CHAPTER 3

Discrete Stable-Linnik Processes and Generalizations

3.1 Introduction

In many applications of probability theory certain specific classes of distributions, usually called “fat tailed” or “heavy tailed” distributions have become very useful. The Stable distributions are probably most popular among the heavy tailed distributions. Stable distributions play a vital role in statistical theory as a natural generalization of normal distribution. A necessary and sufficient condition for a distribution function to be stable is given by Gnedenko and Kolmogorov (1968). Properties of geometric stable distributions are given in Feller (1966). Pillai (1988) introduced and studied generalized geometric stable distributions. Steutel and van Harn (1979) introduced the concept of discrete stability for

Some results included in this chapter form part of the paper Jose and Mariyamma (2012c).

integer-valued random variables. Distributional representations for discrete stable distribution, discrete Linnik distribution and Sibuya distribution are available in Devroye (1993). Christoph and Schreiber (1998a) considered explicit and asymptotic formulae for the tail probabilities of the discrete stable distribution. Semi stable distributions can be viewed as a rich alternative to stable laws in stochastic modeling.

Let W be an $\mathbb{Z}_+$-valued random variable and $\rho \in (0,1)$, then the thinning operator '$*$' is defined as

$$\rho * W = \sum_{i=1}^{W} Y_i, \tag{3.1.1}$$

where Y_i's are i.i.d. Bernoulli random variables with $P(Y_i = 1) = 1 - P(Y_i = 0) = \rho$, that is independent of W. The operation '$*$' incorporates the discrete nature of the variates and acts as the analogue of the standard multiplication used in the standard ARMA models. With this operator, the INAR(1) model is defined as

$$W_n = \rho * W_{n-1} + \epsilon_n,\ n \geq 1, \tag{3.1.2}$$

where $\{\epsilon_n\}$ will be referred to as innovation sequence, throughout the rest of the Chapter. Discrete time series modelling is based on binomial thinning which utilizes the concept of discrete self-decomposability introduced by Steutel and van Harn (1979). We consider the alternate probability generating function (*apgf*) $A(s) = G(1-s)$ instead of the *pgf* in $(3.1.2)$, then we get,

$$A(s) = A(\rho s) A_\epsilon(s),\ \text{for every } \rho \in (0,1). \tag{3.1.3}$$

The above equation (3.1.3), is analogous to the definition of self-decomposability for continuous random variables. A discrete stable random variable is infinitely divisible and unimodal and it is discrete self-decomposable in the sense that

$$G(s) = G(1 - \rho + \rho s) G_\epsilon(s), \tag{3.1.4}$$

for $\rho \in (0,1)$ and with $G_{\epsilon}(s)$ a *pgf*, (for more details on self-decomposability, see Steutel and van Harn, 2004, p.246). Bouzar (2002) introduced mixture representations for the discrete Mittag-Leffler and Linnik laws. Based on binomial thinning operator, several stationary integer-valued first order autoregressive models are developed in Bouzar and Jayakumar (2008). A first order observation driven integer-valued autoregressive model and the problem of estimation of its parameters are discussed in Zheng and Basawa (2008). Generalised geometric Mittag-Leffler distribution and autoregressive processes were developed by Jose et al. (2010).

Laplace distribution and its generalizations are of important consideration in statistical literature, because of the nice properties held by them. A widely used generalization of Laplace distribution is Linnik distribution (Linnik, 1953). Pakes (1995) introduced non-negative random variables with Laplace-Stieltjes transform

$$L_{\alpha,c,\beta}(s) = \left(\frac{1}{1+cs^{\alpha}}\right)^{\beta},\ s \geq 0,\ 0 < \alpha \leq 1,\ c > 0,\ \beta > 0.$$

and referred to them as positive Linnik laws. Christoph and Screiber (1998b) introduced a random variable with *pgf*

$$G(s) = \begin{cases} \left[1 + \frac{\lambda(1-s)^{\alpha}}{\beta}\right]^{-\beta} & \text{for} \quad 0 < \beta < \infty, |s| \leq 1 \\ exp\{-\lambda(1-s)^{\alpha}\} & \text{for} \quad \beta = \infty \end{cases} \tag{3.1.5}$$

and referred to it as discrete Linnik distributed with characteristic exponent $\alpha \in (0,1]$, scale parameter $\lambda > 0$ and form parameter $\beta > 0$. If $\beta = 1$, then $(3.1.5)$ gives the *pgf* of discrete Mittag-Leffler distribution. A convolution of discrete stable and discrete Linnik distributions is significant in the same background having the combined properties.

