Chapter 1

Preliminaries

This chapter contains basic results needed for the work. The first section deals with the concept of Integral equations. The subsequent sections discuss the concept on the metric spaces, fuzzy set theory and Interval arithmetic.

1.1. Concept of Integral equations

The definitions and results are taken from Ram P. Kanwal [24].

Definition. An integral equation is an equation in which an unknown function appears under one or more integral signs. Naturally in such an equation there can occur other terms aswell.

For example, for \( a \leq x \leq b, a \leq y \leq b \), the equations
\[ g(x) = \int_a^b K(x, y) f(y) dy. \]  
\[ f(x) = g(x) + \int_a^b K(x, y) f(y) dy \]  
\[ f(x) = \int_a^b K(x, y) [f(y)]^2 dy \]

where the function \( f(x) \) is the unknown function, while all other functions are known, are integral equations. These functions may be complex-valued functions of the real variables \( x \) and \( y \).

One can also consider integral equations in which the unknown function is dependent not only on one variable but on several variables. Such, for example, is the equation

\[ f(x) = g(x) + \int_{\Omega} K(x, y) f(y) dy \]

where \( x \) and \( y \) are \( n \)-dimensional vectors in \( \mathbb{R}^n \) and \( \Omega \) is a region of \( \mathbb{R}^n \). Similarly one can also consider systems of integral equations with several unknown functions.

An integral equation is called linear if only linear operations are performed in it upon the unknown function. The equations (1.1.1) and (1.1.2) are linear, while (1.1.3) is non-linear.

The most general type of linear integral equation is of the form

\[ h(x) f(x) = g(x) + \lambda \int_a^b K(x, y) f(y) dy \]

where the upper limit may be either variable or fixed. The functions \( h, K \) and \( g \) are
known functions, while $f$ is to be determined, $\lambda$ is a non zero, real or complex parameter.

The functions $K(x, y)$, a function of two variables, defined in the interval $a \leq x \leq b$, $a \leq y \leq b$ is called the kernel. The following special cases of equation (1.1.5) are of main interest.

(i) Fredholm Integral equations.

In all Fredholm integral equations, the upper limit of integration $b$, say is fixed.

(1) If we put $h(x) = 0$ in the equation (1.1.5) we get the Fredholm integral equation of the first kind. That is

$$g(x) + \lambda \int_a^b K(x, y)f(y)dy = 0.$$  \hspace{1cm} (1.1.6)

(2) In equation (1.1.5) put $h(x) = 1$, we get the Fredholm integral equation of the second kind. That is

$$f(x) = g(x) + \lambda \int_a^b K(x, y)f(y)dy.$$ \hspace{1cm} (1.1.7)

(3) The homogeneous Fredholm integral equation of the second kind is a special case of (1.1.2) above. In this case $g(x) = 0$. That is

$$f(x) = \lambda \int_a^b K(x, y)f(y)dy.$$ \hspace{1cm} (1.1.8)

(ii) Volterra Equations.

Volterra equations of the first, homogeneous and second kinds are defined
precisely as above except that \( b = x \) is a variable upper limit of integration. Equation (1.1.5) itself is called an integral equation of the third kind.

(iii) Singular integral equations.

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular.

For example, the integral equations

\[
  f(x) = g(x) + \int_{-\infty}^{\infty} \exp(-|x - y|) f(y) \, dy \quad \text{and}
\]

\[
  g(x) = \int_{0}^{x} \left[ \frac{1}{|x - y|^2} \right] f(y) \, dy, \quad 0 < \alpha < 1
\]

are singular integral equations.

1.2. Metric space and some of its properties

Metric spaces are fundamental in functional analysis because they play a role similar to that of the real line \( \mathbb{R} \) in calculus. The following definitions and results obtained from Kreyszig [26].

**Definition 1.2.1.** A metric space is a pair \((X, d)\) where \( X \) is a set and \( d \) is a metric on \( X \) (or distance function on \( X \)), that is, a function defined on \( X \times X \) such that for all \( x, y, z \in X \) we have

1. \( d \) is real valued, finite and non negative
2. \( d(x, y) = 0 \) iff \( x = y \).
(3) \( d(x, y) = d(y, x) \).

(4) \( d(x, y) \leq d(x, z) + d(z, y) \).

**Definition 1.2.2** (Ball and Sphere). Given a point \( x_0 \in X \) and a real number \( r > 0 \), we define three types of sets.

(a) \( B(x_0, r) = \{ x \in X | d(x, x_0) < r \} \) (open ball).

