CHAPTER – IV

\(\hat{\eta}^*-\) HOMEOMORPHISMS IN TOPOLOGICAL SPACES

4.1 Introduction

Maki et al [29] introduced \(g\)-homeomorphisms in topological spaces. The content of this chapter is \(\hat{\eta}^*\)-closed maps, pre \(\hat{\eta}^*\)-closed maps, quasi \(\hat{\eta}^*\)-closed maps and strongly \(\hat{\eta}^*\)-closed maps. The respective \(\hat{\eta}^*\)-open maps are also studied. Then \(\hat{\eta}^*\)-homeomorphisms, strongly \(\hat{\eta}^*\)-homeomorphisms (\(S\hat{\eta}^*\)-homeomorphisms) and \(\hat{\eta}^*\)-quotient maps have been introduced. The new concepts namely \(\hat{\eta}^*\)-regular, ultra \(\hat{\eta}^*\)-regular, \(\hat{\eta}^*\)-normal and ultra \(\hat{\eta}^*\)-normal spaces are also investigated.

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4.2 \(\hat{\eta}^*\)-closed maps

T.Noiri, H. Maki and J. Umehara [37] introduced the concepts of \(gp\)-closed and pre \(gp\)-closed map using \(gp\)-closed sets. This section contains \(\hat{\eta}^*\)-closed maps, pre \(\hat{\eta}^*\)-closed maps and their properties in topological spaces have been introduced. Further the properties of these maps are obtained.

**Definition 4.2.1** A map \(f: X \to Y\) is said to be \(\hat{\eta}^*\)-closed, if the image of every closed set of \(X\) is \(\hat{\eta}^*\)-closed in \(Y\).

**Definition 4.2.2** A map \(f: X \to Y\) is said to be pre-\(\hat{\eta}^*\)-closed if the image of every semi-preclosed set of \(X\) is \(\hat{\eta}^*\)-closed in \(Y\).
Remark 4.2.3 It is obvious that both closedness and pre-$\mathcal{H}^*$-closedness imply $\mathcal{H}^*$-closedness. However the converses are false as the following example shows.

Example 4.2.4 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$. Clearly the identity map $f: (X, \tau) \to (Y, \sigma)$ is $\mathcal{H}^*$-closed but not closed, since $\{c\}$ is closed in $X$ and $f(\{c\}) = \{c\}$ is not closed in $Y$.

Example 4.2.5 Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \emptyset, \{b\}\}$. Clearly the identity map $f: (X, \tau) \to (Y, \sigma)$ is $\mathcal{H}^*$-closed but not pre-$\mathcal{H}^*$-closed, since $\{b\}$ is semi-preclosed in $X$ and $f(\{b\}) = b$ is not $\mathcal{H}^*$-closed in $Y$.

Theorem 4.2.6 A surjective map $f: X \to Y$ is $\mathcal{H}^*$-closed if and only if for each subset $S$ of $Y$ and each open set $U$ containing $f^{-1}(S)$, there exists an $\mathcal{H}^*$-open set $V$ of $Y$ such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.

Proof:

Necessity. Suppose that $f$ is $\mathcal{H}^*$-closed. Let $S$ be any subset of $Y$ and $U$ an open set of $X$ containing $f^{-1}(S)$. Put $V = (f(U))^c$. Then $V$ is $\mathcal{H}^*$-open in $Y$ containing $S$ and $f^{-1}(V) \subseteq U$.

Sufficiency. Let $F$ be any closed set of $X$. Put $B = (f(F))^c$, then we have $f^{-1}(B) \subseteq F^c$ and $F^c$ is open in $X$. By hypothesis there exists an $\mathcal{H}^*$-open set $V$ of $Y$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq F^c$ and so $F \subseteq (f^{-1}(V))^c = f^{-1}(V^c)$. Therefore, we obtain $f(F) = V^c$. Since $V^c$ is $\mathcal{H}^*$-closed, $f(F)$ is $\mathcal{H}^*$-closed in $Y$. This implies that $f$ is $\mathcal{H}^*$-closed.

Remark 4.2.7 Necessity of the above theorem is proved without assuming that $f$ is surjective. Therefore we can obtain the following Corollary.
Corollary 4.2.8 If \( f: X \rightarrow Y \) is \( \mathcal{H}^* \)-closed, then for any closed set \( F \) of \( Y \) and for any open set \( U \) of \( X \) containing \( f^{-1}(F) \) there exists a semi-preopen set \( V \) of \( Y \) such that \( F \subseteq V \) and \( f^{-1}(V) \subseteq U \).

Proof: By Theorem 4.2.6, there exists an \( \mathcal{H}^* \)-open set \( W \) of \( Y \) such that \( F \subseteq W \) and \( f^{-1}(W) \subseteq U \). Since \( F \) is closed, \( F \) is \( \omega \)-closed. By Theorem 2.4.7, \( F \subseteq \text{spint}(W) \). Put \( V = \text{spint}(W) \) then \( V \) is semi-preopen in \( Y \) such that \( F \subseteq V \) and \( f^{-1}(\text{spint}(W)) \subseteq f^{-1}(W) \subseteq U \) and hence \( f^{-1}(V) \subseteq U \).

Proposition 4.2.9 If \( f: X \rightarrow Y \) is \( \omega \)-irresolute pre-\( \mathcal{H}^* \)-closed and \( A \) is \( \mathcal{H}^* \)-closed in \( X \), then \( f(A) \) is \( \mathcal{H}^* \)-closed in \( Y \).

Proof: Let \( U \) be any \( \omega \)-open set of \( Y \) containing \( f(A) \). Then \( A \subseteq f^{-1}(U) \) and \( f^{-1}(U) \) is \( \omega \)-open in \( X \). Since \( A \) is \( \mathcal{H}^* \)-closed in \( X \), \( \text{spcl}(A) \subseteq f^{-1}(U) \) and hence \( f(A) \subseteq f(\text{spcl}(A)) \subseteq U \). Also since \( f \) is pre-\( \mathcal{H}^* \)-closed and \( \text{spcl}(A) \) is semi-pre closed in \( X \), \( f(\text{spcl}(A)) \) is \( \mathcal{H}^* \)-closed in \( Y \) and hence \( \text{spcl}(f(A)) \subseteq \text{spcl}(f(\text{spcl}(A))) \subseteq U \). This shows that \( f(A) \) is \( \mathcal{H}^* \)-closed in \( Y \).

Remark 4.2.10 The following example shows that the composition of two \( \mathcal{H}^* \)-closed maps is not \( \mathcal{H}^* \)-closed.

Example 4.2.11 Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \) and \( \eta = \{Z, \phi, \{c\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be identity maps. Then clearly \( f \) and \( g \) are \( \mathcal{H}^* \)-closed maps but \( g \circ f: X \rightarrow Z \) is not \( \mathcal{H}^* \)-closed, since \( \{c\} \) is closed in \( X \) and \( g \circ f(\{c\}) = g(f(\{c\})) = \{c\} = \{c\} \) is not \( \mathcal{H}^* \)-closed in \( Z \).

Proposition 4.2.12 If \( f: X \rightarrow Y \) and \( g: Y \rightarrow Z \) are \( \mathcal{H}^* \)-closed maps with \( Y \) is a \( T_{\mathcal{H}^*} \)-space, then \( g \circ f: X \rightarrow Z \) is also an \( \mathcal{H}^* \)-closed map.

Proof: Clearly follows from Definitions.
Proposition 4.2.13 Let \( f: X \to Y \) be a map from a space \( X \) to a \( T_{\hat{\eta}^*} \)-space \( Y \).

Then the following are equivalent:

1) \( f \) is \( \hat{\eta}^* \)-closed,

2) \( f \) is closed.

Proof: Follows by Definition 2.5.1.

Proposition 4.2.14 Let \( f: X \to Y \) and \( g: Y \to Z \) be two maps such that \( g \circ f: X \to Z \) is \( \hat{\eta}^* \)-closed.

1) If \( f \) is continuous surjection, then \( g \) is \( \hat{\eta}^* \)-closed;

2) If \( g \) is \( \hat{\eta}^* \)-irresolute and injective, then \( f \) is \( \hat{\eta}^* \)-closed;

3) If \( f \) is \( \hat{\eta}^* \)-continuous surjective and \( X \) is a \( T_{\hat{\eta}^*} \)-space, then \( g \) is \( \hat{\eta}^* \)-closed.

