by (3.2.4). Hence if \( u \in \tilde{S} \), then \( \tau \in J \) and \((\tau, u(\tau)) \in R \). Since \( f \) is continuous on \( R \), the integral exist.

Now we can show that \( T \) maps \( \tilde{S} \) into itself. Consider

\[ D_2(Tu(t), u_0) = D_2 \left( \int_{t_0}^t (f(\tau, u(\tau))d\tau, \hat{0}) \right) \]
\[ \leq |t - t_0| \hat{0}, m] \]
\[ < \beta \hat{0}, m] = \hat{0}, m\beta \]
\[ < \hat{0}, q]. \]

Next show that \( T \) is a contraction on \( \tilde{S} \). For, consider
\[ D_2(Tu, Tv) = D_2 \left( \int_{t_0}^{t} f(\tau, u(\tau))d\tau, \int_{t_0}^{t} f(\tau, v(\tau))d\tau \right) \]
\[ \leq \int_{t_0}^{t} D_2(f(\tau, u(\tau)), f(\tau, v(\tau)))d\tau \]
\[ \leq K \left[ \int_{t_0}^{t} \inf \min_r \min \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\}, \right. \]
\[ \int_{t_0}^{t} \sup \max_r \max \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\} dt \]
\[ \leq K \left[ \inf \min_r \min \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\} \int_{t_0}^{t} dt \right. \]
\[ \sup \max_r \max \{|a(r, t) - c(r, t)|, |b(r, t) - d(r, t)|\} \int_{t_0}^{t} dt \]
\[ = K|t - t_0|D_2(u, v) \]
\[ \leq K\beta D_2(u, v). \]

Since \( K\beta < 1 \), \( T \) is a contraction on \( \tilde{S} \). Hence \( T \) has a unique fixed point \( u \) such that \( Tu = u \), by Banach fixed point theorem.

\[ u(t) = u(t_0) + \int_{t_0}^{t} f(\tau, u(\tau))d\tau \quad (3.2.6) \]

\( u(t) \) is differentiable since \( f \) is continuous and \( (\tau, u(\tau)) \in R \). Thus \( u \) satisfies (3.2.1).

Conversely every solution of (3.2.1) satisfy (3.2.6). Hence the proof. \( \square \)

### 3.2.2. Application of Banach theorem to fuzzy Integral equations.

Here we find that Banach fixed point theorem is useful for proving the existence and uniqueness theorems for fuzzy Fredholm integral equation of the second kind.
\[ u(t) = v(t) + \mu \int_a^b K(t, s)u(s)ds \] (3.2.7)

where \( u \) is a fuzzy function continuous on \([a, b]\), which is unknown, \( \mu \) is a parameter, \( K \) is a continuous function on \([a, b] \times [a, b]\) and \( v \) is a given continuous fuzzy function on \([a, b]\) that is on \( S'[a, b] \) and the metric on \( S'[a, b] \) is \( D_2 \).

**Theorem 3.2.1.** Consider the fuzzy Fredholm integral equation (3.2.7) where \( K \) is continuous on \( R = J \times J \) where \( J = [a, b] \). Assume that \( \mu \) satisfies \( |\mu| < \frac{1}{c(b-a)} \). Then the integral equation (3.2.7) has a unique solution \( u(t) \) in \( S'(J) \) and is the limit of the iterative sequence \((u_n)\) where \( u_0 \) is any continuous fuzzy function on \( J \) and for \( n = 0, 1, 2, \ldots \)

\[ u_{n+1}(t) = v(t) + \mu \int_a^b K(t, s)u_n(s)ds. \]

**Proof.** Since the fuzzy function space \( S'(J) \) is complete we can apply Banach fixed point theorem. \( \square \)

**Theorem 3.2.2.** Consider the fuzzy Volterra Integral equation

\[ u(t) = v(t) + \mu \int_0^t k(t, s)u(s)ds. \] (3.2.8)

where \( v \) is continuous and fuzzy function on \([a, b]\) and the kernel \( K \) is continuous on the triangular region \( R \) in the \((t, s)\) plane given by \( a \leq s \leq t, a \leq t \leq b \). Then (3.2.8) has a unique fuzzy solution \( u \) on \([a, b]\) for every \( \mu \).

**Proof.** This can be proved by the use of Banach fixed point theorem. Equation (3.2.8) can be written \( u = Tu \) with \( T : S'[a, b] \rightarrow S'[a, b] \), where \( S'[a, b] = \{u/u : \)}
\([a, b] \rightarrow E', u\) is a continuous function \}. \(T\) is defined by

\[
Tu(t) = v(t) + \mu \int_a^t K(t, S)u(S)dS.
\]

Since \(K\) is continuous on \(R\) and \(R\) is closed and bounded, hence \(K\) is a bounded function on \(R\), say \(|K(t, S)| \leq c, \forall (t, S) \in R\).

Using the metric \(D_2\) in \(S'[a, b]\) we obtain for all fuzzy functions \(u, w \in S'[a, b]\).

\[
D_2[Tu, Tw] \leq |\mu| c(t - a) D_2(u, w). \quad (3.2.9)
\]

We show by induction that

\[
D_2(T^m u, T^m w) \leq \frac{|\mu| c^m (t - a)^m}{m!} D_2(u, w). \quad (3.2.10)
\]

For \(m = 1\), this is (3.2.9). For proving (3.2.10) for \(m + 1\), consider

\[
D_2(T^{m+1} u, T^{m+1} w) = |\mu| \int_a^t K(t, S)D_2(T^m u, T^m w) dS
\]

\[
\leq |\mu| c \int_a^t \frac{|\mu| c^m (t - a)^m}{m!} D_2(u, w)dS
\]

\[
= \frac{|\mu| c^{m+1} (t - a)^{m+1}}{(m+1)!} D_2(u, w).
\]

This is the induction proof of (3.2.10). \(\square\)

Taking \(t - a \leq b - a\) on the right hand side of (3.2.10) we obtain \(D_2(T^m u, T^m w) \leq \alpha_m D_2(u, w)\), where \(\alpha_m = \frac{|\mu| c^{m} (b-a)^m}{m!}\) for any fixed \(\mu\) and for large \(m\), we have \(\alpha_m < 1\).
Hence we get $T^m$ is a contraction on $S'[a,b]$. Then $T$ has a unique fixed point by following Banach fixed point theorem and the following result.

**Result 3.2.3.** Let $T : S' \to S'$ be a mapping on a complete metric space $S' = (S', D_2)$, and suppose that $T^m$ is a contraction on $S'$ for some positive integer $m$. Then $T$ has a unique fixed point.

**Proof.** Given that $T^m$ is a contraction on $S'$. Hence by Banach Fixed point theorem $T^m = R$ has a fixed point $a$. i.e., $R\hat{u} = \hat{u}$. Then $R^n\hat{u} = \hat{u}$, since $R^n\hat{u} = R^{n-1}(R\hat{u}) = R^{n-1}(\hat{u}) = \ldots = \hat{u}$. Using Banach theorem for every $u \in S'$. $R^n u \to \hat{u}$ as $n \to \infty$. For the particular $u = T\hat{u}$, since $R^n = T^m$, we thus obtain

$$\hat{u} = \lim_{n \to \infty} R^n T\hat{u} = \lim_{n \to \infty} TR^n \hat{u}$$

$$= \lim_{n \to \infty} T\hat{u}.$$  

Thus $\hat{u}$ is a fixed point of $T$. We see that $T$ cannot have more than one fixed point, since every fixed point of $T$ is also a fixed point of $R$. This gives the proof.  

\[\square\]