(b) \( B(x_0, r) = \{ x \in X | d(x, x_0) \leq r \} \) (closed ball).

(c) \( S(x_0, r) = \{ x \in X | d(x, x_0) = r \} \) (sphere).

In all three cases \( x_0 \) is called the centre and \( r \) the radius.

**Definition 1.2.3** (open set, closed set). A subset \( M \) of a metric space \( X \) is said to be open if it contains a ball about each of its points. A subset \( K \) of \( X \) is said to be closed if its complement (in \( X \)) is open, i.e., \( K^c = X - K \) is open.

**Definition 1.2.4** (Continuous mapping). Let \( X = (X, d) \) and \( Y = (Y, \tilde{d}) \) be metric spaces. A mapping \( T : X \to Y \) is said to be continuous at a point \( x_0 \in X \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( \tilde{d}(Tx, Tx_0) < \epsilon \) for all \( x \) satisfying \( d(x, x_0) < \delta \). \( T \) is said to be continuous if it is continuous at every point of \( X \).

The following theorem gives the characterization of continuous mappings in terms of open subset.

**Theorem 1.2.5** (Continuous mapping). A mapping \( T \) of a metric space \( X \) into a metric space \( Y \) is continuous if and only if the inverse image of any open subset of \( Y \) is an open subset of \( X \).
**Definition 1.2.6.** Let $M$ be a subset of a metric space $X$. Then a point $x_0$ of $X$ (which may or may not be a point of $M$) is called an accumulation point of $M$ (or limit point of $M$) if every neighbourhood of $x_0$ contains at least one point $y \in M$ distinct from $x_0$.

**Definition 1.2.7.** The set consisting of the points of $M$ and the accumulation points of $M$ is called the closure of $M$ and is denoted by $\overline{M}$.

**Definition 1.2.8** (Convergence of a sequence, limit). A sequence $(x_n)$ in a metric space $(X,d)$ is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0;$$

$x$ is called the limit of $(x_n)$ and we write $\lim_{n \to \infty} x_n = x$ or simply $x_n \to x$. We say that $(x_n)$ converges to $x$ or has the limit $x$.

Hence if $x_n \to x$, an $\epsilon > 0$ being given there is an $N = N(\epsilon)$ such that all $x_n$ with $n > N$ lie in the $\epsilon$-neighbourhood $B(x, \epsilon)$ of $x$.

**Lemma 1.2.9** (Boundedness, limit). Let $X = (X,d)$ be a metric space. Then

(a) A convergent sequence in $X$ is bounded and its limit is unique.

(b) If $x_n \to x$ and $y_n \to y$ in $X$, then $d(x_n, y_n) \to d(x, y)$.

**Definition 1.2.10** (Cauchy sequence, completeness). A sequence $(x_n)$ in a metric space $X = (X,d)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $d(x_m, x_n) < \epsilon \forall m, n > N$. The space $X$ is said to be complete if for every Cauchy sequence in $X$ converges (that is, has a limit which is an element of $X$).

**Theorem 1.2.11.** The real line and the complex plane are complete metric spaces.

**Theorem 1.2.12.** Every convergent sequence in a metric space is a Cauchy sequence.
Theorem 1.2.13 (Closure, closed set). Let $M$ be a non empty subset of a metric space $(X, d)$ and $\overline{M}$ its closure. Then

(a) $x \in \overline{M}$ iff there is a sequence $(x_n)$ in $M$ such that $x_n \to x$.

(b) $M$ is closed iff the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.

Theorem 1.2.14. A subspace $M$ of a complete metric space $X$ is itself complete iff the set $M$ is closed in $X$.

Theorem 1.2.15. A mapping $T : X \to Y$ of a metric space $(X, d)$ into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ iff $x_n \to x_0$ implies $Tx_n \to Tx_0$.

Example 1.2.16. The function space $C[a, b]$ is complete where $[a, b]$ is any given closed interval on $R$.

Theorem 1.2.17. Convergence $x_m \to x$ in the space $C[a, b]$ is uniform convergence, that is, $(x_m)$ converges uniformly on $[a, b]$ to $x$.

Definition 1.2.18 (Compactness). A metric space $X$ is said to be compact, if every sequence in $X$ has a convergent subsequence. A subset $M$ of $X$ is said to be compact if $M$ is compact considered as a subspace of $X$, that is, if every sequence in $M$ has a convergent subsequence whose limit is an element of $M$.

Result 1.2.19.

(1) A compact subset $M$ of a metric space is closed and bounded.