Proof:

i) Let \( A \) be a closed set of \( Y \). Since \( f \) is continuous, \( f^{-1}(A) \) is closed in \( X \). Also since \( g \circ f \) is \( \hat{\eta}^* \)-closed and \( f \) is surjective, \( (g \circ f)^{-1}(A) = g(A) \) is \( \hat{\eta}^* \)-closed in \( Z \). Hence \( g \) is \( \hat{\eta}^* \)-closed.

ii) Let \( B \) be a closed set of \( X \). Since \( g \circ f \) is \( \hat{\eta}^* \)-closed, \( (g \circ f)(B) \) is \( \hat{\eta}^* \)-closed in \( Z \). Also since \( g \) is \( \hat{\eta}^* \)-irresolute, \( g^{-1}(g \circ f)(B) \) is \( \hat{\eta}^* \)-closed in \( Y \). Since \( g \) is injective, \( f(B) \) is \( \hat{\eta}^* \)-closed in \( Y \). Hence, \( f \) is \( \hat{\eta}^* \)-closed.

iii) Let \( A \) be a closed set of \( Y \). Since \( f \) is \( \hat{\eta}^* \)-continuous, \( f^{-1}(A) \) is \( \hat{\eta}^* \)-closed in \( X \). Also since \( X \) is a \( T_{\hat{\eta}^*} \)-space, we have \( f^{-1}(A) \) is closed in \( X \). Since \( g \circ f \) is closed and \( f \) is surjective, then \( (g \circ f)^{-1}(A) = g(A) \) is \( \hat{\eta}^* \)-closed in \( Z \). Hence, \( g \) is \( \hat{\eta}^* \)-closed.
Definition 4.2.15  A space $X$ is said to be ultra $\hat{n}^*$-regular if for each closed set $F$ of $X$ and each point $x \not\in F$ there exist disjoint $\hat{n}^*$-open sets $U$ and $V$ such that $F \subset U$ and $x \in V$.

Theorem 4.2.16  In a topological space $X$, assume that $\hat{n}^*$-$\text{o}(\tau)$ is closed under any union. Then the following statements are equivalent:

a) $X$ is ultra $\hat{n}^*$-regular,

b) for every point $x$ of $X$ and every open set $V$ containing $x$, there exists an $\hat{n}^*$-open set $A$ such that $x \in A \subset \hat{n}^* \text{cl}(A) \subset V$.

Proof:

$a \Rightarrow b$. Let $x \in X$ and $V$ be an open set containing $x$. Then $V^c$ is closed and $x \not\in V^c$. By (a) there exist disjoint $\hat{n}^*$-open sets $A$ and $B$ such that $x \in A$ and $V^c \subset B$. That is $B^c \subset V$. Since every open set is $\hat{n}^*$-open, $V$ is $\hat{n}^*$-open. Since $B$ is $\hat{n}^*$-open, $B^c$ is $\hat{n}^*$-closed. Therefore, $\hat{n}^* \text{cl}(B^c) \subset V$. Since $A \cap B = \emptyset$, $A \subset B^c$. Therefore, $x \in A \subset \hat{n}^* \text{cl}(A) \subset \hat{n}^* \text{cl}(B^c) \subset V$. Hence, $x \in A \subset \hat{n}^* \text{cl}(A) \subset V$.

$b \Rightarrow a$. Let $F$ be a closed set and $x \not\in F$. This implies that $F^c$ is an open set containing $x$. By (b) there exists an $\hat{n}^*$-open set $A$ such that $x \in A \subset \hat{n}^* \text{cl}(A) \subset F^c$. That is, $F \subset (\hat{n}^* \text{cl}(A))^c$. By Proposition 3.2.19, $\hat{n}^* \text{cl}(A)$ is $\hat{n}^*$-closed. Hence, $(\hat{n}^* \text{cl}(A))^c$ is $\hat{n}^*$-open. Therefore, $A$ and $(\hat{n}^* \text{cl}(A))^c$ are the required $\hat{n}^*$-open sets.

Theorem 4.2.17  Assume that $\hat{n}^*$-$\text{o}(\tau)$ is closed under any union. If $f: X \to Y$ is a continuous open $\hat{n}^*$-closed surjective map and $X$ is a regular space, then $Y$ is ultra $\hat{n}^*$-regular.

Proof: Let $y \in Y$ and $V$ be an open set containing $y$ of $Y$. Let $x$ be a point of $X$ such that $y = f(x)$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. Since $X$ is regular,
there exists an open set \( U \) such that \( x \in U \subset \text{cl}(U) \subset f^{-1}(V) \). Hence, \( y = f(x) \in f(U) \subset f(\text{cl}(U)) \subset V \). Since \( f \) is an \( \tilde{\eta}^* \)-closed map, \( f(\text{cl}(U)) \) is an \( \tilde{\eta}^* \)-closed set contained in the open set \( V \). Since every open set is \( \omega \)-open, \( V \) is \( \omega \)-open. Hence, \( \text{spcl}(f(\text{cl}(U))) \subset V \). Therefore, \( y \in f(U) \subset \tilde{\eta}^*\text{cl}(f(U)) \subset \tilde{\eta}^*\text{cl}(f(\text{cl}(U))) \subset \text{spcl}(f(\text{cl}(U))) \subset V \). This implies that \( y \in f(U) \subset \tilde{\eta}^*\text{cl}(f(U)) \subset V \).

Since \( f \) is an open map and \( U \) is open in \( X \), \( f(U) \) is open in \( Y \). Since every open set is \( \tilde{\eta}^* \)-open, \( f(U) \) is \( \tilde{\eta}^* \)-open in \( Y \). Thus for every point \( y \) of \( Y \) and every open set \( V \) containing \( y \) there exists an \( \tilde{\eta}^* \)-open set \( f(U) \) such that \( y \in f(U) \subset \tilde{\eta}^*\text{cl}(f(U)) \subset V \). Hence by Theorem 4.2.16, \( Y \) is ultra \( \tilde{\eta}^* \)-regular.

**Definition 4.2.18**  A space \( X \) is said to be ultra \( \tilde{\eta}^* \)-normal if for disjoint closed sets \( A \) and \( B \) of \( X \) there exist disjoint \( \tilde{\eta}^* \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

**Theorem 4.2.19**  Assume that \( \tilde{\eta}^*o(\tau) \) is closed under any union. If \( f: X \to Y \) is a continuous \( \tilde{\eta}^* \)-closed surjection and \( X \) is a normal space, then \( Y \) is ultra \( \tilde{\eta}^* \)-normal.

**Proof:** Let \( A \) and \( B \) be disjoint closed sets of \( Y \). Since \( X \) is normal there exist disjoint open sets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subset U \) and \( f^{-1}(B) \subset V \). By Theorem 4.2.6, there exist \( \tilde{\eta}^* \)-open sets \( G \) and \( H \) such that \( A \subset G \), \( B \subset H \) and \( f^{-1}(G) \subset U \) and \( f^{-1}(H) \subset V \). Then we have \( f^{-1}(G) \cap f^{-1}(H) = \emptyset \) and hence \( G \cap H = \emptyset \). Since \( G \) is \( \tilde{\eta}^* \)-open and \( A \) is closed, \( A \subset G \) implies that \( A \subset \text{spint}(G) \subset \tilde{\eta}^*\text{int}(G) \). Similarly \( B \subset \tilde{\eta}^*\text{int}(H) \). Therefore, \( \tilde{\eta}^*\text{int}(G) \cap \tilde{\eta}^*\text{int}(H) = \emptyset \). Thus \( Y \) is ultra \( \tilde{\eta}^* \)-normal.

Regarding the restriction \( f_A \) of a map \( f: X \to Y \) to a subset \( A \) of \( X \) we have the following:
Theorem 4.2.20  Let X and Y be any two topological spaces. Then

(i) If $f: X \rightarrow Y$ is $\hat{\eta}^*$-closed and $A$ is a closed subset of $X$ then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is $\hat{\eta}^*$-closed.

(ii) If $f: X \rightarrow Y$ is $\omega$- irresolute pre-$\hat{\eta}^*$-closed and $A$ is clopen in $X$, then $f_A : (A, \tau_A) \rightarrow (Y, \sigma)$ is $\hat{\eta}^*$-closed.

Proof (i):  Let $F$ be a closed set of $A$. Since $A$ is closed in $X$, $F$ is closed in $X$ and $f(F)$ is $\hat{\eta}^*$-closed in $Y$. But $f(F) = f_A(F)$. Hence, $f_A$ is $\hat{\eta}^*$-closed.

(ii) Let $F$ be a closed set of $A$. Hence $F$ is $\hat{\eta}^*$-closed in $A$ and $A$ is clopen in $X$. By Theorem 2.2.28, $F$ is $\hat{\eta}^*$-closed in $X$. By Proposition 4.2.9 $f(F) = f_A(F)$ is $\hat{\eta}^*$-closed in $Y$ and hence $f_A$ is $\hat{\eta}^*$-closed.

Theorem 4.2.21  If $f: X \rightarrow Y$ is a bijective $\hat{\eta}^*$-closed map of a space $X$ onto an $\hat{\eta}^*$-connected space $Y$, then $X$ is connected.

