Chapter 4

Further Applications of Banach Fixed point Theorem for Fuzzy sets

Existence theorems for Volterra Integral equations have wide applications to predator – prey models and in medical diagnosis. Here such existence theorems have been generalized to fuzzy valued mappings. In this chapter the conditions needed for the existence of a unique solution in different models of Volterra Integral equations are studied using Banach fixed point theorem for fuzzy sets.

4.1. Existence and Uniqueness of solution for some Fuzzy Volterra Integral equations

Here we find some fuzzy Volterra Integral equations and conditions by which it possess unique solution.
Theorem 4.1.1. Consider the fuzzy valued Volterra Integral equation

\[ u(t) = f(t) + \int_{t_0}^{t} g(t, s, u(s))ds. \]  \hspace{1cm} (4.1.1)

Here \( f \) and \( g \) are continuous fuzzy functions from \([t_0, t_1] \to E^n\) and \( g \) satisfies Lipschitz condition on the 3rd parameter. That is \( D_2 [g(t, s, u_1), g(t, s, u_2)] \leq LD_2(u_1, u_2) \) where \( L \) is the Lipschitz constant. Then the integral equation has a unique solution.

**Proof.** Let the parametric representation of the fuzzy function

\[ u_j(s) = [a_j(r, s), b_j(r, s)] \quad \text{and} \]

\[ g(t, s, u_j(s)) = [g_{a_j}(r, t, s), g_{b_j}(r, t, s)] \]

where \( a_j(r, s) = (a_{j1}(r, s), a_{j2}(r, s), \ldots, a_{jn}(r, s)) \in R^n \) and \( j = 1, 2 \) also

\[ |g_{a1}^i(r, t, s) - g_{a2}^i(r, t, s)| \leq L |a_{1}^i(r, s) - a_{2}^i(r, s)| \]

\[ |g_{b1}^i(r, t, s) - g_{b2}^i(r, t, s)| \leq L |b_{1}^i(r, s) - b_{2}^i(r, s)| \]

and \( S^n = \{ f/f : [t_0, t_1] \to E^n \text{ is a continuous} \} \)

and let \( M : S^n \to S^n \) by \( Mu(t) = f(t) + \int_{t_0}^{t} g(t, s, u(s))ds \)

thus \( Mu_j(t) = [c(r, t), d(r, t)] + \int_{t_0}^{t} (g_{a_j}(r, t, s), g_{b_j}(r, t, s)) ds. \)
$$D_2(u_1, u_2) = \left[ \inf_t \min_r \min_i \left\{ |a_1^i(r, s) - a_2^i(r, s)|, |b_1^i(r, s) - b_2^i(r, s)| \right\},
\sup_t \max_r \max_i \left\{ |b_1^i(r, s) - b_2^i(r, s)|, |a_1^i(r, s) - a_2^i(r, s)| \right\} \right]. \quad (4.1.2)$$

Now consider

$$D_2(Mu_1, Mu_2)$$

$$= D_2 \left\{ [c(r, t), d(r, t)] + \int_{t_0}^t [g_{a_1}(r, t, s), g_{u_1}(r, t, s)] ds \right\},
\left\{ [c(r, t), d(r, t)] + \int_{t_0}^t [g_{a_2}(r, t, s), g_{u_2}(r, t, s)] ds \right\}$$

$$= D_2 \left\{ \int_{t_0}^t [g_{a_1}(r, t, s), g_{u_1}(r, t, s)], \int_{t_0}^t [g_{a_2}(r, t, s), g_{u_2}(r, t, s)] \right\}
\leq \int_{t_0}^t D_2 \left\{ [g_{a_1}(r, t, s), g_{u_1}(r, t, s)], [g_{a_2}(r, t, s), g_{u_2}(r, t, s)] \right\} ds$$

$$\leq \int_{t_0}^t \left[ \inf_t \min_r \min_i \left\{ |g_{a_1}^i(r, t, s) - g_{a_2}^i(r, t, s)|, |g_{u_1}^i(r, t, s) - g_{u_2}^i(r, t, s)| \right\},
\sup_t \max_r \max_i \left\{ |g_{a_1}^i(r, t, s) - g_{a_2}^i(r, t, s)|, |g_{u_1}^i(r, t, s) - g_{u_2}^i(r, t, s)| \right\} \right] ds$$

$$\leq \int_{t_0}^t \left[ \inf_t \min_r \min_i \left\{ L|a_1^i(r, s) - a_2^i(r, s)|, |b_1^i(r, s) - b_2^i(r, s)| \right\},
\sup_t \max_r \max_i \left\{ L|a_1^i(r, s) - a_2^i(r, s)|, |b_1^i(r, s) - b_2^i(r, s)| \right\} \right] ds$$