Generalized Laplacian distributions and autoregressive processes are developed by Jose and Manu (2011). Recently, Lishamol and Jose (2011) introduced geometric normal-Laplace distributions and autoregressive processes. Lishamol and Jose (2009) introduced autoregressive models with generalized normal-Laplace stationary marginals. Here we extend it to the discrete case.

In this chapter we introduce the convolution of discrete stable and discrete Linnik distributions and various properties like infinite divisibility, discrete self-decomposability, etc. are studied. Discrete semi stable-Linnik distribution and its generalizations are also discussed. We introduce first order integer-valued autoregressive processes with discrete stable-Linnik marginals. We develope the joint distribution and study the regression behaviour of the processes. The geometric discrete stable-Linnik distribution and geometric discrete semi stable-Linnik distribution are introduced and their properties are studied. We derived the geometric discrete stable-Linnik processes. The corresponding INAR(p) processes are also described.

3.2 Discrete Stable-Linnik Distribution

Consider the convolution of a discrete stable random variable $U(DS(\nu, \delta))$ and a discrete generalized Linnik random variable $V(DL(\lambda, \alpha, \beta))$, given by

$$W = U + V$$

where U and V are independent. The *pgf* of W is given by

$$G_W(s) = e^{-\nu(1-s)^\delta} \left[1 + \lambda(1-s)^\alpha\right]^{-\beta} \tag{3.2.1}$$

where $|s| \leq 1, \nu > 0, \delta \in (0,1), \lambda > 0, \alpha \in (0,1), \beta > 0$. The random variable having the *pgf* in $(3.2.1)$ is named as Discrete Stable-Linnik (DSL($\nu, \delta, \lambda, \alpha, \beta$)) random variable.

The discrete stable-Linnik family of distributions consist of a number of distributions and can be used to model a wide range of data. For various combinations of values of $\delta,\ \lambda,\ \alpha$ and β, we get many distributions as special cases.

Parameters	Distribution
$\delta = 1, \beta = 1,$	Poisson-discrete Mittag-Leffler
$\lambda = \frac{q}{p},\ \alpha = 1,\ \beta = 1$	Discrete stable geometric
$\delta = 1,\ \lambda = \frac{q}{p},\ \alpha = 1,\ \beta = 1,$	Poisson-geometric
$\lambda = \frac{q}{p},\ \alpha = 1$	Discrete stable negative binomial

Theorem 3.2.1. *Discrete stable-Linnik random variables are closed under linear transformation.*

Proof: Consider the linear transformation $T = aW + b$. The *pgf* of T can be obtained as

$$\begin{aligned} G_T(s) &= E[s^{aW+b}] \\ &= s^b G_W(s^a) \\ &= e^{-\nu(1-s^a)^\delta + b\ln s}\left[1 + \lambda(1 - s^a)^\alpha\right]^{-\beta} \end{aligned}$$

which is of the form of the *pgf* in $(3.2.1)$.

Remark 3.2.1. *DSL distributions are infinitely divisible.*

Theorem 3.2.2. *The DSL$(\nu, \delta, \lambda, \alpha, \beta)$ distribution is discrete self-decomposable.*

Proof: Consider the *pgf* of DSL$(\nu, \delta, \lambda, \alpha, \beta)$,

$$\begin{aligned} G(s) &= e^{-\nu(1-s)^\delta}\left[1 + \lambda(1 - s)^\alpha\right]^{-\beta} \\ &= \left\{\frac{e^{\frac{-\nu\rho^\delta(1-s)^\delta}{\beta}}}{1 + \lambda\rho^\alpha(1 - s)^\alpha}\right\}^\beta \left\{e^{\frac{-\nu(1-\rho^\delta)(1-s)^\delta}{\beta}}\left(\rho^\alpha + (1 - \rho^\alpha)\frac{1}{1 + \lambda(1 - s)^\alpha}\right)\right\}^\beta \\ &= G(1 - \rho + \rho s)G_\rho(s) \end{aligned}$$

where $G_\rho(s)$ is a *pgf* as in $(3.1.4)$. Hence the DSL$(\nu, \delta, \lambda, \alpha, \beta)$ is discrete self-decomposable.

