Chapter 6

Application of Fuzzy Integral Equations

Fuzzy Linear integral equations arise frequently in physical problems as a result of the possibility of super-imposing the effects due to several reasons. The most important contribution of the theory of fuzzy integral equations consists in the solution of fuzzy initial and boundary value problems. The fuzzy boundary value problems for equations of elliptic type can be reduced to fuzzy Fredholm integral equations while the study of parabolic and hyperbolic fuzzy differential equations leads to fuzzy Volterra integral equations.

The theory of fuzzy Volterra Integral equations makes it possible to solve an initial value problem for a linear fuzzy ordinary differential equation of an arbitrary order.

Consider a linear fuzzy differential equation of order $n$.

$$P_0(t)\frac{d^n s}{dt^n} + P_1(t)\frac{d^{n-1} s}{dt^{n-1}} + P_2(t)\frac{d^{n-1} s}{dt^{n-2}} + \cdots + P_{n-1}(t)\frac{ds}{dt} + P_n(t)s = F(t) \quad (6.0.1)$$
where $P_0(t), P_1(t), P_2(t), \ldots, P_n(t)$ and $F(t)$ are defined functions and continuous in the closed interval $[a, b]$, $P_0(t) \neq 0$ on $[a, b]$.

The solution of (6.0.1) at a point $t_0 \in [a, b]$ satisfies certain given conditions

$$s(t_0) = [c_0, c_0], \quad s'(t_0) = [c_1, c_1], \quad s''(t_0) = [c_{n-2}, c_{n-2}], \ldots$$

$$\ldots, s^n(t_0) = [c_{n-1}, c_{n-1}], \quad (6.0.2)$$

where the prime denotes differentiation with respect to $t$ and $s \in E^n$. Consider $\frac{d^n s}{dt^n} = \left[ \phi(r, t), \overline{\phi}(r, t) \right]$. By integrating and using the initial conditions we have

$$\frac{d^{n-1} s}{dt^{n-1}} = \int_{t_0}^{t} \left[ \phi(r, \xi) d\xi, \overline{\phi}(r, \xi) d\xi \right] + \left[ c_{n-1}, c_{n-1} \right].$$

Integrating again and using the conditions we get a non-homogeneous Volterra integral equation of second kind (see Appendix).

Conversely if $\left[ \phi(r, t), \overline{\phi}(r, t) \right]$ satisfies the integral equation in the interval $[a, b]$ then the function

$$s(t) = \int_{t_0}^{t} \frac{(t - \xi)^{n-1}}{(n-1)!} \left[ \phi(r, \xi), \overline{\phi}(r, \xi) \right] d\xi$$

$$+ \left[ c_{n-1}, c_{n-1} \right] \frac{(t - t_0)^{n-1}}{(n-1)!} + \cdots + \left[ c_1, c_1 \right] (t - t_0) + c_0$$

satisfies the differential equation (6.0.1) with the initial condition (6.0.2) (see Appendix).
Thus from the solution of a fuzzy Volterra integral equation we get the solution of
the fuzzy initial value problem. The following fuzzy boundary value problem is used in
the theory of determination of an elastic rod supported at two points

Consider a fuzzy boundary value problem from ordinary fuzzy differential equations

\[
\frac{d^2 \varphi}{dt^2} + \lambda P(t) \varphi = q(t) \tag{6.0.3}
\]

where \( \varphi(a) = \varphi(b) = 0. \tag{6.0.4} \)

\( p(t) \) and \( q(t) \) are defined and continuous in the closed interval \([a, b]\).

Consider \( \frac{d^2 \varphi}{dt^2} = F(t) \) where \( F(t) \) is a fuzzy continuous function defined on \([a, b]\) i.e.,

\[
\frac{d^2 \varphi}{dt^2} = \left( F(r, t), F^*(r, t) \right). \tag{6.0.5}
\]

The general solution of (6.0.5) is of the form

\[
\frac{d\varphi}{dt} = \left[ \int_a^t F(r, z) dz, \int_a^t F^*(r, z) dz \right] + \left[ c_1, c_1^* \right] + \left[ c_2, c_2^* \right]. \tag{6.0.6}
\]

\[
\varphi(t) = \left[ \int_a^t \int_a^t F(r, \xi) d\xi, \int_a^t \int_a^t F^*(r, \xi) d\xi \right] + \left[ c_1 t, c_1^* t \right] + \left[ c_2, c_2^* \right].
\]