$$\leq L \left[ \inf_t \min_r \min_{i=1,2,...,n} \left\{ |a_1^i(r, s) - a_2^i(r, s)|, |b_1^i(r, s) - b_2^i(r, s)| \right\},
\sup_t \max_r \max_{i=1,2,...,n} \left\{ |a_1^i(r, s) - a_2^i(r, s)|, |b_1^i(r, s) - b_2^i(r, s)| \right\} \right] ds$$

$$\leq L(t - t_0) D_2(u_1, u_2). \quad (4.1.3)$$
Now we prove
\[ D_2(M^n u_1, M^n u_2) \leq \frac{L^n(t - t_0)^n}{n!} D_2(u_1, u_2) \] (4.1.4)
with mathematical induction.

For that we prove (4.1.5) for \( n + 1 \) by assuming it is true for \( n \).

Consider
\[
D_2(M^{n+1} u_1, M^{n+1} u_2)
\]
\[
= D_2 \left\{ \int_{t_0}^{t} M^n(g_{a_1}(r,t,s), g_{a_2}(r,t,s)), M^n(g_{b_1}(r,t,s), g_{b_2}(r,t,s)) \right\} ds
\]
\[
\leq L \int_{t_0}^{t} D_2(M^n u_1, M^n u_2) \, dt
\]
\[
\leq L \int_{t_0}^{t} \frac{L^n(t - t_0)^n}{n!} D_2(u_1, u_2) \, dt
\]
\[
= L^{n+1} \frac{D_2(u_1, u_2)(t - t_0)^{n+1}}{(n + 1)!}.
\]

Since \( \lim_{n \to \infty} \frac{(t - t_0)^n}{n!} = 0 \), there exist a natural number \( N \) such that for \( n = N \), \( h = \frac{(t - t_0)^N}{N!} < 1 \). This means that

\[ D_2(M^N u_1, M^N u_2) \leq h D_2(u_1, u_2) \text{ where } u_1, u_2 \in S^n, 0 < h < 1. \] (4.1.5)

From that \( S^n \) is a complete space and the mapping \( M \) is a contraction mapping on \( S^n \) hence by Banach fixed point theorem \( M \) has and only has a fixed point in \( S^n \) such that \( Mu_1(s) = u_1(s), s \in [t_0, t_1] \).
Thus
\[ u(t) = f(t) + \int_{t_0}^{t} g(t, s, u(s))ds. \]

Hence the proof. □

**Theorem 4.1.2.** Consider the fuzzy Volterra Integral equation
\[ u(t) = f(t) + \int_{t_0}^{t} K(t, s)g(s, u(s))ds, \tag{4.1.6} \]

(1) \( f : [t_0, t_1] \to E^n \) is continuous and bounded
(2) \( K : \Delta \to R \) is continuous where
\[ \Delta = \{(t, s) : t_0 \leq s \leq t \leq t_1 \} \]
and there exists \( M > 0 \) such that \( \int_{t_0}^{t} |K(t, s)|ds \leq M. \)
(3) \( g : [t_0, t_1] \times S^n \to S^n \) is continuous and satisfies the Lipschitz condition with respect to the second parameter. That is
\[ D_2[g(t, u_1(t)), g(t, u_2(t))] \leq LD_2[u_1(t), u_2(t)]. \]
\( t_0 \leq t \leq t_1, \) where \( L < M^{-1} \) and \( u_1, u_2 : [t_0, t_1] \to E^n. \)

Thus (4.1.6) has a unique solution provided the integral \( \int_{t_0}^{t} K(t, s)g(s, u(s))ds \) exist.

**Proof.** Let \( S^n = \{ f | f \) is a continuous fuzzy function on \([t_0, t_1]. \)

(1) Suppose that
\[ u_1(t) = (a_1(r, t), b_1(r, t)) \in S^n. \]
\[ u_2(t) = (a_2(r, t), b_2(r, t)) \in S^n. \]
\[ g(t, u_i) = (g_{a_i}(r, s), g_{b_i}(r, s)), \quad \text{where} \]
\[ g_{a_i}(r, s) = (g^1_{a_i}, g^2_{a_i}, \ldots, g^n_{a_i}) \in \mathbb{R}^n. \]

Consider the fuzzy Volterra Integral equation (4.1.6)
\[ u(t) = f(t) + \int_{t_0}^t K(t, s) g(s, u(s)) ds, \quad t \geq 0. \]