Remark 3.2.2. *As DSL distribution is a convolution of discrete stable and discrete Linnik distribution which is not geometrically infinitely divisible(gid), it follows that the DSL distribution is not gid.*

3.2.1 Discrete Semi Stable-Linnik Distribution

Discrete semi stable-Linnik random variable is obtained as the convolution of discrete semi stable and discrete semi Linnik random variables and is having the *pgf* of the form

$$G_W(s) = e^{-\psi_1(s)}\left[\frac{1}{1+\psi_2(s)}\right]. \tag{3.2.2}$$

where $\psi_i(s)$ satisfies certain conditions.

Definition 3.2.1. *A distribution function F of a random variable X with pgf in (3.2.1) is called discrete semi stable-Linnik if $\psi_i(s)$ of (3.2.2) satisfies the property*

$$\psi_i(s) = \frac{1}{a}\psi_i(a^{\frac{1}{\alpha_i}}, s), \quad 0 < \alpha_i \leq 1, 0 < a < 1, i = 1,2 \tag{3.2.3}$$

where $\psi(s)$ satisfies the functional equation $a\psi(s) = \psi(a^{\alpha}s)$ for all $0 < s < 1$. The solution of this equation is given by $\psi_i(s) = s^{\alpha_i}h_i(s), \;\; i = 1,2$ where $h_i(s)$ is a periodic function in $\ln s$ with period $\frac{2\pi\alpha_i}{-\ln a_i}$.

3.2.2 Bivariate Discrete Semi Stable-Linnik Distribution

The bivariate discrete semi stable-Linnik distribution is obtained as the convolution of bivariate discrete semi stable and bivariate discrete semi-Linnik distributions.

Definition 3.2.2. *The distribution function of a random variable X is called bivariate discrete semi stable-Linnik if its pgf is given by,*

$$\psi(s_1, s_2) = e^{-\psi_1(s_1,s_2)}\frac{1}{1+\psi_2(s_1,s_2)}$$

if $\psi_i(s_1, s_2)$ *satisfies the property*

$$a\psi_i(s_1, s_2) = \psi_i(a^{\frac{1}{\alpha_1}} s_1, a^{\frac{1}{\alpha_2}} s_2),\ \alpha_1, \alpha_2 \in (0,1),\ 0 < a < 1,\ i = 1, 2.$$

3.3 Discrete Stable-Linnik Processes

In this section, we develop first order integer-valued autoregressive processes with discrete stable-Linnik marginals.

Theorem 3.3.1. *Let* $G(s)$ *be the pgf of a DSL*$(\nu, \delta, \lambda, \alpha, \beta)$ *distribution with* $\nu > 0, \beta > 0, \lambda > 0, \delta \in (0,1), \alpha \in (0,1), \rho \in (0,1)$. *Then there exists a stationary INAR(1) process* $\{W_n, n \in \mathbb{Z}\}$, *given in* (3.1.2) *with* $G(s)$ *as the pgf of its marginal distribution. Also the marginal distribution of the innovation sequences* $\{\epsilon_n, n \in \mathbb{Z}\}$ *has apgf* $A_\epsilon(s)$ *given by*

$$A_\epsilon(s) = \frac{e^{\nu \rho^\delta s^\alpha}(1 + \lambda \rho^\alpha s^\alpha)^\beta}{e^{\nu s^\delta}(1 + \lambda s^\alpha)^\beta}. \tag{3.3.1}$$