Since \( \varphi(a) = 0 \Rightarrow (c_1 a, c_1^* a) + (c_2, c_2^*) = 0 \) and

\[
\varphi(b) = 0 \Rightarrow \left[ \int_a^b (b - \xi) F(r, \xi) d\xi, \int_a^b (b - \xi) F^*(r, \xi) d\xi \right] + \left[ c_1 b, c_1^* b \right] + \left[ c_2, c_2^* \right] = 0 \tag{6.0.7}
\]
i.e., \[ [c_1, c_1] = \frac{1}{a-b} \int_a^b (b - \xi) \left[ F(r, \xi), F(r, \xi) \right] d\xi \]

and \[ [c_2, c_2] = \frac{-a}{b-a} \left[ \int_a^b (b - \xi) F(r, \xi) d\xi, \int_a^b (b - \xi) F(r, \xi) d\xi \right] \] \hspace{1cm} (6.0.8)

From relations (6.0.6) and (6.0.8) we have

\[
\varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) d\xi, \int_a^t (t - \xi) F(r, \xi) d\xi \right]
\]

\[
+ \frac{t}{a-b} \left[ \int_a^b (b - \xi) F(r, \xi) d\xi, \int_a^b (b - \xi) F(r, \xi) d\xi \right]
\]

\[
- \frac{a}{a-b} \left[ \int_a^b (b - \xi) F(r, \xi) d\xi, \int_a^b (b - \xi) F(r, \xi) d\xi \right]
\]

i.e.,

\[
\varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) d\xi, \int_a^t (t - \xi) F(r, \xi) d\xi \right]
\]

\[
+ \left[ \int_a^b (b - \xi) F(r, \xi) d\xi + (b - \xi) F(r, \xi) d\xi \right] \left[ \frac{t-a}{a-b} \right].
\]

i.e.,

\[
\varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) d\xi, \int_a^t (t - \xi) F(r, \xi) d\xi \right]
\]

\[
+ \left[ \int_a^b (b - \xi) F(r, \xi) d\xi, (b - \xi) F(r, \xi) d\xi \right] \left[ \frac{t-a}{a-b} \right].
\]

i.e.,

\[
\varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi), \int_a^t (t - \xi) F(r, \xi) d\xi \right]
\]

\[
+ \left[ \int_a^b \frac{t-a}{a-b} (b - \xi) F(r, \xi) d\xi, \frac{b}{a-b} (b - \xi) F(r, \xi) d\xi \right]
\]
i.e.,

\[
\varphi(t) = \left[ \int_a^t \frac{(t-b)(\xi-a)}{b-a} F(r, \xi) d\xi \right] F(r, \xi) + \left[ \int_t^b \frac{(t-a)(\xi-a)}{b-a} F(r, \xi) d\xi \right] F(r, \xi)
\]

or

\[
\varphi(t) = \int_a^b G(t, \xi) \left[ F(r, \xi), F(r, \xi) \right] d\xi.
\]

Where \(G(t, \xi)\) is known as the Green’s function and is defined by the relations

\[
G(t, \xi) = \begin{cases} 
\frac{(t-a)(\xi-b)}{(b-a)} & \text{if } t \leq \xi \\
\frac{(t-b)(\xi-a)}{b-a} & \text{if } t \geq \xi.
\end{cases}
\]

(6.0.9)

Thus the fuzzy function \(\varphi(t)\) which is a solution of the fuzzy differential equation (6.0.3) with the boundary condition (6.0.4) satisfies the equation

\[
\varphi(t) = \lambda \int_a^b G(t, \xi) p(\xi) \varphi(\xi) d\xi + \int_a^b G(t, \xi) q(\xi) d\xi
\]

(6.0.10)

which is a fuzzy Fredholm integral equation.

The function \(p(t)\) and \(q(t)\) one known and continuous in the interval \([a, b]\), \(\lambda\) is a non zero numerical parameter.