Let mapping \( M : S^m_{[t_0, t_1]} \rightarrow S^n_{[t_0, t_1]} \) such that for any \( u(t) \in S^n_{[t_0, t_1]} \).
\[ Mu(t) = f(t) + \int_{t_0}^t K(t, s) g(s, u(s)) ds. \]

For any \( u_1, u_2 \in S^m_{[t_0, t_1]} \) and any \( t \in [t_0, t_1] \).
\[ D(Mu_1, Mu_2) \]
\[ = D_2 \left( f(t) + \int_{t_0}^t K(t, s) g(s, u_1(s)) ds, f(t) + \int_{t_0}^t k(t, s) g(s, u_2(s)) ds \right) \]
\[ = D_2 \left( \int_{t_0}^t k(t, s) g(s, u_1(s)) ds, \int_{t_0}^t k(t, s) g(s, u_2(s)) ds \right) \]
\[ = \left[ \inf \min_{t} \min_{r} \min_{i} \left\{ \left| \int_{t_0}^t k(t, s) \left[ g_{a_i}^1(r, s) - g_{a_2}^i(r, s) \right] ds \right|, \right. \right. \]
\[ \left. \left. \left| \int_{t_0}^t k(t, s) \left[ g_{b_i}^1(r, s) - g_{b_2}^i(r, s) \right] ds \right| \right\} \right] \]
\[ \sup \max_{t} \max_{r} \max_{i} \left\{ \left| \int_{t_0}^t k(t, s) \left[ g_{a_i}^1(r, s) - g_{a_2}^i(r, s) \right] ds \right|, \right. \right. \]
\[ \left. \left. \left| \int_{t_0}^t k(t, s) \left[ g_{b_i}^1(r, s) - g_{b_2}^i(r, s) \right] ds \right| \right\} \right] \]
\[\begin{align*}
&\leq \left[ \inf_t \min_r \min_i \int_{t_0}^t |k(t, s)||g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|ds, \int_{t_0}^t |k(t, s)||g_{b_1}^i (r, s) - g_{b_2}^i (r, s)|ds, \\
&\quad \sup_t \max_r \max_i \int_{t_0}^t |k(t, s)||g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|ds, \int_{t_0}^t |k(t, s)||g_{b_1}^i (r, s) - g_{b_2}^i (r, s)|ds \right] \\
&\leq \left[ \inf_t \min_r \min_i \left\{ |g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|, |g_{b_1}^i (r, s) - g_{b_2}^i (r, s)| \right\} \int_{t_0}^t k(t, s)ds, \\
&\quad \sup_t \max_r \max_i \left\{ |g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|, |g_{b_1}^i (r, s) - g_{b_2}^i (r, s)| \right\} \int_{t_0}^t k(t, s)ds \right] \\
&\leq M \left[ \inf_t \min_r \min_i \left\{ |g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|, |g_{b_1}^i (r, s) - g_{b_2}^i (r, s)| \right\} \\
&\quad \sup_t \max_r \max_i \left\{ |g_{a_1}^i (r, s) - g_{a_2}^i (r, s)|, |g_{b_1}^i (r, s) - g_{b_2}^i (r, s)| \right\} \right] \\
&\leq MD_2 (g(t, u_1(t)), g(t, u_2(t))) \\
&= MLD_2(u_1, u_2).
\end{align*}\]

Since \(LM < 1\), and using Banach fixed point theorem the mapping \(M\) has and only has a fixed point \(u(t)\). i.e.,

\[u(t) = f(t) + \int_{t_0}^t K(t, s)g(s, u(s))ds\]

which is the unique solution of the integral equation (4.1.6). \(\square\)

### 4.2. Initial value problem of Fuzzy Differential Equations (FDE)

In this section we study the initial value of fuzzy Differential equation

\[
\frac{d}{ds}u(s) = K(t, s)g(s, u(s)), u(t_0) = m, \quad \text{where} \quad u(s) \in E^n
\] (4.2.1)
is an unknown continuous fuzzy function. The mapping $g : [t_0, t_1] \times E^n \to E^n$ is a continuous function.

**Theorem 4.2.1.** Consider the initial value problem (4.2.1) of fuzzy differential equation, where $u(t_0) = m$ is a fuzzy number and

1. the integral $\int_{t_0}^t g(s, u(s))ds$, exists for any fuzzy function $u(s)$.
2. $g$ satisfies the Lipschitz condition with respect to the 2nd variable. i.e.,
   $$D_2 ([g(s, u_1(s)), g(s, u_2(s))]) \leq LD_2 [u_1(s), u_2(s)].$$
3. $K(t, s)$ is a real continuous function on $\Delta = \{(t, s)|t_0 \leq t \leq t_1, t_0 \leq s \leq t\}$ and there is a constant $M > 0$ such that $0 \leq K(t, s) \leq M$ and $M < (t_1 - t_0)^{-1}$. Then the initial value problem (4.2.1) has a unique continuous solution on $[t_0, t_1]$.