Proof: In terms of *apgf* defined as $A(s) = G(1-s)$, the INAR(1) model defined in (3.1.2) can be rewritten as

$$A_{W_n}(s) = A_{W_{n-1}}(\rho s) A_{\epsilon_n}(s).$$

Under stationary equilibrium, it reduces to

$$A_W(s) = A_W(\rho s) A_\epsilon(s).$$

Hence

$$A_\epsilon(s) = \frac{A_W(s)}{A_W(\rho s)}.$$

The INAR(1) with DSL$(\nu, \delta, \lambda, \alpha, \beta)$ marginals is defined, if there exists an innovation sequence $\{\epsilon_n\}$ such that $A_\epsilon(s)$ is an *apgf*.
From (3.2.1)

$$A_W(s) = e^{-\nu s^\delta} \left[1 + \lambda s^\alpha\right]^{-\beta}.$$

Then we have,

$$\begin{aligned} A_\epsilon(s) &= \frac{e^{\nu\rho^\delta s^\delta}(1+\lambda\rho^\alpha s^\alpha)^\beta}{e^{\nu s^\delta}(1+\lambda s^\alpha)^\beta} \\ &= e^{-\nu(1-\rho^\delta)s^\delta}\left(\rho^\alpha+(1-\rho^\alpha)\frac{1}{1+\lambda s^\alpha}\right)^\beta. \end{aligned}$$

Therefore, the innovation sequence $\{\epsilon_n\}$ has the convolution structure,

$$\epsilon_n = U_n + M_n$$

where U_n follows discrete stable DS($\nu(1-\rho^\delta),\delta$) and M_n is a β-fold convolutions of DML(λ,α,β).

Remark 3.3.1. *If W_0 is distributed arbitrarily, then also the process is asymptotically Markovian with discrete stable-Linnik marginal distribution.*

Proof: We have

$$\begin{aligned} W_n &= \rho * W_{n-1} + \epsilon_n \\ &= \rho^n * W_0 + \sum_{k=0}^{n-1}\epsilon_{n-k}. \end{aligned}$$

Writing in terms of *apgf*,

$$\begin{aligned} A_{W_n}(s) &= A_{W_0}(\rho^n s)\prod_{k=0}^{n-1}A_{\epsilon_n}(\rho^k s) \\ &= A_{W_0}(\rho^n s)\prod_{k=0}^{n-1}\left[\frac{e^{\frac{\nu}{\beta}(\rho^{k+1}s)^\delta}(1+\lambda(\rho^{k+1}s)^\alpha)}{e^{\frac{\nu}{\beta}(\rho^k s)^\delta}(1+\lambda(\rho^k s)^\alpha)}\right]^\beta \\ &\to e^{-\nu s^\delta}\left[1+\lambda s^\alpha\right]^{-\beta} \text{ as } n\to\infty. \end{aligned}$$

Hence it follows that even if W_0 is arbitrarily distributed the process is asymptotically stationary Markovian with discrete stable-Linnik marginals.

3.4 Joint Distribution of (W_{n-1}, W_n)

The joint *pgf* of (W_{n-1}, W_n) is given by

$$\begin{aligned} G_{W_{n-1},W_n}(s_1, s_2) &= E[s_1^{W_{n-1}} s_2^{\rho * W_{n-1}+\epsilon_n}] \\ &= G_{\epsilon_n}(s_2) G_{W_{n-1}}[s_1(1-\rho+\rho s_2)] \\ &= e^{-\nu(1-\rho^\delta)s_2^\delta} \left(\frac{1+\lambda\rho^\alpha s_2^\alpha}{1+\lambda s_2^\alpha}\right)^\beta \frac{e^{-\nu(1-s_1(1-\rho+\rho s_2))^\delta}}{[1+\lambda(1-s_1(1-\rho+\rho s_2))^\alpha]^\beta}. \end{aligned}$$

Since $G_{W_{n-1},W_n}(s_1, s_2) \neq G_{W_{n-1},W_n}(s_2, s_1)$, the process is not time reversible.

3.4.1 Regression Behaviour of DSL(1) Process

Study on regression of the model is in effect the forecasting of the process. Regression in the forward direction explains forecasting future values of W_n while the prediction of past values of W_n can be done through regression in the backward direction. As stated in Lawrance (1978), the practical implication of regression will be in the statistical analysis of direction-dependent data, since GDSL($\nu, \delta, \lambda, \alpha, \beta$) process is not time reversible.