Proof:  Let us assume that $X$ is not connected. Then there exist nonempty open sets $U$ and $V$ such that $U \cap V = \emptyset$ and $X = U \cup V$. Therefore $U$ and $V$ are clopen in $X$ and $f(U)$ and $f(V)$ are $\hat{\eta}^*$-closed. Moreover, we have $f(U) \cap f(V) = \emptyset$ and $f(U) \cup f(V) = Y$. Since $f$ is bijective, $f(U)$ and $f(V)$ are non empty. This indicates that $Y$ is not $\hat{\eta}^*$-connected. This is a contradiction.

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4.3 Strongly $\hat{\eta}^*$-closed and quasi-$\hat{\eta}^*$-closed maps.

G. B. Navalagi [35] introduced the concepts of strongly $\alpha$-closed maps and quasi $\alpha$-closed maps in topological spaces by using $\alpha$-closed sets in topological spaces. This section contains strongly $\hat{\eta}^*$-closed maps and quasi $\hat{\eta}^*$-closed maps and the relationships between these maps.
Definition 4.3.1 A map \( f: X \to Y \) is said to be strongly \( \hat{\eta}^* \)-closed if for each \( \hat{\eta}^* \)-closed set \( F \) of \( X \), \( f(F) \) is \( \hat{\eta}^* \)-closed in \( Y \).

Definition 4.3.2 A map \( f: X \to Y \) is said to be quasi-\( \hat{\eta}^* \)-closed if for each \( \hat{\eta}^* \)-closed set \( F \) of \( X \), \( f(F) \) is closed in \( Y \).

Proposition 4.3.3 Every quasi-\( \hat{\eta}^* \)-closed map is strongly \( \hat{\eta}^* \)-closed.

Proof: Obvious.

Proposition 4.3.4 Every quasi-\( \hat{\eta}^* \)-closed map is pre-\( \hat{\eta}^* \)-closed.

Proof: The proof follows from the fact that every semi-preclosed set is \( \hat{\eta}^* \)-closed.

Proposition 4.3.5 Every quasi-\( \hat{\eta}^* \)-closed map is closed.

Proof: The proof follows from the fact that every closed set is \( \hat{\eta}^* \)-closed.

Proposition 4.3.6 Every strongly \( \hat{\eta}^* \)-closed map is pre-\( \hat{\eta}^* \)-closed (resp. \( \hat{\eta}^* \)-closed).

Proof: Clearly follows from Definitions.

Example 4.3.7 Let \( X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \). Clearly the identity map \( f: (X, \tau) \to (Y, \sigma) \) is strongly \( \hat{\eta}^* \)-closed map (resp. pre \( \hat{\eta}^* \)-closed) but not quasi-\( \hat{\eta}^* \)-closed, since \( \{b\} \) is \( \hat{\eta}^* \)-closed in \( X \) but \( f(\{b\}) = \{b\} \) is not closed in \( Y \).

Example 4.3.8 Let \( X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity map. Clearly \( f \) is a closed map but not quasi-\( \hat{\eta}^* \)-closed, since \( \{b\} \) is \( \hat{\eta}^* \)-closed in \( X \) but \( f(\{b\}) = \{b\} \) is not closed in \( Y \). However \( f \) is pre-\( \hat{\eta}^* \)-closed but not strongly \( \hat{\eta}^* \)-closed, since \( \{a, b\} \) is \( \hat{\eta}^* \)-closed in \( X \) but \( f(\{a, b\}) = \{a, b\} \) is not \( \hat{\eta}^* \)-closed in \( Y \).
Remark 4.3.9 The map defined in Example 4.3.7 is strongly $\mathring{\eta}^*$-closed but not closed. The map defined in Example 4.3.8 is closed but not strongly $\mathring{\eta}^*$-closed. Thus strongly $\mathring{\eta}^*$-closed maps and closed maps are independent of each other. From the above propositions and examples we get the following diagram.

\[
\begin{array}{c}
\text{pre-} \mathring{\eta}^* \text{- closed map} \\
\text{quasi-} \mathring{\eta}^* \text{- closed map} \\
\text{strongly } \mathring{\eta}^* \text{- closed map} \\
\text{closed map}
\end{array}
\]

Theorem 4.3.10 A surjective mapping $f: X \to Y$ is quasi-$\mathring{\eta}^*$-closed if and only if for any subset $B$ of $Y$ and for each $\mathring{\eta}^*$-open set $U$ of $X$ containing $f^{-1}(B)$, there is an open set $V$ of $Y$ containing $B$ and $f^{-1}(V) \subset U$.

Proof: It is similar to the proof of the Theorem 4.2.6.

Theorem 4.3.11 In a topological space $X$, assume that $\mathring{\eta}^* o(\tau)$ is closed under any union. A map $f: X \to Y$ is quasi-$\mathring{\eta}^*$-closed if and only if for every subset $U$ of $X$, $\text{cl}(f(U)) \subset f(\mathring{\eta}^* \text{cl}(U))$.

Proof: Let $f$ be quasi- $\mathring{\eta}^*$-closed. We have $U \subset \mathring{\eta}^* \text{cl}(U)$ and also $\mathring{\eta}^* \text{cl}(U)$ is an $\mathring{\eta}^*$-closed set. Hence we obtain $f(U) \subset f(\mathring{\eta}^* \text{cl}(U))$ and $f(\mathring{\eta}^* \text{cl}(U))$ is closed. Hence $\text{cl}(f(U)) \subset f(\mathring{\eta}^* \text{cl}(U))$.

Conversely, assume that the given condition holds. If $U$ is an $\mathring{\eta}^*$-closed in $X$, then $\text{cl}(f(U)) \subset f(\mathring{\eta}^* \text{cl}(U)) = f(U)$. Consequently, $f(U) = \text{cl}(f(U))$ and hence $f$ is quasi-$\mathring{\eta}^*$-closed.
**Theorem 4.3.12** In a topological space $X$, assume that $\hat{\eta}^*\circ(\tau)$ is closed under any union. A map $f: X \to Y$ is strongly $\hat{\eta}^*$-closed if and only if for every subset $U$ of $X$, $\hat{\eta}^*\text{cl}(f(U)) \subseteq f(\hat{\eta}^*\text{cl}(U))$.

**Proof:** Similar to the proof of the Theorem 4.3.11.

**Proposition 4.3.13** Let $f: X \to Y$ and $g: Y \to Z$ be two strongly $\hat{\eta}^*$-closed mappings. Then $g \circ f: Y \to Z$ is a strongly $\hat{\eta}^*$-closed mapping.

**Proof:** Obvious.

**Theorem 4.3.14** If $f: X \to Y$ and $g: Y \to Z$ are two mappings such that $g \circ f: Y \to Z$ is strongly $\hat{\eta}^*$-closed. Then

1) $f$ is $\hat{\eta}^*$ irresolute and surjective implies that $g$ is strongly $\hat{\eta}^*$-closed.

2) $g$ is an $\hat{\eta}^*$-irresolute injection implies that $f$ is strongly $\hat{\eta}^*$-closed.

**Proof:** It is similar to the proof of the Proposition 4.2.14.

**Theorem 4.3.15** Assume that $\hat{\eta}^*\circ(\tau)$ is closed under any union. If $f: X \to Y$ is a continuous strongly $\hat{\eta}^*$-closed bijective map and $X$ is an ultra $\hat{\eta}^*$-regular space, then $Y$ is ultra $\hat{\eta}^*$-regular.

**Proof:** Let $y \in Y$ and $V$ be an open set containing $y$. Let $x$ be a point of $X$ such that $y = f(x)$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. By Theorem 4.2.16, there exists an $\hat{\eta}^*$-open set $U$ such that $x \in U \subseteq \hat{\eta}^*\text{cl}(U) \subseteq f^{-1}(V)$. Then $y \in f(U) \subseteq f(\hat{\eta}^*\text{cl}(U)) \subseteq V$. By Proposition 3.2.19, $\hat{\eta}^*\text{cl}(U)$ is $\hat{\eta}^*$-closed. Since $f$ is a strongly $\hat{\eta}^*$-closed map, $f(\hat{\eta}^*\text{cl}(U))$ is an $\hat{\eta}^*$-closed set. Since every open set is $\omega$-open [50], $V$ is $\omega$-open. Therefore, we have $\hat{\eta}^*\text{cl}(f(\hat{\eta}^*\text{cl}(U))) \subseteq \text{spcl}(f(\hat{\eta}^*\text{cl}(U))) \subseteq V$. This implies that $y \in f(U) \subseteq \hat{\eta}^*\text{cl}(f(U)) \subseteq \hat{\eta}^*\text{cl}(f(\hat{\eta}^*\text{cl}(U))) \subseteq V$. That is, $y \in f(U) \subseteq \hat{\eta}^*\text{cl}(f(U)) \subseteq V$. Now, $U$ is $\hat{\eta}^*$-open.
implies that $U^c$ is $\ast$-closed in $X$. Since $f$ is strongly $\ast$-closed, $f(U^c)$ is $\ast$-closed in $Y$. That is, $(f(U))^c$ is $\ast$-closed in $Y$. This implies that $f(U)$ is $\ast$-open in $Y$. Thus for every point $y$ of $Y$ and every open set $V$ containing $y$ there exists an $\ast$-open set $f(U)$ such that $y \in f(U) \subset \ast\text{cl}(f(U)) \subset V$. Hence by Theorem 4.2.16, $Y$ is ultra $\ast$-regular.