**Proof.** The initial value problem is equivalent to the integral equation $u(t) = m + \int_{t_0}^t K(t, s)g(s, u(s))ds$. Let mapping $M : S^n[t_0, t_1] \to S^n[t_0, t_1]$ be such that for any $u(s) \in S^n_{[t_0, t_1]}$, $Mu(t) = m + \int_{t_0}^t K(t, s)g(s, u(s))ds$.

For any $u_1, u_2 \in S^n_{[t_0, t_1]}$, let its parametric representation be

$$u_1(s) = (a_1(r, s), b_1(r, s))$$

$$u_2(s) = (a_2(r, s), b_2(r, s)), \ s \in [t_0, t_1] \quad \text{and}$$

$$g(s, x_j) = (g_{a_j}(r, s), g_{b_j}(r, s)), \ j = 1, 2.$$
\begin{align*}
D_2 (M_{u_1}, M_{u_2}) &= \left[ \inf \min_r \min_i \left\{ \left| \int_{t_0}^t k(t, s) \left( g^i_{a_1}(r, s) - g^i_{a_2}(r, s) \right) \, ds \right|, \right. \right. \\
& \quad \left. \left. \left| \int_{t_0}^t k(t, s) \left( g^i_{b_1}(r, s) - g^i_{b_2}(r, s) \right) \, ds \right| \right\}, \right. \\
& \quad \left. \sup \max_r \max_i \left\{ \left| \int_{t_0}^t k(t, s) \left( g^i_{a_1}(r, s) - g^i_{a_2}(r, s) \right) \, ds \right|, \right. \right. \\
& \quad \left. \left. \left| \int_{t_0}^t k(t, s) \left( g^i_{b_1}(r, s) - g^i_{b_2}(r, s) \right) \, ds \right| \right\} \right], \\
& \leq \left[ \inf \min_r \min_i \left\{ \left| \int_{t_0}^t k(t, s) \left( g^i_{a_1}(r, s) - g^i_{a_2}(r, s) \right) \, ds \right|, \right. \right. \\
& \quad \left. \left. \left| \int_{t_0}^t k(t, s) \left( g^i_{b_1}(r, s) - g^i_{b_2}(r, s) \right) \, ds \right| \right\} \right], \\
& \leq ML(t - t_0)D_2(u_1, u_2) \\
& \leq ML(t_1 - t_0)D_2(u_1, u_2) \\
& \leq \alpha D_2(u_1, u_2)
\end{align*}

where \( \alpha = ML(t_1 - t_0) < 1 \).

Then by Banach’s theorem the initial value problem (4.2.1) has a unique continuous solution on \([t_0, t_1]\) which is the unique fixed point of \( M \). i.e., \( Mu(t) = u(t) \).

Thus \( u(t) = m + \int_{t_0}^t k(t, s)g(s, u(s))ds \). \hfill \end{prooft}

\textbf{Theorem 4.2.2.} Initial value problems of second order linear fuzzy differential equations \( u''(t) = f(t, u(t), u'(t)) \) \( u(t_0) = K_1, u'(t_0) = K_2, t_0 \in [t_0, t_1] \subset [a, b] \) has a
unique solution on \([a, b]\) if \(f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a continuous map and for \(K > 0\),

\[
D_2 (f(t, u(t), u'(t)), f(t, v(t), v'(t))) \leq KD(u(t), v(t))
\]

\[
\forall t \in [t_0, t_1], u, u', v, v' : [t_0, t_1] \rightarrow \mathbb{R}^n.
\]

**Proof.** The above initial value problem can be converted into a fuzzy Volterra integral equation [see Appendix].

\[
u(t) = \int_{t_0}^{t} K(t, s, u(s))ds + g(t)
\]

where \(K : [a, b] \times [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(g(t) = K_2(t - t_0) + K_1\)

and

\[
K(t, s, u(s)) = \int_{t_0}^{t} f(s, u(s), u'(s))ds
\]

where \(u(s)\) and \(u'(s)\) are fuzzy numbers. \(f(s, u(s), u'(s))\) is a fuzzy number. So \(K(t, s, u(s))\) is also a fuzzy number.