Regression in Backward Direction

Since the above INAR(1) is not time reversible, the backward regression would be of interest. By differentiating $G_{W_{n-1},W_n}(s_1, s_2)$ with respect to s_1 and putting $s_1 = 0$,

$$E(W_{n-1}s_2^{W_n}) = (1-\rho+\rho s_2)\left(\nu\delta + \frac{\alpha\beta\lambda}{(1+\lambda)}\right) e^{-\nu\left[(1-\rho^\delta)s_2^\delta+1\right]} \left(\frac{1+\lambda\rho^\alpha s_2^\alpha}{(1+\lambda s_2^\alpha)(1+\lambda)}\right)^\beta. \tag{3.4.1}$$

But

$$E(W_{n-1}s_2^{W_n}) = \sum_{t=0}^{\infty} s_2^t E(W_{n-1}|W_n = t) P(W_n = t).$$

Hence $E(W_{n-1}|W_n = t)$ can be obtained from the coefficient of s_2^t in the expansion of right hand side of $(3.4.1)$.

3.5 Geometric Discrete Stable-Linnik Distribution

In this section geometric discrete stable-Linnik distribution is introduced and some of its properties are studied.

Definition 3.5.1. *A random variable W on $\mathbb{Z}_+$ is said to follow geometric discrete stable-Linnik distribution with parameters $\nu, \delta, \lambda, \alpha$ and β, if its apgf is given by*

$$A(s) = \frac{1}{1 + \nu s^\delta + \beta \ln(1 + \lambda s^\alpha)}, |s| \leq 1, \nu > 0, \beta > 0, \lambda > 0, \delta \in (0,1), \alpha \in (0,1). \tag{3.5.1}$$

We will write $W \stackrel{d}{=} GDSL(\nu, \delta, \lambda, \alpha, \beta)$.

Remark 3.5.1. *Geometric discrete stable-Linnik distribution is infinitely divisible.*

Theorem 3.5.1. *Let $W_1, W_2, \cdots,$ are independently and identically distributed geometric discrete stable-Linnik random variables where $T = W_1 + W_2 + \cdots + W_{N(p)}$ and $N(p)$ be geometric with mean $\frac{1}{p}$,*

$$P[N(p) = k] = p(1-p)^{k-1}, \; k = 1, 2, \cdots, 0 < p < 1.$$

Then $T \stackrel{d}{=} GDSL(\frac{\nu}{p}, \delta, \lambda, \alpha, \frac{\beta}{p})$.

Proof: The *apgf* of T is

$$\begin{aligned} A_T(s) &= \sum_{k=1}^{\infty} \{A_W(s)\}^k p(1-p)^{k-1} \\ &= \frac{pA_W(s)}{1-(1-p)A_W(s)} \\ &= \frac{1}{[1 + \frac{\nu}{p} s^\delta + \frac{\beta}{p} \ln(1 + \lambda s^\alpha)]}. \end{aligned}$$

Theorem 3.5.2. *Geometric discrete stable-Linnik distribution is the limit distribution of geometric sum of $GDSL(\nu, \delta, \lambda, \alpha, \frac{\beta}{n})$ random variables.*

Proof: From (3.2.1) we have,

$$\left[e^{\frac{\nu}{\beta}s^{\delta}}(1+\lambda s^{\alpha})\right]^{-\beta} = \left\{1+\left[e^{\frac{\nu}{\beta}s^{\delta}}(1+\lambda s^{\alpha})\right]^{\frac{\beta}{n}}-1\right\}^{-n}$$

is the *apgf* of a probability distribution since discrete stable-Linnik distribution is infinitely divisible. Hence by lemma 3.2 of Pillai (1990)

$$A_n(s) = \left\{1+n\left[e^{\frac{\nu}{\beta}s^{\delta}}(1+\lambda s^{\alpha})\right]^{\frac{\beta}{n}}-1\right\}^{-n}$$

is the *apgf* of a geometric sum of independently and identically distributed discrete stable-Linnik random variables. Taking limit as $n \to \infty$

$$\begin{aligned} A(s) &= \lim_{n\to\infty} A_n(s) \\ &= \left\{1+\lim_{n\to\infty}\left(n\left[e^{\frac{\nu}{\beta}s^{\delta}}(1+\lambda s^{\alpha})\right]^{\frac{\beta}{n}}-1\right)\right\}^{-1} \\ &= \left\{1+\beta\ln\left[e^{\frac{\nu}{\beta}s^{\delta}}(1+\lambda s^{\alpha})\right]\right\}^{-1}. \end{aligned}$$