**Theorem 4.3.16** If $f: X \to Y$ is a continuous quasi-$\ast$-closed surjection and $X$ is an ultra $\ast$-normal space, then $Y$ is normal.

**Proof:** Let $A$ and $B$ be disjoint closed sets in $Y$. Since $X$ is ultra $\ast$-normal, there exist disjoint $\ast$-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Theorem 4.3.10, there exist open sets $G$ and $H$ of $Y$ such that $A \subset G$, $B \subset H$ and $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and hence $G \cap H = \emptyset$. Thus $Y$ is normal.

**Theorem 4.3.17** Let $f: X \to Y$ be a bijective map. Then following hold:

1) If $f$ is a strongly $\ast$-closed map and $Y$ is an $\ast$-connected space, then $X$ is $\ast$-connected.

2) If $f$ is a quasi-$\ast$-closed map and $Y$ is a connected space, then $X$ is $\ast$-connected.

**Proof:** Similar to the proof of the Theorem 4.2.21.

**Proposition 4.3.18** Let $f: X \to Y$ from a space $X$ to a $T\ast$-space $Y$. Then the following are equivalent:

1) $f$ is strongly $\ast$-closed,

2) $f$ is quasi-$\ast$-closed.

**Proof:** Follows by Proposition 4.3.3 and by Definition 2.5.1.

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4.4 $\hat{n}^*$-homeomorphisms

M. Lellis Thivagar [23] introduced the concepts of quasi $\alpha$-open and strongly $\alpha$-open mappings using $\alpha$-sets. In this section $\hat{n}^*$-open maps, quasi-$\hat{n}^*$-open maps and strongly $\hat{n}^*$-open maps in topological spaces have been introduced and also obtained the characterizations of these maps. Further two new homeomorphisms namely $\hat{n}^*$-homeomorphisms and strongly $\hat{n}^*$-homeomorphisms ($S\hat{n}^*$-homeomorphisms) have been studied. The set of all $S\hat{n}^*$-homeomorphisms form a group under the operation composition of maps and $S\hat{n}^*$-homeomorphisms is an equivalence relation between topological spaces have been proved.

**Definition 4.4.1** A map $f : X \rightarrow Y$ is said to be an $\hat{n}^*$-open map if the image $f(A)$ is $\hat{n}^*$-open in $Y$ for each open set $A$ in $X$.

**Example 4.4.2** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an identity map. Here $\hat{n}^*o(\sigma) = P(X) - \{b, c\}$. Then $f$ is an $\hat{n}^*$-open map.

**Theorem 4.4.3** A surjective map $f : X \rightarrow Y$ is $\hat{n}^*$-open if and only if for any subset $S$ of $Y$ and for any closed set $F$ containing $f^{-1}(S)$, there exists an $\hat{n}^*$-closed set $K$ of $Y$ containing $S$ such that $f^{-1}(K) \subset F$.

**Proof:** Similar to the proof of the Theorem 4.2.6.

**Theorem 4.4.4** For any bijection $f : X \rightarrow Y$, the following conditions are equivalent:

i) $f^{-1} : Y \rightarrow X$ is $\hat{n}^*$-continuous,

ii) $f$ is an $\hat{n}^*$-open map,
iii) \( f \) is an \( \tilde{\eta}^* \)-closed map.

Proof:

(i) \( \Rightarrow \) (ii): Let \( U \) be an open set of \( X \).

By assumption \( (f^{-1})^{-1}(U) = f(U) \) is \( \tilde{\eta}^* \)-open in \( Y \) and so \( f \) is \( \tilde{\eta}^* \)-open.

(ii) \( \Rightarrow \) (iii): Let \( F \) be a closed set of \( X \). Then \( F^c \) is open in \( X \). By (ii) \( f(F^c) \) is \( \tilde{\eta}^* \)-open in \( Y \) and therefore \( f(F^c) = (f(F))^c \) is \( \tilde{\eta}^* \)-open in \( Y \). Thus \( f(F) \) is \( \tilde{\eta}^* \)-closed in \( Y \) implies that \( f \) is \( \tilde{\eta}^* \)-closed.

(iii) \( \Rightarrow \) (i): Let \( F \) be a closed set in \( X \). By (iii), \( f(F) \) is \( \tilde{\eta}^* \)-closed in \( Y \). But \( f(F) = (f^{-1})^{-1}(F) \) and therefore \( f^{-1} \) is \( \tilde{\eta}^* \)-continuous.

Definition 4.4.5 A map \( f: X \to Y \) is said to be strongly \( \tilde{\eta}^* \)-open if the image of every \( \tilde{\eta}^* \)-open set in \( X \) is \( \tilde{\eta}^* \)-open in \( Y \).

Definition 4.4.6 A map \( f: X \to Y \) is said to be quasi-\( \tilde{\eta}^* \)-open if the image of every \( \tilde{\eta}^* \)-open set in \( X \) is open in \( Y \).

Theorem 4.4.7 A surjective map \( f: X \to Y \) is quasi-\( \tilde{\eta}^* \)-open if and only if for any subset \( B \) of \( Y \) and any \( \tilde{\eta}^* \)-closed set \( F \) of \( X \) containing \( f^{-1}(B) \), there exists a closed set \( G \) of \( Y \) containing \( B \) such that \( f^{-1}(G) \subset F \).

Proof: Suppose that \( f \) is quasi-\( \tilde{\eta}^* \)-open. Let \( B \subset Y \) and \( F \) be an \( \tilde{\eta}^* \)-closed set of \( X \) containing \( f^{-1}(B) \). Now, put \( G = (f(F))^c \). Then \( G \) is a closed set of \( Y \) containing \( B \) such that \( f^{-1}(G) \subset F \).

Conversely, let \( U \) be an \( \tilde{\eta}^* \)-open set of \( X \) and put \( B = (f(U))^c \). Then \( U^c \) is an \( \tilde{\eta}^* \)-closed set in \( X \) containing \( f^{-1}(B) \). By hypothesis, there exists a closed set \( F \) of \( Y \) such that \( B \subset F \) and \( f^{-1}(F) \subset U^c \). Hence, we obtain \( f(U) \subset F^c \). On the other hand it follows that \( B \subset F, F^c \subset B^c = f(U) \). Thus we obtain \( f(U) = F^c \) which is open in \( Y \) and hence \( f \) is quasi-\( \tilde{\eta}^* \)-open map.
Remark 4.4.8  From the above definitions we obtain the following implications.

quasi-$\mathcal{H}^*$-open $\Rightarrow$ strongly $\mathcal{H}^*$-open $\Rightarrow$ $\mathcal{H}^*$-open. However the reverse implications are not true by the following examples.

Example 4.4.9  Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$, $\sigma = \{X, \emptyset, \{a\}, \{b, c\}\}$. Here $\mathcal{H}^*o(\sigma) = P(X)$ and $\mathcal{H}^*o(\tau) = P(X) - \{b, c\}$. Define a map $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then $f$ is a strongly $\mathcal{H}^*$-open map but $f$ is not a quasi-$\mathcal{H}^*$-open map since $\{b\}$ is $\mathcal{H}^*$-open in $X$, but $f(\{b\}) = \{c\}$ is not open in $Y$.

Example 4.4.10  Let $(X, \tau)$ and $(Y, \sigma)$ be defined as in Example 4.4.2. Here $\mathcal{H}^*o(\tau) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be defined as $f(a) = b$, $f(b) = a$ and $f(c) = c$. Clearly $f$ is $\mathcal{H}^*$-open but $f$ is not strongly $\mathcal{H}^*$-open, since $\{a, c\}$ is $\mathcal{H}^*$-open in $X$ but $f(\{a, c\}) = \{b, c\}$ is not $\mathcal{H}^*$-open in $Y$.

Theorem 4.4.11  For any bijection $f: X \rightarrow Y$, the following conditions are equivalent:

i) $f^{-1}: Y \rightarrow X$ is $\mathcal{H}^*$-irresolute,

ii) $f$ is a strongly $\mathcal{H}^*$-open map,

iii) $f$ is a strongly $\mathcal{H}^*$-closed map.

Proof: Similar to the proof of the Theorem 4.4.4.

Definition 4.4.12  A bijection $f: X \rightarrow Y$ is called $\mathcal{H}^*$-homeomorphisms if $f$ is both $\mathcal{H}^*$-continuous and $\mathcal{H}^*$-open.

Proposition 4.4.13  Every homeomorphism is an $\mathcal{H}^*$-homeomorphism but not conversely.
Proof: Follows from Definitions.