Suppose that the fuzzy integral

\[
\int_{t_0}^{t} \int_{t_0}^{t} f(s, u(s), u'(s))dsds
\]

exists. For any
\[ u(s) = [a_1(r, s), b_1(r, s)] \]
\[ v(s) = [a_2(r, s), b_2(r, s)]. \]

where \( a_j(r, s), b_j(r, s) \in \mathbb{R}^n, j = 1, 2. \)

So \( a_j(r, s) = (a_{1j}(r, s), a_{2j}(r, s), \ldots, a_{nj}(r, s)). \)

\[
D_2(f(t, u(t), u'(t)), f(t, v(t), v'(t)) \leq KD_2(u(t), v(t)), \quad K > 0.
\]

Where \( f(t, u(t), u'(t)) = [f_{a_1}(r, t, s), f_{b_1}(r, t, s)] \)
\[
f(t, v(t), v'(t)) = [f_{a_2}(r, t, s), f_{b_2}(r, t, s)]
\]

Let \( M : S^n_{[t_0, t_1]} \rightarrow S^n_{[t_0, t_1]} \) such that for any \( t \in [t_0, t_1] \)

\[
u(t) \in S^n_{[t_0, t_1]}
\]

\[
Mu(t) = g(t) + \int_{t_0}^{t} \int_{t_0}^{t} f(s, u(s), u'(s))ds.
\]

For any \( u_1, v_1 \in S^n_{[t_0, t_1]} \) and any \( t \in [t_0, t_1] \).

\[
D_2(Mu, Mv) = D_2 \left( \int_{t_0}^{t} \int_{t_0}^{t} f(s, u(s), u'(s))ds ds, \int_{t_0}^{t} \int_{t_0}^{t} f(s, v(s), v'(s))ds ds \right)
\]
\[
\leq \int_{t_0}^{t} \int_{t_0}^{t} D_2(f(s, u(s), u'(s)), f(s, v(s), v'(s)))ds ds
\]
\[
\leq \int_{t_0}^{t} \int_{t_0}^{t} K D_2(u(s), v(s))
\]

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\[ \leq K(D_2(u(s), v(s))) \int_{t_0}^{t} \int_{t_0}^{s} ds \, ds \]

\[ \leq K(t_1 - t_0)^2 D_2(u, v). \]

If \( K \) is in such a way that \( K(t_1 - t_0)^2 < 1 \), then \( D_2(Mu, Mv) \leq h D_2(u, v) \). Where \( u, v \in S^n \), \( 0 < h < 1 \). Since \( S^n \) is a complete space. We know that the mapping \( M \) is a contraction mapping. So \( M \) has and only has a fixed point in \( S^n \), such that

\[ Mu(t) = u(t) \quad \forall t \in [t_0, t_1]. \]

So

\[ u(t) = g(t) + \int_{t_0}^{t} \int_{t_0}^{s} f(s, u(s), u'(s)) ds \, ds. \]

\( u(t) \) is the unique continuous solution of the initial value problem. \( \square \)

The above theorem is a modified form of Georgiou’s result in [16]. There \([t_0, t_1]\) is considered as the union of finite family of intervals with length of each interval less than \( n \).

### 4.3. Bounded solutions for Fuzzy Integral Equations

We are following the approach of D. N. Georgious and I. E. Kougias did in their paper [15]. Here the conditions under which some singular integral equations possesses only bounded solutions is examined.
**Definition 4.3.1.** A mapping \( u : T \to E^n \) is bounded, where \( T \) is an interval of the real line, if there exists an element \( r > 0 \) such that \( D_2(u(t), \hat{0}) \leq [0, r], \forall t \in T \).

**Theorem 4.3.2.** Suppose that \( f : [0, \infty) \to E^n \) with \( D(f(t), \hat{0}) \leq [0, M] \) and \( G : \Delta \to R \) is continuous, where \( \Delta = \{(t, s) : 0 \leq s \leq t \leq \infty\} \). If there exist \( m < 1 \) with \( \int_0^t |G(t, s)| \, ds \leq m \) for \( t \in [0, \infty] \), then all solutions of the fuzzy integral equation

\[
u(t) = \lambda \int_0^t G(t, s)u(s) \, ds + f(t)
\]

are bounded where \( |\lambda| < \frac{1}{m} \).

