Chapter 2

Fuzzy function space

2.1. Introduction

A fuzzy function can be considered as the generalization of a classical function. A classical function $f$ is a mapping from the domain $D$ to a space $S$, where $f(D) \subseteq S$ is called the range of $f$. A classical function can be fuzzified in different ways. First is a crisp mapping from a fuzzy set which generates a fuzzy set. In the second type, the mapping itself is fuzzy. Thus the crisp domain is mapped on to fuzzy range. In the third type the ordinary functions can have fuzzy properties or can be constrained by fuzzy constraints.
2.2. Fuzzy number space $E^n$

The fuzzy number space $E^n$ is an extension of the $n$-dimensional Euclidean space $\mathbb{R}^n$.

**Definition 2.2.1.** $P_k(\mathbb{R}^n)$ denote the family of all non empty compact convex subsets of $\mathbb{R}^n$ and define the addition and scalar multiplication in $P_k(\mathbb{R}^n)$ as usual.

**Definition 2.2.2 (Fuzzy number space $E^n$).** By $E^n$, we denote the family of all fuzzy sets $u$ of $\mathbb{R}^n$ for which $u$ satisfies the following properties

1. $u$ is normal, that is there exist an element $t_0 \in \mathbb{R}^n$ such that $u(t_0) = 1$.
2. $u$ is fuzzy convex, that is, for any $t, s \in \mathbb{R}^n$ and $\lambda \in [0, 1]$,

$$u(\lambda t + (1 - \lambda)s) \geq \min\{u(t), u(s)\}.$$  \hfill (2.2.1)

3. $u$ is upper-semi continuous, i.e., $\lim_{t \to t_0^+} u(t) = u(t_0)$.
4. $[u]^0 = \text{cl}\{t \in \mathbb{R}^n | u(t) > 0\}$ is compact.

Let $u \in E^n$ then $u$ is called a fuzzy number and $E^n$ is said to be a fuzzy number space.

For $0 < r \leq 1$ denote $[u]^r = \{t \in \mathbb{R}^n / u(t) \geq r\}$, then from 1 to 4 it follows that the $r$-level set $[u]^r \in P_k(\mathbb{R}^n)$ for all $0 \leq r \leq 1$. A real vector $a \in \mathbb{R}^n$ is a special fuzzy number as follows.

$$u(t) = \begin{cases} 
1 & \text{if } t = a \\
0 & \text{if } t \neq a.
\end{cases}$$

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Definition 2.2.3. **Parametric representation of fuzzy number.** (Yunqing Wang, 1990)

If

\[ a(r) = (a^1(r), a^2(r), \ldots, a^n(r)) \quad \text{and} \quad b(r) = (b^1(r), b^2(r), \ldots b^n(r)) \in \mathbb{R}^n \]

also \( a(r) < b(r) \) whenever \( a^i(r) < b^i(r) \) for \( i = 1, 2, \ldots n \). We can identify a fuzzy number \( u \) with parameterized pairs

\[ u = [a(r), b(r)], \ r \in I = [0, 1] \quad (2.2.2) \]

where \( [a(r), b(r)] = cl\{t \in \mathbb{R}^n / u(t) \geq r\} \) if \( 0 \leq r \leq 1 \).

The following facts are clear

(5) \( a(r) \) is a bounded increasing function, \( b(r) \) is a bounded decreasing function

(6) \( a(1) \leq b(1) \)

(7) For \( 0 < p \leq 1 \), \( a(r) \) and \( b(r) \) are left hand continuous at \( p \), \( a(r) \) and \( b(r) \) are right hand continuous at \( r = 0 \).

If \( a(r) \) and \( b(r) \) satisfy the conditions 5,6 and 7 then \( u : \mathbb{R}^n \rightarrow I \) is a fuzzy number with parameterization given by (2.2.2).

**Definition 2.2.4.** Let \( V = \{u = (a(r), b(r))/r \in I, a(r) \text{ and } b(r) \text{ are bounded functions}\} \).
The addition $\,\oplus\,\,$, scalar multiplication $(\cdot\,)$, and metric $D$ are defined in $V$ as follows.

\begin{align*}
(a(r), b(r)) + (c(r), d(r)) &= (a(r) + c(r), b(r) + d(r)) \\
K(a(r), b(r)) &= (Ka(r), Kb(r)) \\
D[(a(r), b(r)), (c(r), d(r))] &= [p(r), q(r)],
\end{align*}

where

\begin{align*}
p(r) &= \min_{i=1,2,\ldots,n} \{|a_i^1(r) - c_i^1(r)|, |b_i^1(r) - d_i^1(r)|\}, \\
q(r) &= \max_{i=1,2,\ldots,n} \{|a_i^1(r) - c_i^1(r)|, |b_i^1(r) - d_i^1(r)|\}.
\end{align*}