Example 4.4.14 Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a map defined by \( f(a) = b \), \( f(b) = c \) and \( f(c) = a \). Clearly \( f \) is bijective, \( \hat{\eta}^* \)-continuous and \( \hat{\eta}^* \)-open. Hence \( f \) is an \( \hat{\eta}^* \)-homeomorphism but \( f \) is not a homeomorphism, since \( \{b, c\} \) is open in \( X \), \( f(\{b, c\}) = \{a, c\} \) is not open in \( Y \), hence \( f \) is not an open map.

Theorem 4.4.15 Let \( f: X \rightarrow Y \) be a bijective, \( \hat{\eta}^* \)-continuous map. Then the following are equivalent:

i) \( f \) is an \( \hat{\eta}^* \)-open map,

ii) \( f \) is an \( \hat{\eta}^* \)-homeomorphism,

iii) \( f \) is an \( \hat{\eta}^* \)-closed map.

Proof:

(i) \( \iff \) (ii): Obvious from definition.

(ii) \( \iff \) (iii): Suppose that \( f \) is an \( \hat{\eta}^* \)-open map and let \( F \) be a closed set in \( X \). Then \( F^c \) is open in \( X \), hence \( f(F^c) = (f(F))^c \) is \( \hat{\eta}^* \)-open in \( Y \) implies that \( f \) is a closed map. Converse follows by the same technique.

Remark 4.4.16 The composition of two \( \hat{\eta}^* \)-homeomorphisms need not be an \( \hat{\eta}^* \)-homeomorphism as seen from the following example.

Example 4.4.17 Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \), \( \sigma = \{Y, \phi, \{a\}, \{b, c\}\} \) and \( \eta = \{Z, \phi, \{c\}\} \). Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be identity maps. Clearly \( f \) and \( g \) are \( \hat{\eta}^* \)-homeomorphisms but their composition \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is not an \( \hat{\eta}^* \)-homeomorphism, because for
the open set \{a, b\} in X, \(g \circ f\)\({a, b}\) = \{a, b\} which is not an \(\hat{\eta}^*\)-open map and so \(g \circ f\) is not an \(\hat{\eta}^*\)-homeomorphism.

**Definition 4.4.18** A bijection \(f: X \rightarrow Y\) is said to be strongly \(\hat{\eta}^*\)-homeomorphism if both \(f\) and \(f^{-1}\) are \(\hat{\eta}^*\)-irresolute.

**Example 4.4.19** Let \(X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}\) and \(\sigma = \{Y, \phi, \{a, b\}\}\). Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be an identity map. Then \(f\) is a strongly \(\hat{\eta}^*\)-homeomorphism.

We denote the family of all \(\hat{\eta}^*\)-homeomorphisms (resp. strongly \(\hat{\eta}^*\)-homeomorphism) of a topological space \(X\) onto itself by \(\hat{\eta}^*\)-h\((X)\) (resp. \(S\hat{\eta}^*\)-h\((X)\)).

**Proposition 4.4.20** Every strongly \(\hat{\eta}^*\)-homeomorphism is an \(\hat{\eta}^*\)-homeomorphism but not conversely. In otherwords for any space \(X\), \(S\hat{\eta}^*\)-h\((X)\) \subset \(\hat{\eta}^*\)-h\((X)\).

**Proof:** Since every \(\hat{\eta}^*\)-irresolute map is \(\hat{\eta}^*\)-continuous and also from remark 4.4.8, we get the proof.

**Example 4.4.21** Let \(X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}\}\) and \(\sigma = \{Y, \phi, \{a\}\}\). Here \(\hat{\eta}^*\)o\((\tau)\) = \(P(X)\) and \(\hat{\eta}^*\)o\((\sigma)\) = \(P(X) - \{b, c\}\). Let \(f: (X, \tau) \rightarrow (Y, \sigma)\) be the map defined by \(f(a) = b, f(b) = c\) and \(f(c) = a\). Clearly \(f\) is an \(\hat{\eta}^\ast\)-homeomorphism. But \(f\) is not a strongly \(\hat{\eta}^*\)-homeomorphism, since \(\{a, b\}\) is \(\hat{\eta}^*\)-open in \(X\) but \((f^{-1})^{-1}\)\((\{a, b\}) = \{b, c\}\) is not \(\hat{\eta}^*\)-open in \(Y\). Hence \(f^{-1}\) is not a \(\hat{\eta}^*\)-irresolute and so \(f\) is not a strongly \(\hat{\eta}^*\)-homeomorphism.
Proposition 4.4.22 If \( f: X \to Y \) and \( g: Y \to Z \) are two strongly \( \tilde{\eta}^* \)-homeomorphisms then their composition \( g \circ f: X \to Z \) is also a strongly \( \tilde{\eta}^* \)-homeomorphism.

Proof: Let \( U \) be an \( \tilde{\eta}^* \)-open set in \( Z \). Now \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V) \) where \( V = g^{-1}(U) \). By hypothesis, \( V \) is \( \tilde{\eta}^* \)-open in \( Y \) and so again by hypothesis \( f^{-1}(V) \) is \( \tilde{\eta}^* \)-open in \( X \). Thus \( g \circ f \) is \( \tilde{\eta}^* \)-irresolute. Also for an \( \tilde{\eta}^* \)-open set \( G \) in \( X \), we have \( (g \circ f)(G) = g(f(G)) = g(W) \) where \( W = f(G) \), by hypothesis \( f(G) \) is \( \tilde{\eta}^* \)-open in \( Y \) and so again by hypothesis, \( g(f(G)) \) is \( \tilde{\eta}^* \)-open in \( Z \). Thus \( (g \circ f)^{-1} \) is \( \tilde{\eta}^* \)-irresolute. Hence, \( g \circ f \) is a strongly \( \tilde{\eta}^* \)-homeomorphism.

Theorem 4.4.23 The set \( S\tilde{\eta}^* \cdot h(X) \) is a group under the composition of maps.

Proof: Define a binary operation \( \circ : S\tilde{\eta}^* \cdot h(X) \times S\tilde{\eta}^* \cdot h(X) \to S\tilde{\eta}^* \cdot h(X) \), by \( f \circ g = g \circ f \) for all \( f \) and \( g \) in \( S\tilde{\eta}^* \cdot h(X) \) and \( \circ \) is the usual operation of composition of maps. Then by Proposition 4.4.22, \( g \circ f \in S\tilde{\eta}^* \cdot h(X) \). We know that the composition of maps is associative and the identity map \( i: X \to X \) belonging to \( S\tilde{\eta}^* \cdot h(X) \) serves as the identity element. If \( f \in S\tilde{\eta}^* \cdot h(X) \) then \( f^{-1} \in S\tilde{\eta}^* \cdot h(X) \) such that \( f \circ f^{-1} = f^{-1} \circ f = i \) and so inverse exists for each element of \( S\tilde{\eta}^* \cdot h(X) \). Therefore, \( (S\tilde{\eta}^* \cdot h(X), \circ) \) is a group under the composition of maps.

Theorem 4.4.24 Let \( f: X \to Y \) be an \( S\tilde{\eta}^* \)-homeomorphism. Then \( f \) induces an isomorphism from the group \( S\tilde{\eta}^* \cdot h(X) \) onto the group \( S\tilde{\eta}^* \cdot h(Y) \).

Proof: Using the map \( f \), we define a map \( \psi_f : S\tilde{\eta}^* \cdot h(X) \to S\tilde{\eta}^* \cdot h(Y) \) by \( \psi_f(h) = f \circ h \circ f^{-1} \) for each \( h \in S\tilde{\eta}^* \cdot h(X) \). Then \( \psi_f \) is a bijection, further for all \( h_1, h_2 \in S\tilde{\eta}^* \cdot h(X) \), \( \psi_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \psi_f(h_1) \circ \psi_f(h_2) \). Therefore, \( \psi_f \) is a homomorphism and so it induces an isomorphism induced by \( f \).
Theorem 4.4.25  $\mathring{\eta}^*$-homeomorphism is an equivalence relation on the collection of all topological spaces.

Proof:  Reflexivity and symmetry are immediate and transitivity follows from Proposition 4.4.22.

4.5 $\mathring{\eta}^*$-quotient map

Lellis Thivagar [22] introduced the concepts of $\alpha$-quotient map, semi quotient map and pre quotient map. In this section, we introduce the concepts of $\mathring{\eta}^*$-quotient map which is weaker than $\mathring{\eta}^*$-homeomorphism. Further strongly $\mathring{\eta}^*$-quotient map and completely $\mathring{\eta}^*$-quotient map are introduced and the relationships between these maps are obtained.

Definition 4.5.1  A surjective map $f: X \to Y$ is said to be an $\mathring{\eta}^*$-quotient map if $f$ is $\mathring{\eta}^*$-continuous and $f^{-1}(V)$ is open in $X$ implies that $V$ is $\mathring{\eta}^*$-open in $Y$.

The following proposition is an easy consequence from the definitions

Proposition 4.5.2  Every quotient map is $\mathring{\eta}^*$-quotient but not conversely.

Proof:  The proof follows from the Definitions.