**Proof.** Let

\[Tu(t) = f(t) + \lambda \int_0^t G(t, s)u(s) \, ds, \]

where \( u_i(t) = [a_i(r, t), b_i(r, t)]\).

\[Tu(t) = \left( \left\{ f_a(r, t) + \lambda \int_0^t G(t, s)a_i(r, t) \, ds \right\}, \left\{ f_b(r, t) + \lambda \int_0^t G(t, s)b_i(r, t) \, ds \right\} \right).\]

Since

\[
D_2(u_1, u_2) = \left[ \inf_t \min_r \min_{i=1,2,...,n} \left\{ \left| a^i_1(r, t) - a^i_2(r, t) \right|, \left| b^i_1(r, t) - b^i_2(r, t) \right| \right\} , \sup_t \max_r \max_{i=1,2,...,n} \left\{ \left| a^i_1(r, t) - a^i_2(r, t) \right|, \left| b^i_1(r, t) - b^i_2(r, t) \right| \right\} \right],
\]

We have
\[ D_2(Tu_1, Tu_2) \]
\[ = \left[ \inf_t \min_r \min_i \left\{ \left| \lambda \int_0^t G(t, s)a_i^1(r, t)dt - \lambda \int_0^t G(t, s)a_i^2(r, t)dt \right|, \right. \right. \]
\[ \left. \left. \left| \lambda \int_0^t G(t, s)b_i^1(r, t)dt - \lambda \int_0^t G(t, s)b_i^2(r, t)dt \right| \right\}, \right. \]
\[ \sup_{t \in ]0, \infty[} \max_r \max_i \left\{ \left| \lambda \int_0^t G(t, s)a_i^1(r, t)dt - \lambda \int_0^t G(t, s)a_i^2(r, t)dt \right|, \right. \right. \]
\[ \left. \left. \left| \lambda \int_0^t G(t, s)b_i^1(r, t)dt - \lambda \int_0^t G(t, s)b_i^2(r, t)dt \right| \right\} \right]\]
\[ \leq \left[ \inf_t \min_r \min_i \left\{ \left| a_i^1(r, t) - a_i^2(r, t) \right|, \left| b_i^1(r, t) - b_i^2(r, t) \right| \right\} \lambda \int_0^t |G(t, s)|ds, \right. \]
\[ \sup_{t \in ]0, \infty[} \max_r \max_i \left\{ \left| a_i^1(r, t) - a_i^2(r, t) \right|, \left| b_i^1(r, t) - b_i^2(r, t) \right| \right\} \lambda \int_0^t |G(t, s)|ds \]
\[ \leq \lambda_m D_2(u_1, u_2). \]

Since \( \lambda_m < 1 \), \( T \) is a contraction map. This shows the existence of a unique solution which is the fixed point of \( T \). This solution is also bounded. To show the boundedness of the solution, let \( u(t) \) be an unbounded solution of the integral equation. Then for every \( r > 0 \), there exist an element \( t_1 \in (0, \infty) \) such that \( D_2(u(s), \hat{0}) < [0, r] \) for every \( s \in [0, t_1] \), and \( D_2(u(t_1), \hat{0}) = [0, r] \). Clearly we can find a positive number \( r \) with \( M + \lambda mr < r \). We have

\[ [0, r] = D_2(u(t_1), \hat{0}) = D_2 \left( \lambda \int_0^{t_1} G(t_1, s)u(s)ds + f(t), \hat{0} \right) \]
\[ \leq D_2 \left( \lambda \int_0^{t_1} G(t_1, s)u(s)ds, \hat{0} \right) + D_2 \left( f(t_1), \hat{0} \right) \]
≤ \lambda \int_0^{t_1} D_2(G(t_1, s)u(s), \hat{0})ds + [0, M]
≤ |\lambda| \int_0^t |G(t_1, s)|D_2(u(s), \hat{0}) + [0, M]
≤ |\lambda|m[0, r] + [0, M]
= [0, |\lambda|mr] + [0, M] = [0, M + \lambda mr] < [0, r]

which is a contradiction, shows that \( u(t) \) is a bounded solution for the integral equation. \( \square \)

**Theorem 4.3.3.** Suppose that \( f : [0, \infty) \rightarrow E^n \) and \( K : [0, \infty) \rightarrow \mathbb{R} \) are continuous and that there exists constants \( A, B \) such that \( A > 0 \) with \( 0 < B < 1 \) and

\[ D_2(f(t), \hat{0}) < \left[ 0, \frac{A}{p(t)} \right], \quad p(t) \in C(R^+, R^+) \]

and

\[ \int_0^t |K(t - s)|ds \leq \frac{B}{p(t)}. \]

Then every solution of fuzzy integral equation

\[ u(t) = \int_0^t K(t - s)u(s)ds + f(t) \quad (4.3.1) \]

is bounded.