3.5.1 Geometric Discrete Semi Stable-Linnik Distribution

We have the $DSL(\nu,\alpha_1,\lambda,\alpha_2,\beta)$ distribution is infinitely divisible and hence the discrete semi stable-Linnik distributions are infinitely divisible. Now,

$$e^{-\psi_1(s)}\frac{1}{1+\psi_2(s)} = \exp\left\{1-\frac{1}{[1+\psi_1(s)+\ln(1+\psi_2(s))]^{-1}}\right\}.$$

Hence $\frac{1}{[1+\psi_1(s)+\ln(1+\psi_2(s))]}$ is geometrically infinitely divisible (Klebanov et al.,1984).

Definition 3.5.2. *A random variable W on $\mathbb{Z}_+$ is said to follow geometric discrete semi stable-Linnik distribution and write $W \stackrel{d}{=} DSL(\nu,\alpha_1,\lambda,\alpha_2,\beta)$, if it has the apgf,*

$$A(s) = \frac{1}{[1+\psi_1(s)+\ln(1+\psi_2(s))]};\ |s|\leq 1, 0<\alpha\leq 1, c>0, \beta>0. \qquad (3.5.2)$$

Like discrete semi stable-Linnik distribution mentioned above, the geometric generalized discrete semi stable-Linnik distribution corresponding to generalized discrete semi stable-Linnik distribution is obtained as one with *apgf*,

$$\frac{1}{[1+\psi_1(s)+\beta\ln(1+\psi_2(s))]}.$$

Theorem 3.5.3. *Geometric discrete semi stable-Linnik distribution is the limit distribution of geometric sum of discrete semi stable-Linnik variables.*

Proof:

$$A_n(s)=\left[1+n\left\{\left(\frac{1+\psi_2(s)}{e^{-\psi_1(s)}}\right)^{\frac{1}{n}}-1\right\}\right]^{-1}. \tag{3.5.3}$$

By Lemma 3.2 of Pillai (1990), $(3.5.3)$ is the *pgf* of the geometric sum of iid discrete semi stable-Linnik variables. Taking limit as $n\to\infty$,

$$\begin{aligned} A(s) &= \lim_{n\to\infty} A_n(s) \\ &= \left[1+\lim_{n\to\infty} n\left\{\left(\frac{1+\psi_2(s)}{e^{-\psi_1(s)}}\right)^{\frac{1}{n}}-1\right\}\right]^{-1} \\ &= \frac{1}{1+\psi_1(s)+\ln[1+\psi_2(s)]}. \end{aligned}$$

3.6 Geometric Discrete Stable-Linnik Processes

In this section, we develop a first order integer-valued autoregressive process with geometric discrete stable-Linnik marginals.

Theorem 3.6.1. *Let* $\{W_n, n\geq 1\}$ *be defined as*

$$W_n = I_n W_{n-1}+\epsilon_n \tag{3.6.1}$$

where $\{I_n\}$ *is a Bernoulli sequence with* $P(I_n=0)=1-P(I_n=1)=p,\ 0<p<1$ *and* $\{\epsilon_n\}$ *is a sequence of i.i.d. random variables. A necessary and sufficient condition* *that $\{W_n\}$ is a strictly stationary Markov process with GDSL($\nu, \delta, \lambda, \alpha, \beta$) marginals is that ϵ_n are distributed as geometric discrete stable-Linnik provided W_0 is distributed as geometric discrete stable-Linnik.*

Proof: Taking the *apgf* of $(3.6.1)$ we have,

$$A_{W_n}(s) = pA_{\epsilon_n}(s) + (1-p)A_{W_{n-1}}(s)A_{\epsilon_n}(s). \tag{3.6.2}$$

Under stationary equilibrium, it becomes,

$$A_W(s) = A_\epsilon(s)\{p + (1-p)A_W(s)\}.$$

That is,

$$A_\epsilon(s) = \frac{A_W(s)}{p + (1-p)A_W(s)}$$

where

$$A_W(s) = \frac{1}{1 + \nu s^\delta + \beta \ln(1 + \lambda s^\alpha)}.$$

On simplification we get,

$$A_\epsilon(s) = \frac{1}{1 + p\nu s^\delta + p\beta \ln(1 + \lambda s^\alpha)}$$

and hence $\epsilon_n \stackrel{d}{=} GDSL(p\nu, \delta, \lambda, \alpha, p\beta)$.