(2) In a finite dimensional normed space $X$, any subset $M \subset X$ is compact iff $M$ is closed and bounded.
Let $X$ and $Y$ be metric spaces and $T : X \rightarrow Y$ a continuous mapping. Then the image of a compact subset $M$ of $X$ under $T$ is compact.

A continuous mapping $T$ of a compact subset $M$ of a metric space $X$ into $R$ assumes a maximum and a minimum at some points of $M$.

### 1.3. Fuzzy set theory

In Fuzzy set theory everything has elasticity. It is a theory of graded concepts. There are two lines along which fuzzy set theory has developed. The first is generalizing or fuzzifying classical mathematical results and the second is modeling real life situations having uncertainties.

**Definition 1.3.1** (Fuzzy sets). If $X$ is a collection of objects, then a fuzzy set $u$ in $X$ is a set of ordered pairs, $u = \{(x, u(x)/x \in X)\}$, $u(x)$ is called the membership function of $x$ in $u$ which maps $X$ to the membership space (usually membership space contain values between 0 and 1). [51].

A fuzzy set is a generalization of a classical set and the membership function a generalization of characteristic function.

**Definition 1.3.2.** Support of a fuzzy set $u$ is the set of all $x \in X$ such that $u(x) > 0$.

**Definition 1.3.3** ($r$-level set). The set of elements that belong to the fuzzy set $u$ at least to the degree $r$ is called the $r$-level set $[u]^r = \{x \in X/u(x) \geq r\}$.

**Definition 1.3.4.** A fuzzy set $u$ is convex if
\[ u(\lambda x_1 + (1 - \lambda)x_2) \geq \min \left( u(x_1), u(x_2) \right) \cdot \]

Thus a fuzzy set is convex if all \( r \)-level sets are convex.

**Fuzzy number 1.3.5.** A fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is thus a special case of a convex fuzzy set. Since fuzzy numbers is an extension of real numbers, calculations with fuzzy numbers allow the incorporation of uncertainty on parameters, properties, geometry, initial conditions etc.

**Definition 1.3.6.** A fuzzy number is a fuzzy set \( u : R \to I = [0, 1] \) with the properties

1. \( u \) is upper semi continuous
2. \( u(x) = 0 \), outside of some interval \([c, d]\)
3. there are real numbers \( a \) and \( b, c \leq a \leq b \leq d \) such that \( u \) is increasing on \([c, a]\), decreasing on \([b, d]\) and \( u(x) = 1 \) for each \( x \in [a, b] \).

### 1.4. Interval arithmetic

Interval Arithmetic found applications in various fields during the past centuries. In the 3\(^{rd}\) century B.C, Archimedes calculated the lower and upper bounds \( \frac{223}{71} < \pi < \frac{22}{7} \). Although the actual calculations using intervals have not been very popular, they have never been completely ignored. Rosalind Cicely Young, a doctoral candidate at the university of Cambridge published the rules for calculating with intervals and other subsets of the real numbers in 1931. Modern interval arithmetic is considered to have
been born by the appearance of the book Interval Analysis by Raman E. Moore in 1966. An year later, an article about computer interval arithmetic was published by the same author. Since 1996, the journal Reliable Computing is being published with its focus on computer aided computations. Robust control theory and the estimation of pre images of parameterized functions form the topics for scientific work in recent years. Calculation of upper and lower end points for the range of values of a function in one or more variables is performed in a simple way using interval arithmetic.

Simple arithmetic operations are

**Addition.** \([x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2]\).

**Subtraction.** \([x_1, x_2] - [y_1, y_2] = [x_1 - y_2, x_2 - y_1]\).

**Multiplication**
\([x_1, x_2][y_1, y_2] = \left[ \min(x_1y_1, x_1y_2, x_2y_1, x_2y_2), \max(x_1y_1, x_1y_2, x_2y_1, x_2y_2) \right]\).

**Division**
\([x_1, x_2]/[y_1, y_2] = [x_1, x_2][1/[y_1, y_2]]\) where \(1/[y_1, y_2] = [1/y_2, 1/y_1]\) if \(0 \notin [y_1, y_2]\).

Here each interval operation is reduced to real operations and comparisons.

If \(A, B, C, D\) are intervals and \(A \subseteq B, C \subseteq D\), then \(A \ast C \subseteq B \ast D\) for \(\ast \in \{+, -, \cdot, /\}\). This property is called as the inclusion isotony of interval operations. It is termed as the fundamental principle of interval analysis.