Example 4.5.3  Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. Here $\mathring{\eta}^*(\tau) = P(X) - \{c, d\}$ and $\mathring{\eta}^*(\sigma) = P(X) - \{b, c\}$. Then the map $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = a$, $f(b) = b$ and $f(c) = f(d) = c$ is an $\mathring{\eta}^*$-quotient map but not a quotient map.

Proposition 4.5.4  If a map $f: X \to Y$ is surjective, $\mathring{\eta}^*$-continuous and $\mathring{\eta}^*$-open, then $f$ is an $\mathring{\eta}^*$-quotient map.
Proof: We only need to prove that \( f^{-1}(V) \) is open in \( X \) implies that \( V \) is an \( \hat{\eta}^* \)-open set in \( Y \). Let \( f^{-1}(V) \) be open in \( X \). Then \( f(f^{-1}(V)) \) is an \( \hat{\eta}^* \)-open set, since \( f \) is \( \hat{\eta}^* \)-open. Hence, \( V \) is an \( \hat{\eta}^* \)-open set, as \( f \) is surjective and \( f(f^{-1}(V)) = V \). Thus \( f \) is an \( \hat{\eta}^* \)-quotient map.

**Proposition 4.5.5** If a map \( f: X \to Y \) is a homeomorphism, then \( f \) is a quotient map but not conversely.

**Proof:** Clearly follows from definition.

**Example 4.5.6** Let \((X, \tau), (Y, \sigma)\) and \( f \) be defined as in Example 4.5.3. Here \( f \) is an \( \hat{\eta}^* \)-quotient map but not a homeomorphism, since \( f \) is not injective.

**Proposition 4.5.7** Let \( f: X \to Y \) be an open surjective \( \hat{\eta}^* \)-irresolute map and \( g: Y \to Z \) be an \( \hat{\eta}^* \)-quotient map. Then the composition \( g \circ f: X \to Z \) is an \( \hat{\eta}^* \)-quotient map.

**Proof:** Let \( V \) be any open set in \( Z \). Then \( g^{-1}(V) \) is an \( \hat{\eta}^* \)-open set, since \( g \) is an \( \hat{\eta}^* \)-quotient map. Since \( f \) is \( \hat{\eta}^* \)-irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is an \( \hat{\eta}^* \)-open set in \( X \), which implies that \( (g \circ f)^{-1}(V) \) is an \( \hat{\eta}^* \)-open set. This shows that \( g \circ f \) is \( \hat{\eta}^* \)-continuous. Also, assume that \( (g \circ f)^{-1}(V) \) is open in \( X \) for \( V \subset Z \), that is, \( f^{-1}(g^{-1}(V)) \) is open in \( X \). Since \( f \) is open, \( f(f^{-1}(g^{-1}(V))) \) is open in \( Y \). It follows that \( g^{-1}(V) \) is open in \( Y \), because \( f \) is surjective. Since \( g \) is a \( \hat{\eta}^* \)-quotient map, \( V \) is an \( \hat{\eta}^* \)-open set. Thus \( g \circ f: X \to Z \) is an \( \hat{\eta}^* \)-quotient map.

**Proposition 4.5.8** If \( h: X \to Y \) is an \( \hat{\eta}^* \)-quotient map and \( g: X \to Z \) is a continuous map where \( Z \) is a space that is constant on each set \( h^{-1} \{ \{y\} \} \), for each \( y \in Y \), then \( g \) induces an \( \hat{\eta}^* \)-continuous map \( f: Y \to Z \) such that \( f \circ h = g \).

**Proof:** Since \( g \) is constant on \( h^{-1} \{ \{y\} \} \), for each \( y \in Y \), the set \( g(h^{-1}(\{y\})) \) is an one point set in \( Z \). If we let \( f(y) \) to denote this point, then it is clear that \( f \) is well
defined and for each \( x \in X \), \( f(h(x)) = g(x) \). We claim that \( f \) is \( \hat{\eta}^* \)-continuous. For if we let \( V \) be any open set in \( Z \), then \( g^{-1}(V) \) is an open set as \( g \) is continuous. But \( g^{-1}(V) = h^{-1}(f^{-1}(V)) \) is open in \( X \). Since \( h \) is an \( \hat{\eta}^* \)-quotient map, \( f^{-1}(V) \) is an \( \hat{\eta}^* \)-open set in \( Y \).

**Definition 4.5.9** A surjective map \( f: X \to Y \) is said to be a strongly \( \hat{\eta}^* \)-quotient map if \( f \) is \( \hat{\eta}^* \)-continuous and \( f^{-1}(V) \) is \( \hat{\eta}^* \)-open in \( X \) implies that \( V \) is \( \hat{\eta}^* \)-open in \( Y \).

**Proposition 4.5.10** Every strongly \( \hat{\eta}^* \)-quotient map is an \( \hat{\eta}^* \)-quotient map.

**Proof:** Let \( f: X \to Y \) be a strongly \( \hat{\eta}^* \)-quotient map. Let \( f^{-1}(V) \) be open in \( X \). Then \( f^{-1}(V) \) is an \( \hat{\eta}^* \)-open in \( X \). Since \( f \) is a strongly \( \hat{\eta}^* \)-quotient map, \( V \) is \( \hat{\eta}^* \)-open in \( Y \). This shows that \( f \) is an \( \hat{\eta}^* \)-quotient map.

**Remark 4.5.11** The converse of the above Proposition need not be true in general as shown in the following example.

**Example 4.5.12** The map \( f: (X, \tau) \to (Y, \sigma) \) is defined as in Example 4.5.3 is an \( \hat{\eta}^* \)-quotient map but not a strongly \( \hat{\eta}^* \)-quotient map, since \( f^{-1}(V) = \{b, c, d\} \) is \( \hat{\eta}^* \)-open in \( X \) but \( V = \{b, c\} \) is not \( \hat{\eta}^* \)-open in \( Y \).

**Definition 4.5.13** Let \( f: X \to Y \) be a surjective map. Then \( f \) is called a completely \( \hat{\eta}^* \)-quotient map if \( f \) is \( \hat{\eta}^* \)-irresolute and \( f^{-1}(U) \) is \( \hat{\eta}^* \)-open in \( X \) implies that \( U \) is open in \( Y \).

**Theorem 4.5.14** Let \( f: X \to Y \) be a surjective strongly \( \hat{\eta}^* \)-open and \( \hat{\eta}^* \)-irresolute map and \( g: Y \to Z \) be a completely \( \hat{\eta}^* \)-quotient map. Then \( g \circ f \) is a completely \( \hat{\eta}^* \)-quotient map.

86
Proof: Since $f$ and $g$ are $\tilde{\eta}^*$-irresolute, $g \circ f$ is $\tilde{\eta}^*$-irresolute, by Proposition 3.3.6(a). Suppose that $(g \circ f)^{-1}(V)$ is an $\tilde{\eta}^*$-open set in $X$ for $V \subseteq Z$, that is, $f^{-1}(g^{-1}(V))$ is an $\tilde{\eta}^*$-open set in $X$. Since $f$ is surjective and strongly $\tilde{\eta}^*$-open, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $\tilde{\eta}^*$-open in $Y$. Also $g$ is completely $\tilde{\eta}^*$-quotient implies that $V$ is open in $Z$. Thus $g \circ f$ is a completely $\tilde{\eta}^*$-quotient map.

Proposition 4.5.15 Every completely $\tilde{\eta}^*$-quotient map is a strongly $\tilde{\eta}^*$-quotient map.

Proof: Let $f: X \to Y$ be a completely $\tilde{\eta}^*$-quotient map. By Proposition 3.3.3, $f$ is $\tilde{\eta}^*$-irresolute implies that $f$ is $\tilde{\eta}^*$-continuous. Hence the proof follows.

Remark 4.5.16 The converse of the above Proposition need not be true in general as shown in the following example.

Example 4.5.17 Let $X = \{a, b, c\} = Y$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Here $\tilde{\eta}^*o(\tau) = \{ X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tilde{\eta}^*o(\sigma) = P(X) - \{b, c\}$. Define a map $f: (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Clearly $f$ is strongly $\tilde{\eta}^*$-quotient map. But $f$ is not a completely $\tilde{\eta}^*$-quotient map, since $\{a, b\}$ is $\tilde{\eta}^*$-closed in $(Y, \sigma)$. But $f^{-1}(\{a, b\}) = \{a, c\}$ is not $\tilde{\eta}^*$-closed in $(X, \tau)$, implies that $f$ is not $\tilde{\eta}^*$-irresolute.

Theorem 4.5.18 Let $f: X \to Y$ be a surjective map and both $X$ and $Y$ be $\tilde{T}_{\tilde{\eta}^*}$-spaces. Then the following are equivalent:

(i) $f$ is a completely $\tilde{\eta}^*$-quotient map;

(ii) $f$ is a strongly $\tilde{\eta}^*$-quotient map;

(iii) $f$ is a $\tilde{\eta}^*$-quotient map.