**Proof.** Let \( f(t) = (c^i(r, t), d^i(r, t)) \in E^n, i = 1, 2, \cdots n \) and \( u(t) = (a^i(r, t), b^i(r, t)) \) be an unbounded solution of equation (4.3.1) then for every \( r > 0 \), \( \exists \) an element
$t_1 \in (0, \infty)$ such that $D_2(u(s), \hat{0}) < [0, r]$ for every $s \in [0, t_1)$ and

$$D_2(u(t_1), \hat{0}) = [0, r].$$ \hfill (4.3.2)

By considering $Tu(t) = f(t) + \int_0^t K(t - s)u(s)ds$. We can show that $T$ is a contraction mapping and hence there exist a fixed point for $T$ which is the solution of equation (4.3.1) and it is bounded also, since we can find a positive number $r$ with $A + Br < r$. For proving it consider

$$[0, r] = D_2u(t_1, \hat{0}).$$

$$= D_2 \left( \int_0^{t_1} K(t_1 - s)u(s)ds + f(t_1), \hat{0} \right)$$

$$\leq D_2 \left( \int_0^{t_1} K(t_1 - s)u(s)ds, \hat{0} \right) + D_2(f(t_1), \hat{0})$$

$$\leq \int_0^{t_1} D_2 \left( K(t_1 - s)u(s), \hat{0} \right) ds + \left[ 0, \frac{A}{p(t_1)} \right]$$

$$\leq \int_0^{t_1} |K(t_1 - s)|D_2(u(s), \hat{0}) ds + \left[ 0, \frac{A}{p(t_1)} \right]$$

$$\leq \frac{B[0, r]}{p(t_1)} + \left[ 0, \frac{A}{p(t_1)} \right]$$

$$= \left[ 0, \frac{Br}{p(t_1)} \right] + \left[ 0, \frac{A}{p(t_1)} \right]$$

$$= \left[ 0, \frac{A + Br}{p(t_1)} \right] < [0, r]$$

and thus $p(t_1)r < A + Br < r$ which is a contradiction. Thus $u(t)$ is bounded. \hfill $\Box$
Theorem 4.3.4. Consider the initial value problem of first order fuzzy differential equation \( u^1 = f(t, u), \ u(0) = u_0, \ t \in [0, \infty) \) has a unique bounded solution on \([0, \infty)\) if \( f : [0, \infty) \times E^n \to E^n \) is a continuous map and \( \int_0^t f(s, u(s))ds \leq mu(s) \) where \( 0 < m < 1 \).

**Proof.** The given differential equation is equivalent to the Volterra Integral equation \( Tu(t) = u(0) + \lambda \int_0^t f(s, u(s))ds. \)

Let \( Tu(t) = u(0) + \lambda \int_0^t f(s, u(s))ds. \) Then we can show that \( T \) is a contraction mapping hence it has a fixed point. Thus the integral equation has a unique solution. Now we can show that it is bounded. For, assume that \( u(t) \) is unbounded then for every \( r > 0 \), there exist \( t_1 \in [0, \infty) \) such that \( D_2(u(s), \hat{0}) < [0, r] \) and \( D_2(u(t_1), \hat{0}) = [0, r] \).

Then

\[
\begin{align*}
[0, r] &= D_2(u(t_1), \hat{0}) = D_2 \left( u(0) + \int_0^{t_1} f(s, u(s))ds, \hat{0} \right) \\
&\leq D_2(u(0), \hat{0}) + \int_0^{t_1} D_2 \left( f(s, u(s)), \hat{0} \right) ds. \\
&\leq [0, r] + mD_2(u(s), \hat{0}) \\
&< [0, r] + m[0, r] \\
&< [0, r + mr], \quad 0 < m < 1.
\end{align*}
\]

So solution of the integral equation is bounded. \( \square \)
Theorem 4.3.5. Suppose that $f : [0, \infty) \times E^n \to E^n$ and $u(t) : [0, \infty) \to E^n$ are continuous. Then the fuzzy initial value problem $u''(t) = f(t, u(t), u'(t))$, $u(t_0) = K_1$, $u'(t_0) = K_2$, $t \in [0, \infty)$ can be converted into a fuzzy Volterra integral equation $u(t) = \int_{t_0}^{t} K(t, s)u(s)ds + g(t)$ where $K : [t_0, t] \times [t_0, t] \times E^n \to E^n$ be a map such that

$$K(t, s, u(s)) = \int_{t_0}^{t} f(s, u(s), u'(s))ds.$$

has a bounded solution when $D_2(g(t), 0) \leq \left[0, \frac{A}{p(t)}\right]$. 

$$D_2\left(0, \int_{t_0}^{t} K(t, s, u(s))ds\right) \leq \left[\frac{B}{p(t)}\right] D_2(0, u(t))$$

where $A$ and $B$ are constants with $0 < B < 1$ and $p(t) \in c(R^+, R^+)$, $g(t) = K_2(t - t_0) + K_1$.