The converse part can be proved by the method of mathematical induction as follows. Now assume that $W_{n-1} \stackrel{d}{=} GDSL(\nu, \delta, \lambda, \alpha, \beta)$. Then

$$\begin{aligned} A_{W_{n-1}}(s) &= A_{\epsilon_n}(s)\{p + (1-p)A_{W_{n-2}}(s)\} \\ &= \frac{1}{1 + p\nu s^\delta + p\beta \ln(1 + \lambda s^\alpha)} \left[p + (1-p) \left\{ \frac{1}{1 + \nu s^\delta + \beta \ln(1 + \lambda s^\alpha)} \right\} \right] \\ &= \frac{1}{1 + \nu s^\delta + \beta \ln(1 + \lambda s^\alpha)}. \end{aligned}$$

The rest follows similarly.

3.7 INAR(p) Process of GDSL Distribution

Here we define p^{th} order integer-valued AR (INAR(p)) with probability structure,

$$W_n = \begin{cases} \rho_1 * W_{n-1} + \epsilon_n, & \text{with probability} \quad \xi_1 \\ \rho_2 * W_{n-2} + \epsilon_n, & \text{with probability} \quad \xi_2 \\ \cdots & \cdots \qquad\qquad\qquad \cdots \\ \cdots & \cdots \qquad\qquad\qquad \cdots \\ \rho_p * W_{n-p} + \epsilon_n, & \text{with probability} \quad \xi_p \end{cases} \tag{3.7.1}$$

where $0 < \rho_i, \xi_i \leq 1,\ i = 1, 2, \cdots, p;\ \sum_{i=1}^{p} \xi_i = 1.$
In terms of *apgf* the above equation can be written as

$$A_{W_n}(s) = A_{\epsilon_n}(s) \left(\sum_{i=1}^{p} \xi_i A_{W_{n-i}}(\rho_i s) \right).$$

Assuming stationarity it reduces to

$$A_W(s) = A_\epsilon(s) \left(\sum_{i=1}^{p} \xi_i A_W(\rho_i s) \right).$$

Hence

$$A_\epsilon(s) = \frac{A_W(s)}{\sum_{i=1}^{p} \xi_i A_W(\rho_i s)}.$$

For the GDSL marginals, the innovation sequence of the process has *apgf*,

$$A_\epsilon(s) = \frac{\left[1 + \nu s^\delta + \beta \ln(1 + \lambda s^\alpha)\right]^{-1}}{\sum_{i=1}^{p} \xi_i \left[1 + \nu (\rho_i s)^\delta + \beta \ln(1 + \lambda (\rho_i s)^\alpha)\right]^{-1}}. \tag{3.7.2}$$

For the particular case of $\rho_i = \rho$, for $i = 1, 2, \cdots, p$, $(3.7.2)$ yields the similar pattern of *apgf* defined in $(3.3.1)$. Hence with an error sequence $\{\epsilon_n\}$ distributed as GDSL random variables, the p^{th} order GDSL autoregressive processes are properly defined.

3.7.1 Geometric Discrete Semi Stable-Linnik Processes

For the first order autoregressive processes with geometric discrete semi stable-Linnik as marginals, the following structure for $\{W_n\}$ is considered.

$$W_n = \begin{cases} \epsilon_n, & \text{with probability} \quad p \\ W_{n-1} + \epsilon_n, & \text{with probability} \quad (1-p) \end{cases}. \tag{3.7.3}$$

In terms of *apgf* the model defined in $(3.7.3)$ can be given as

$$A_{W_n}(s) = pA_{\epsilon_n}(s) + (1-p)A_{W_{n-1}}(s)A_{\epsilon_n}(s). \tag{3.7.4}$$

Assuming stationarity, we have,

$$\begin{aligned} A_\epsilon(s) &= \frac{A_W(s)}{p + (1-p)A_W(s)} \\ &= \frac{1}{[1 + p\psi_1(s) + p\ln(1+\psi_2(s))]}. \end{aligned}$$

Hence the innovation sequence $\{\epsilon_n\}$ is distributed as geometric discrete semi stable-Linnik if and only if $\{W_n\}$ is distributed marginally as geometric discrete semi stable-Linnik.