(2.2.3)

Also

\begin{align*}
a(r) &= (a^1(r), a^2(r), \ldots a^n(r)) \quad b(r) = (b^1(r), b^2(r), \ldots b^n(r)) \\
& \quad c(r) = (c^1(r), c^2(r), \ldots c^n(r)) \quad d(r) = (d^1(r), d^2(r), \ldots d^n(r)) \in \mathbb{R}^n.
\end{align*}

Now define $D_1$ as

$$D_1(u, v) = \left[\min_{r \in I} p(r), \max_{r \in I} q(r)\right]$$

(2.2.4)

where $u = [a(r), b(r)], v = [c(r), d(r)], a(r), b(r), c(r), d(r)$ are as in (2.2.3).

Also $D_1(u, v) < D_1(w, x)$ whenever $D_1(u, v) \subseteq D_1(w, x)$.

Here the metric is taken as an interval. It is also clear that $V$ is a linear topological vector space and a complete metric space.

**Theorem 2.2.5.** $V \subset E^n$ is a metric space with the metric $D_1$ defined by
$D_1(u,v) = \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\}, \right.
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\} \right]$

**Proof.** $D_1$ is a metric for, $D_1$ is real valued, finite and non-negative.

$D_1(u,v) = \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\}, \right.$
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\} \right]

$= \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\}, \right.$
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\} \right] = \hat{0} = [0,0]

$\Rightarrow \left\{ |a^i(r) - c^i(r)|, |b^i(r) - d^i(r)| \right\} = 0$ for all $i$ and $r$

$\Rightarrow u = v$ for every $r$

$D_1(u,v) = D_1(v,u)$ clear from the definition.

If $u = [a(r), b(r)], v = [c(r), d(r)]$ and $w = [e(r), f(r)],$

$D_1(u,v) = \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - d^i(r)| \right\}, \right.$
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - d^i(r)| \right\} \right]

$= \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - d^i(r)| \right\}, \right.$
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - d^i(r)| \right\} \right]

$= \left[ \min_{r \in I} \min_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r) + e^i(r) - c^i(r)|, |b^i(r) - f^i(r) + f^i(r) - d^i(r)| \right\}, \right.$
\left. \max_{r \in I} \max_{i=1,2,...,n} \left\{ |a^i(r) - e^i(r) + e^i(r) - c^i(r)|, |b^i(r) - f^i(r) + f^i(r) - d^i(r)| \right\} \right]$
\[
\leq \left[ \min_{r \in I} \min_{i=1,2,\ldots,n} \left\{ |a^i(r) - e^i(r)| + |e^i(r) - c^i(r)|, |b^i(r) - f^i(r)| + |f^i(r) - d^i(r)| \right\} \right.
\max_{r \in I} \max_{i=1,2,\ldots,n} \left\{ |a^i(r) - e^i(r)| + |e^i(r) - c^i(r)|, |b^i(r) - f^i(r)| + |f^i(r) - d^i(r)| \right\}
\]

\[
= \left[ \min_{r \in I} \min_{i=1,2,\ldots,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - f^i(r)| \right\} \right.
\max_{r \in I} \max_{i=1,2,\ldots,n} \left\{ |a^i(r) - e^i(r)|, |b^i(r) - f^i(r)| \right\}
+ \left[ \min_{r \in I} \min_{i=1,2,\ldots,n} \left\{ |e^i(r) - c^i(r)|, |f^i(r) - d^i(r)| \right\} \right.
\max_{r \in I} \max_{i=1,2,\ldots,n} \left\{ |e^i(r) - c^i(r)|, |f^i(r) - d^i(r)| \right\}
\]

\[
= D_1(u, w) + D_1(w, v).
\]

Now we can show that every Cauchy sequence in \( V \) converges. So \( V \) is a complete metric space.

\[\square\]

### 2.3. Fuzzy function space

The definition of fuzzy function space and some properties of this function space are discussed in this section.