Conversely the solution \(\varphi(t)\) determined by the relation (6.0.10) satisfies the given differential equation (6.0.3) with the prescribed boundary conditions (6.0.4)
6.1. Longitudinal vibrations of a rod

The appropriate fuzzy differential equation for modeling the above problem is

\[ \frac{d^2 \varphi}{dt^2} = F(t) \quad (6.1.1) \]

where \( F(t) \) is a known continuous fuzzy function satisfying the boundary condition \( \varphi(a) = 0, \)

\[ \left[ \frac{d\varphi}{dt} \right]_{t=b} = 0. \quad (6.1.2) \]

The general solution of the equation (6.1.1) is of the form

\[ \frac{d\varphi}{dt} = \left[ \int_a^t F(r, \xi) d\xi, \int_a^r F(r, \xi) d\xi \right] + \left[ c_1, c_1 \right] \]

\[ \varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi), \int_a^r (t - \xi) F(r, \xi) \right] d\xi + \left[ c_1 t, c_1 t \right] + \left[ c_2, c_2 \right] \quad (6.1.3) \]

using the boundary conditions (6.1.2), the equation (6.1.3) becomes

\[ \left[ c_1 a, c_1 a \right] + \left[ c_2, c_2 \right] = 0 \]

\[ \left[ \int_a^b F(r, \xi) d\xi, \int_a^b F(r, \xi) d\xi \right] + \left[ c_1, c_1 \right] = 0 \quad (6.1.4) \]

\[ \Rightarrow \left[ c_1, c_1 \right] = \left[ a \int_a^b F(r, \xi) d\xi, a \int_a^b F(r, \xi) d\xi \right]. \quad (6.1.5) \]

Substituting the value of \( c_1 \) and \( c_2 \) in the relation (6.1.3), we have
\[ \varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) \, d\xi, \int_a^t (t - \xi) \overline{F}(r, \xi) \, d\xi \right] - t \left[ \int_a^b F(r, \xi) d\xi, \int_a^b \overline{F}(r, \xi) d\xi \right] \\
+ a \left[ \int_a^b F(r, \xi) d\xi, \int_a^b \overline{F}(r, \xi) d\xi \right] \\
or \varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) \, d\xi, \int_a^t (t - \xi) \overline{F}(r, \xi) \, d\xi \right] \\
+ \left[ \int_a^b (a - t) F(r, \xi) d\xi, \int_a^b (a - t) \overline{F}(r, \xi) d\xi \right] \\
or \varphi(t) = \left[ \int_a^t (t - \xi) F(r, \xi) d\xi, \int_a^t (t - \xi) \overline{F}(r, \xi) d\xi \right] \\
+ \left[ \int_a^t (a - t) F(r, \xi) d\xi, \int_a^t (a - t) \overline{F}(r, \xi) d\xi \right] \\
+ \left[ \int_a^b (a - t) F(r, \xi) d\xi, \int_a^b (a - t) \overline{F}(r, \xi) d\xi \right]. \\
\]

i.e., \[ \varphi(t) = \left[ \int_a^t (a - \xi) F(r, \xi) d\xi, \int_a^t (a - \xi) \overline{F}(r, \xi) d\xi \right] \\
+ \left[ \int_a^b (a - t) F(r, \xi) d\xi, \int_a^b (a - t) \overline{F}(r, \xi) d\xi \right] \\
or \varphi(t) = \int_a^b G(t, \xi) \left[ f(r, \xi), \overline{f(r, \xi)} \right] \, d\xi \\
\]

where

\[ G(t, z) = \begin{cases} 
    a - t & t \leq \xi \\
    a - \xi & t \geq \xi.
\end{cases} \quad (6.1.6) \]

This shows that the function \( G(t, \xi) \) is symmetric. Thus the solution \( \varphi(t) \) which satisfies

the boundary conditions (6.1.2) is a fuzzy solution of the fuzzy integral equation

\[ \varphi(t) = \lambda \int_a^b G(t, \xi) p(y) \varphi(\xi) d\xi + \int_a^b G(t, \xi) q(\xi) d\xi \]

and conversely.
The approach followed in this thesis is useful for obtaining a mathematical evaluation of situation involving fuzziness. Since the interval arithmetic is being widely used in computer applications this approach can further widen the area also.