Proof:
(i) $\Rightarrow$ (ii) : Follows by Proposition 4.5.15.

(ii) $\Rightarrow$ (iii) : Follows by Proposition 4.5.10.

(iii) $\Rightarrow$ (i) : Since $Y$ is a $T_{\eta^*}$ space, $f$ is $\eta^*$-irresolute, by Proposition 3.3.7.

Suppose that $f^{-1}(V)$ is $\eta^*$-open in $X$. Since $X$ is a $T_{\eta^*}$ space, $f^{-1}(V)$ is open in $X$. By (iii), $V$ is $\eta^*$-open in $Y$. Since $Y$ is a $T_{\eta^*}$ space, $V$ is open in $Y$. Hence, we get (i).

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4.6 $\eta^*$-regular and $\eta^*$-normal spaces

Munshi [34] introduced $g$-regular and $g$-normal spaces using $g$-closed sets in topological spaces. Noiri and Popa [40] have further investigated the result of Munshi. In this section some new spaces namely $\eta^*$-regular and $\eta^*$-normal spaces in topological spaces are introduced and some of their characterizations are obtained.

**Definition 4.6.1** A space $X$ is said to be $\eta^*$-regular if for every $\eta^*$-closed set $F$ and each point $x \in F$, there exist disjoint semi-preopen sets $U$ and $V$ such that $F \subseteq U$ and $x \in V$.

**Example 4.6.2** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$. Here $\eta^*c(\tau) = P(X) - \{a, b, d\}$. SPO($X$) = $P(X) - \{c\}$. Clearly $X$ is $\eta^*$-regular.

**Example 4.6.3** Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Here $\eta^*c(\tau) = P(X) - \{a\}$. SPO($X$) = $\{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Clearly $(X, \tau)$ is not $\eta^*$-regular, since $b \not\in \{a, c\}$ and $\{a, c\}$ is an $\eta^*$-closed set, but there are no disjoint semi-preopen sets containing the point $b$ and the set $\{a, c\}$. 

88
Theorem 4.6.4  
If $X$ is an $\mathcal{H}^*$-regular space and $Y$ is a clopen subset of $X$, then the subspace $Y$ is $\mathcal{H}^*$-regular.

Proof: Let $F$ be any $\mathcal{H}^*$-closed subset of $Y$ and $y \in F^c$. By Theorem 2.2.28, $F$ is $\mathcal{H}^*$-closed in $X$. Since $X$ is $\mathcal{H}^*$-regular, there exist disjoint semi-preopen sets $U$ and $V$ of $X$ such that $y \in U$ and $F \subseteq V$. By Lemma 1.1.16 and also $Y$ is open hence $\alpha$-open, we get $U \cap Y$ and $V \cap Y$ are disjoint semi-preopen sets of the subspace $Y$ such that $y \in U \cap Y$ and $F \subseteq V \cap Y$. Hence, the subspace $Y$ is $\mathcal{H}^*$-regular.

Theorem 4.6.5  
Let $X$ be a topological space. Then the following statements are equivalent:

(i). $X$ is $\mathcal{H}^*$-regular.

(ii). For each point $x \in X$ and for each $\mathcal{H}^*$-open neighbourhood $W$ of $x$ (there exists an $\mathcal{H}^*$-open set $G$ such that $x \in G \subseteq W$), there exists a semi-preopen set $U$ of $X$ such that $spcl(U) \subseteq W$.

(iii). For each point $x \in X$ and for each $\mathcal{H}^*$-closed set $F$ not containing $x$, there exists a semi-preopen set $V$ of $x$ such that $spcl(V) \cap F = \emptyset$.

Proof:

(i) $\Rightarrow$ (ii): Let $W$ be any $\mathcal{H}^*$-open neighbourhood of $x$. Then there exists an $\mathcal{H}^*$-open set $G$ such that $x \in G \subseteq W$. Since $G^c$ is $\mathcal{H}^*$-closed and $x \in G^c$, by hypothesis there exist semi-preopen sets $U$ and $V$ such that $G^c \subseteq U$, $x \in V$ and $U \cap V = \emptyset$ and so $V \subseteq U^c$. By Lemma 1.1.5, $spcl(V) \subseteq spcl(U^c) = U^c$ and $G^c \subseteq U$ implies that $U^c \subseteq G \subseteq W$. Therefore, $spcl(V) \subseteq W$.

(ii) $\Rightarrow$ (i): Let $F$ be any $\mathcal{H}^*$-closed set and $x \in F$. Then $x \in F^c$ and $F^c$ is $\mathcal{H}^*$-open and so $F^c$ is an $\mathcal{H}^*$-neighbourhood of $x$. By hypothesis, there exists a semi-preopen set $V$ of $x$ such that $x \in V$ and $spcl(V) \subseteq F^c$, which implies that $F \subseteq (spcl(V))^c$. 

89
Then $(\text{spcl}(V))^c$ is a semi-preopen set containing $F$ and $V \cap (\text{spcl}(V))^c = \emptyset$. Therefore, $X$ is $\hat{\eta}^*$-regular.

(ii) $\Rightarrow$ (iii): Let $x \in X$ and $F$ be an $\hat{\eta}^*$-closed set such that $x \notin F$. Then $F^c$ is an $\hat{\eta}^*$-open neighbourhood of $x$ and by hypothesis, there exists a semi-preopen set $V$ of $x$ such that $\text{spcl}(V) \subseteq F^c$ and hence $\text{spcl}(V) \cap F = \emptyset$.

(iii) $\Rightarrow$ (ii): Let $x \in X$ and $W$ be an $\hat{\eta}^*$-open neighbourhood of $x$. Then there exists an $\hat{\eta}^*$-open set $G$ such that $x \in G \subseteq W$. Since $G^c$ is $\hat{\eta}^*$-closed and $x \notin G^c$, by hypothesis there exists a semi-preopen set $U$ of $x$ such that $\text{spcl}(U) \cap G^c = \emptyset$. Therefore, $\text{spcl}(U) \subseteq G \subseteq W$.

**Theorem 4.6.6** Assume that $\hat{\eta}^*\circ(\tau)$ is closed under any union. Then the following are equivalent:

(i) $X$ is $\hat{\eta}^*$-regular;

(ii) $\text{spcl}_\theta(A) = \hat{\eta}^*\text{cl}(A)$ for every subset $A$ of $X$;

(iii) $\text{spcl}_\theta(A) = A$ for every $\hat{\eta}^*$-closed set $A$.

**Proof:**

(i) $\Rightarrow$ (ii): For any subset $A$ of $X$, we always have $A \subseteq \hat{\eta}^*\text{cl}(A) \subseteq \text{spcl}_\theta(A)$.

Let $x \in (\hat{\eta}^*\text{cl}(A))^c$. Then there exists an $\hat{\eta}^*$-closed set $F$ such that $x \in F^c$ and $A \subseteq F$. By assumption, there exist disjoint semi-preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$. Now $x \in U \subseteq \text{spcl}(U) \subseteq V^c \subseteq F^c \subseteq A^c$ and therefore $\text{spcl}(U) \cap A = \emptyset$. Thus $x \in (\text{spcl}_\theta(A))^c$ and hence $\text{spcl}_\theta(A) = \hat{\eta}^*\text{cl}(A)$.

(ii) $\Rightarrow$ (iii): Clearly follows by (ii) and by Proposition 3.2.18.

(iii) $\Rightarrow$ (i): Let $F$ be any $\hat{\eta}^*$-closed set and $x \in F^c$. Since $F$ is an $\hat{\eta}^*$-closed set, by assumption $x \in (\text{spcl}_\theta(F))^c$ and so there exists a semi-preopen set $U$ such that $x \in U$ and $\text{spcl}(U) \cap F = \emptyset$. Then $F \subseteq (\text{spcl}(U))^c$. Let $V = (\text{spcl}(U))^c$. Then $V$ is
semi-pre open such that $F \subseteq V$. Also the sets $U$ and $V$ are disjoint and hence $X$ is $\hat{\eta}^*$-regular.

**Theorem 4.6.7** If $X$ is a $\hat{\eta}^*$-regular space and $f: X \to Y$ is bijective, $\omega^*$-open, $\beta$-irresolute and pre $\beta$-open, then $(Y, \sigma)$ is $\hat{\eta}^*$-regular.

**Proof:** Let $F$ be an $\hat{\eta}^*$-closed subset of $Y$ and $y \notin F$. By Theorem 3.3.8, the map $f$ is $\hat{\eta}^*$-irresolute and hence $f^{-1}(F)$ is $\hat{\eta}^*$-closed in $(X, \tau)$. Since $f$ is bijective, let $f(x) = y$, then $x \notin f^{-1}(F)$. By hypothesis, there exist disjoint semi-preopen sets $U$ and $V$ such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since $f$ is pre-$\beta$-open and bijective, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. This shows that the space $Y$ is also $\hat{\eta}^*$-regular.