Now we construct an AR(1) model with geometric discrete semi stable-Linnik marginals. Consider an autoregressive process $\{W_n\}$ with structure given in (3.7.3). Suppose that $\{W_n\}$ has the geometric discrete semi stable-Linnik distribution. Then with the assumption of stationarity (3.7.4) gives

$$A_\epsilon(s) = \frac{1}{[1 + p\psi_1(s) + p\beta\ln(1+\psi_2(s))]}.$$

Hence $\{\epsilon_n\}$ is also distributed as geometric discrete semi stable-Linnik.

References

Bouzar, N. (2002). Mixture representations for discrete Linnik laws, *Statistica Neer-*

landica, **56**(3), 295-300.

Bouzar,N., Jayakumar, K. (2008). Time series with discrete semi-stable marginals, *Statistical Papers*, **49**, 619-635.

Christoph, G., Schreiber, K. (1998a). Discrete stable random variables. *Statistics and Probability Letters*, **37**, 243-247.

Christoph, G., Schreiber, K. (1998b). The generalized discrete Linnik distributions. *In Advances in Stochastic Models for Reliability, Quality and Safety* (Eds. W. Kahle et al.), pp.3-18, Boston: Birkhäuser.

Devroye, L. (1993). A triptych of discrete distributions related to the stable law. *Statistics and Probability Letters*, **18**, 349-351.

Feller, W. (1966). *An Introduction to Probability Theory and its Applications- Vol. II*, John Wiley and Sons, New York.

Gnedenko, B.V., Kolmogrov, A.N. (1968). *Limit distributions for sums of independent random variables*, Addison Wesley Publications, New York.

Jose, K.K., Manu, M.T. (2011). Generalized Laplacian distributions and autoregressive processes. *Communications in Statistics-Theory and Methods*, **40**, 4263-4277.

Jose, K.K., Uma, P., Seetha Lekshmi, V., Haubold, H.J. (2010). Generalized Mittag-Leffler distributions and processes for Applications in Astrophysics and Time Series Modeling, *Astrophysics and Space Science Proceedings*, 79-92.

Lawrance, A.J. (1978). Some autoregressive models for point processes. *In Proceedings of Bolyai Mathematical Society Colloquium on point processes and queuing problems*, **24**, Hungary, 257-275.

Linnik, Yu.V. (1953). Linear forms and statistical criteria: I, II, *Ukraïn Mat. Zh.*, 5, 207-243, 247-290. (English translations in Mathematical Statistics and Probability 3, 1-40, 41-90. American mathematical Society, Providence, R.I. (1962)).

Lishamol, T., Jose, K.K. (2009). Generalized normal-Laplace AR process. *Statistics and Probability Letters*, **79**, 1615-1620.

Lishamol, T., Jose, K.K. (2011). Geometric Normal-Laplace Distributions and Autoregressive Processes. *Journal of Applied Statistical Science*, **18**(3), 1-7.

Pakes, A.G. (1995). Characterization of discrete laws via mixed sums and Markov branching processes. *Stochastic Processes and their Applications*, **55**, 285-300.

Pillai, R.N. (1988). Generalized geometric stable distributions. *Report of the Steklov institute of mathematics. Mathematics and Statistics series*, **8**, 155-159.

Pillai, R.N. (1990). Harmonic mixtures and geometric infinite divisibility. *Journal of the Indian Statistical Association*, **28**, 87-98.

Steutel, F.W., van Harn, K. (1979). Discrete analogues of self- decomposability and stability. *Annals of Probability*, **7**, 893-899.

Steutel, F. W., van Harn, K. (2004). *Infinite Divisibility of Probability Distributions in the Real Line*, New York: Dekker.

Zheng, H., Basawa, I. V. (2008). First order observation-driven integer-valued autoregressive processes, *Statistics and Probability Letters*, **78**, 1-9.