**Definition 2.3.1.** A function \( f : X \to E^n \) is said to be fuzzy function where \( X \subset R^n \).

The limits and derivatives of fuzzy functions can be defined as real functions. If \( f(t) = (a(r, t), b(r, t)) \) is differentiable, we have

\[
f'(t) = \left[ \frac{d}{dt} a(r, t), \frac{d}{dt} b(r, t) \right].
\]
Definition 2.3.2 (fuzzy function space). Let $S^n = \{U/U : X \to E^n \text{ is a continuous fuzzy function}\}$. Defining metric in $S^n$ by

$$D_2(U_1, U_2) = \left[ \inf_{t \in X} \min_{r \in I} \min_{i = 1, 2, \ldots, n} \{ |a^i_n(r, t) - a^i_2(r, t)|, |b^i_n(r, t) - b^i_2(r, t)| \}, \right.$$

$$\sup_{t \in X} \max_{r \in I} \max_{i = 1, 2, \ldots, n} \{ |a^i_1(r, t) - a^i_2(r, t)|, |b^i_1(r, t) - b^i_2(r, t)| \} \left. \right]. \tag{2.3.1}$$

where $U_1(t) = [(a^1_1(r, t), a^2_1(r, t), \ldots, a^n_1(r, t)), (b^1_1(r, t), b^2_1(r, t), \ldots, b^n_1(r, t))]$ and $U_2(t) = [(a^1_2(r, t), a^2_2(r, t), \ldots, a^n_2(r, t)), (b^1_2(r, t), b^2_2(r, t), \ldots, b^n_2(r, t))]$.

We call $S^n$ to be a fuzzy function space on $X \subset \mathbb{R}^n$.

Theorem 2.3.3. Fuzzy function space $S^n$ is a complete metric space.

Proof. Let $U_n(t) = [a_n(r, t), b_n(r, t)], n = 1, 2, \ldots$ be a Cauchy sequence in $S^n$. Then

$$\lim_{m,n \to \infty} D_2(U_m, U_n) = \lim_{m,n \to \infty} \left[ \inf_{t \in X} \min_{r \in I} \min_{i = 1, 2, \ldots, n} \{ |a^i_n(r, t) - a^i_m(r, t)|, |b^i_n(r, t) - b^i_m(r, t)| \}, \right.$$

$$\sup_{t \in X} \max_{r \in I} \max_{i = 1, 2, \ldots, n} \{ |a^i_n(r, t) - a^i_m(r, t)|, |b^i_n(r, t) - b^i_m(r, t)| \} \left. \right] < \hat{\epsilon} = [\epsilon, \tau].$$

Hence,

$$\lim_{m,n \to \infty} |a^i_n(r, t) - a^i_m(r, t)| < \tau \text{ uniform on } r,$$

$$\lim_{m,n \to \infty} |b^i_n(r, t) - b^i_m(r, t)| < \tau \text{ uniform on } r.$$
continuous on $t$. Now we prove that $u(t) = [a(r, t), b(r, t)]$ is a fuzzy function. Since $a_n(r, t)$ and $b_n(r, t)$ are fuzzy functions, for any $t \in X$, $a_n(r, t)$ and $b_n(r, t)$ satisfy 5,6 and 7 too. Therefore $U(t) = [a(r, t), b(r, t)]$ is a fuzzy function, and $D_2(U_n, U) \to 0$ as $n \to \infty$. □

**Definition 2.3.4.** Suppose that $f(t)$ is a fuzzy function. If there exist a fuzzy function $F(t)$ such that $F'(t) = f(t)$, $F(t)$ is called anti-derivative of $f(t)$. $F(t) + m$ is called indefinite integral of $f(t)$, where $m$ is an arbitrary fuzzy number. Denote $\int f(t)dt = F(t) + m$.

**Definition 2.3.5.** Let $f(t) = [a(r, t), b(r, t)]$ be a fuzzy function on $[\alpha, \beta]$. If

1. $\int_{\alpha}^{\beta} a(r, t)dt$ and $\int_{\alpha}^{\beta} b(r, t)dt$ exist simultaneously
2. $\int_{\alpha}^{\beta} a(r, t)dt$ and $\int_{\alpha}^{\beta} b(r, t)dt$ is a fuzzy number, we say $f(t)$ is integrable on $[\alpha, \beta]$. Moreover

$$\int_{\alpha}^{\beta} f(t)dt = \left[ \int_{\alpha}^{\beta} a(r, t)dt, \int_{\alpha}^{\beta} b(r, t)dt \right]$$

is called the definite integral of $f(t)$ on $[\alpha, \beta]$.

**Lemma 2.3.6** (Yunqing (1990)). Suppose that $f(t) = [a(r, t), b(r, t)]$ have antiderivatives. Then $F(t) = \left[ \int a(r, t)dt, \int b(r, t)dt \right]$ is an anti-derivative of $f(t)$.

**Lemma 2.3.7** (Yunqing (1990)). Let $f(t)$ is a fuzzy function on $X \subset R^n$. If the definite integral $\int_{t_0}^{t} f(t)dt$ exist for $t_0 \in X (t_0 \leq t)$, then $F(t) = \int_{t_0}^{t} f(t)dt$ is a fuzzy function and $F'(t) = f(t)$.