**Theorem 4.6.8** If $f: X \to Y$ is $\omega$-irresolute pre-$\beta$-closed $\beta$-irresolute injection and $Y$ is $\hat{\eta}^*$-regular, then $X$ is $\hat{\eta}^*$-regular.

**Proof:** Let $F$ be any $\hat{\eta}^*$-closed set of $X$ and $x \notin F$. By hypothesis, $f$ is $\omega$-irresolute and pre-$\beta$-closed, by Theorem 4.2.9, $f(F)$ is $\hat{\eta}^*$-closed in $Y$ and $f(x) \notin f(F)$. Again $Y$ is $\hat{\eta}^*$-regular and so there exist disjoint semi-preopen sets $U$ and $V$ in $Y$ such that $f(x) \in U$ and $f(F) \subseteq V$. Since $f$ is $\beta$-irresolute, $f^{-1}(U)$ and $f^{-1}(V) \in \text{SPO}(X)$ such that $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Therefore, $X$ is $\hat{\eta}^*$-regular.

We conclude this section with the introduction of $\hat{\eta}^*$-normal space in topological spaces.

**Definition 4.6.9** A topological space $X$ is said to be $\hat{\eta}^*$-normal if for any pair of disjoint $\hat{\eta}^*$-closed sets $A$ and $B$, there exist disjoint semi-preopen sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$. 

91
Example 4.6.10  Let $X$ be defined as in Example 4.6.2. Clearly $X$ is $\tilde{\eta}^*$-normal. Also let $X$ be defined as in Example 4.6.3. Here $X$ is not $\tilde{\eta}^*$-normal, since there are no pair of disjoint semi-preopen sets containing disjoint $\tilde{\eta}^*$-closed sets $\{b\}$ and $\{c\}$.

Theorem 4.6.11  If $X$ is a $\tilde{\eta}^*$-normal space and $Y$ is a clopen subset of $X$, then the subspace $Y$ is $\tilde{\eta}^*$-normal.

Proof: Let $A$ and $B$ be any disjoint $\tilde{\eta}^*$-closed sets of $Y$. By Theorem 2.2.28, $A$ and $B$ are $\tilde{\eta}^*$-closed in $X$. Since $X$ is $\tilde{\eta}^*$-normal, there exist disjoint semi-preopen sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. Since $Y$ is open, $Y$ is $\alpha$-open. By Lemma 1.1.6, $U \cap Y$ and $V \cap Y$ are disjoint semi-preopen sets in $Y$ and so the subspace $Y$ is normal.

The next theorem is a characterization of $\tilde{\eta}^*$-normal space.

Theorem 4.6.12  Let $X$ be a topological space. Then the following statements are equivalent:

(i). $X$ is $\tilde{\eta}^*$-normal.

(ii). For each $\tilde{\eta}^*$-closed set $F$ and for each $\tilde{\eta}^*$-open set $U$ containing $F$, there exists a semi-preopen set $V$ containing $F$ such that $\text{spcl}(V) \subseteq U$.

(iii). For each pair of disjoint $\tilde{\eta}^*$-closed sets $A$ and $B$ in $X$, there exists a semi-preopen set $U$ containing $A$ such that $\text{spcl}(U) \cap B = \emptyset$.

(iv). For each pair of disjoint $\tilde{\eta}^*$-closed sets $A$ and $B$ in $X$, there exist semi-preopen sets $U$ containing $A$ and $V$ containing $B$ such that $\text{spcl}(U) \cap \text{spcl}(V) = \emptyset$.

Proof:
(i) ⇒ (ii): Let $F$ be an $\mathcal{H}^*$-closed set and $U$ be an $\mathcal{H}^*$-open set such that $F \subseteq U$. Then $F \cap U^c = \emptyset$. By assumption, there exist semi-preopen sets $V$ and $W$ such that $F \subseteq V$, $U^c \subseteq W$ and $V \cap W = \emptyset$, which implies that $\text{spc}(V) \cap W = \emptyset$.

Now, $\text{spc}(V) \cap U^c \subseteq \text{spc}(V) \cap W = \emptyset$ and so $\text{spc}(V) \subseteq U$.

(ii) ⇒ (iii): Let $A$ and $B$ be disjoint $\mathcal{H}^*$-closed sets of $X$. Since $A \cap B = \emptyset$, $A \subseteq B^c$ and $B^c$ is $\mathcal{H}^*$-open. By assumption, there exists a semi-preopen set $U$ containing $A$ such that $\text{spc}(U) \subseteq B^c$ and so $\text{spc}(U) \cap B = \emptyset$.

(iii) ⇒ (iv): Let $A$ and $B$ be any two disjoint $\mathcal{H}^*$-closed sets of $X$. Then by assumption, there exists a semi-preopen set $U$ containing $A$ such that $\text{spc}(U) \cap B = \emptyset$. Since $\text{spc}(U)$ is semi-preclosed, it is $\mathcal{H}^*$-closed and so $B$ and $\text{spc}(U)$ are disjoint $\mathcal{H}^*$-closed sets in $X$. Therefore again by assumption, there exists a semi-preopen set $V$ containing $B$ such that $\text{spc}(V) \cap \text{spc}(U) = \emptyset$.

(iv) ⇒ (i): Let $A$ and $B$ be any two disjoint $\mathcal{H}^*$-closed sets of $X$. By assumption, there exist semi-preopen sets $U$ containing $A$ and $V$ containing $B$ such that $\text{spc}(U) \cap \text{spc}(V) = \emptyset$, we have $U \cap V = \emptyset$ and thus $X$ is $\mathcal{H}^*$-normal.

**Theorem 4.6.13** If $f: X \to Y$ is a $\mathcal{H}^*$-irresolute bijective pre-$\beta$-open mapping and $X$ is $\mathcal{H}^*$-normal, then $Y$ is $\mathcal{H}^*$-normal.

**Proof:** Let $A$ and $B$ be any two disjoint $\mathcal{H}^*$-closed sets of $Y$. Since the map $f$ is $\mathcal{H}^*$-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\mathcal{H}^*$-closed sets of $X$. Also $X$ is $\mathcal{H}^*$-normal, there exist disjoint semi-preopen sets $U$ and $V$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since $f$ is pre-$\beta$-open and bijective, $f(U)$ and $f(V)$ are semi-preopen in $Y$ such that $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. Therefore, $Y$ is $\mathcal{H}^*$-normal.

**Theorem 4.6.14** If $f: X \to Y$ is $\omega$-irresolute pre-$\beta$-closed $\beta$-irresolute injection and $Y$ is $\mathcal{H}^*$-normal, then $X$ is $\mathcal{H}^*$-normal.
Proof: Let $A$ and $B$ be any two disjoint $\mathcal{Y}^*$-closed subsets of $X$. Since $f$ is $\omega$-irresolute pre-$\beta$-closed, $f(A)$ and $f(B)$ are disjoint $\mathcal{Y}^*$-closed sets of $Y$ by Theorem 4.2.9. Also $Y$ is $\mathcal{Y}^*$-normal, there exist disjoint semi-preopen sets $U$ and $V$ such that $f(A) \subseteq U$ and $f(B) \subseteq V$. Thus $A \subseteq f^{-1}(U)$, $B \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Since $f$ is $\beta$-irresolute $f^{-1}(U)$ and $f^{-1}(V)$ are semi-preopen in $X$. Hence, $X$ is $\mathcal{Y}^*$-normal.

Theorem 4.6.15 [39] A map $f: X \to Y$ is weakly $\beta$-irresolute if and only if $f^{-1}(V) \subseteq \text{spint}(f^{-1}(\text{spcl}(V))).$

Theorem 4.6.16 If $f: X \to Y$ is weakly $\beta$-irresolute $\mathcal{Y}^*$-closed injection and $Y$ is $\mathcal{Y}^*$-normal, then $X$ is sp-normal.

Proof: Let $A$ and $B$ be any two disjoint closed sets of $X$. Since $f$ is injective and $\mathcal{Y}^*$-closed, $f(A)$ and $f(B)$ are disjoint $\mathcal{Y}^*$-closed sets of $Y$. Since $Y$ is $\mathcal{Y}^*$-normal, by Theorem 4.6.12, there exist semi-preopen sets $U$ and $V$ such that $f(A) \subseteq U$, $f(B) \subseteq V$ and $\text{spcl}(U) \cap \text{spcl}(V) = \emptyset$. Since $f$ is weakly $\beta$-irresolute, $A \subseteq f^{-1}(U) \subseteq \text{spint}(f^{-1}(\text{spcl}(U)))$, $B \subseteq f^{-1}(V) \subseteq \text{spint}(f^{-1}(\text{spcl}(V)))$, by Theorem 4.6.15. Thus $\text{spint}(f^{-1}(\text{spcl}(U))) \cap \text{spint}(f^{-1}(\text{spcl}(V))) = \emptyset$. Therefore, $X$ is sp-normal.

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