Proof. The linear fuzzy initial value problem

$$u''(t) = f(t, u(t), u'(t))$$

$u(t_0) = K_1, u'(t_0) = K_2, t \in [0, \infty]$ is equivalent to the Volterra type fuzzy integral equation

$$u(t) = \int_{t_0}^{t} K(t, s, u(s))ds + g(t)$$

where $K : [t_0, t] \times [t_0, t] \times E^n \to E^n$ be a map such that

$$K(t, s, u(s)) = \int_{t_0}^{t} f(s, u(s), u'(s))ds.$$
Then we can show that $u(t)$ is a solution of equation (4.3.3) iff it is a solution of equation (4.3.4). Hence $u(t)$ is a bounded solution for equation (4.3.3) iff it is a bounded solution for (4.3.4). Assume that $u(t)$ is not a bounded solution. Then for every $r > 0$, \( \exists \) an element $t_1 \in (0, \infty)$ such that $D_2(u(s), \hat{0}) \leq [0, r]$ for every $s \in [0, t_1]$ and $D_2(u(t_1), \hat{0}) = [0, r]$.

Clearly we can find a positive number $r$ with $A + Br < r$. We have

\[
\begin{align*}
r &= D_2(u(t_1), \hat{0}) \\
&= D_2 \left( \int_0^{t_1} K(t, s, u(s))ds + g(t_1), \hat{0} \right) \\
&\leq D_2 \left( \int_0^{t_1} K(t, s, u(s)), \hat{0} \right) + D_2(g(t_1), \hat{0}) \\
&\leq \int_0^{t_1} D_2 \left( K(t, s, u(s)), \hat{0} \right) ds + D_2(g(t_1), \hat{0}) \\
&\leq \left[ \frac{B}{p(t_1)} \right] D_2(u(t_1), \hat{0}) + \left[ 0, \frac{A}{p(t_1)} \right] \\
&\leq \left[ \frac{B}{p(t_1)} \right] [0, r] + \left[ 0, \frac{A}{p(t_1)} \right] \\
&\leq \left[ 0, \frac{(A + Br)}{p(t_1)} \right].
\end{align*}
\]

Thus we get $p(t_1)r \leq (A + Br) < r$ which is a contradiction. Thus $u(t)$ is bounded. \( \square \)

**Theorem 4.3.6.** Consider the integral equation

\[
u(t) = f(t) + \int_{t_0}^{t} K(t, s)g(s, u(s))ds
\]

where
(1) \( f : [0, \infty) \to E^n \) is continuous and bounded. i.e., \( D_2(f(t), \hat{0}) \leq [0, M_1] \).

(2) \( K : \Delta \to R \) is continuous, \( \exists M > 0 \) such that \( \int_0^t |K(t, s)|ds \leq M \).

(3) \( g : [t_0, t_1] \times E^n \to E^n \) is continuous and satisfies the Lipchitz condition

\[
D_2[g(t, u_1(t)), g(t, u_2(t))] \leq LD_2[u_1(t), u_2(t)], L < \frac{1}{M}, u_1, u_2 : [t_0, t_1] \to E^n.
\]

**Proof.** Let \( T u(t) = f(t) + \int_0^t K(t, s)g(s, u(s))ds \) and assume that it has an unbounded solution let \( D_2(u(s), \hat{0}) < [0, r] \) and \( D(u(t_1), \hat{0}) = [0, r] \). We can choose \( r \) in such a way that \( M_1 + Mr < r \),

\[
r = D_2(u(t_1), \hat{0}) = D_2 \left( f(t) + \int_0^t K(t, s)g(s, u(s)), \hat{0} \right) \\
\leq D_2 \left( f(t), \hat{0} \right) + D_2 \left( \int_0^t K(t, s)g(s, u(s)), \hat{0} \right) \\
< [0, M_1] + MD_2(u(s), \hat{0}) \\
= [0, M_1] + M[0, r] \\
\leq [0, M_1 + Mr] \\
< [0, r], \text{ a contradiction.}
\]

So bounded solution exists for this integral equation. By proving \( T \) is a contraction mapping and hence it has a fixed point shows the unique solution to the integral equation.

\( \Box \)