CHAPTER II

FUZZY AND ANTI-FUZZY IDEALS

2.1 Introduction:

In this chapter, the theory of fuzzy and anti-fuzzy ideals using homomorphism and anti-homomorphism are studied. Some properties of fuzzy and anti-fuzzy ideals are also discussed.

2.1.1 Definition:

Let \( R \) be a ring. A fuzzy subset \( A \) of \( R \) is called a fuzzy subring of \( R \) if the following conditions are satisfied:

(i) \( A(x - y) \geq \min \{ A(x), A(y) \} \),

(ii) \( A(xy) \geq \min \{ A(x), A(y) \} \), for all \( x, y \in R \).

2.1.1 Example:

Consider the ring \( R = (\mathbb{Z}_p, +_p, \times_p) \), where \( p \) is a prime number, \( \mathbb{Z}_p = \{ 0, 1, 2, ..., (p - 1) \} \) and \( +_p \) and \( \times_p \) denote "addition modulo \( p \)" and "multiplication modulo \( p \)" respectively. Define \( A : \mathbb{Z}_p \to [0, 1] \) by

\[
A(x) =
\begin{cases} 
1, & \text{if } x \text{ is even} \\
0.5, & \text{if } x \text{ is odd}.
\end{cases}
\]

Then \( A \) is a fuzzy subring of \( \mathbb{Z}_p \).

2.1.2 Definition:

Let \( R \) be a ring. A fuzzy subring \( A \) of \( R \) is said to be a fuzzy normal subring of \( R \) if \( A(xy) = A(yx) \), for all \( x, y \in R \).
2.1.2 Example:

The set $\mathbb{Z}$ of integers under ordinary addition and multiplication is a ring. Define $A : \mathbb{Z} \rightarrow [0, 1]$ by

$$A(x) = \begin{cases} 0.8, & \text{if } x \text{ is even} \\ 0.6, & \text{if } x \text{ is odd} \end{cases}.$$ 

Then $A$ is a fuzzy normal subring of $\mathbb{Z}$.

2.1.3 Definition:

Let $R$ be a ring. A fuzzy subset $A$ of $R$ is said to be a **fuzzy ideal** of $R$ if the following conditions are satisfied:

(i) $A(x - y) \geq \min\{A(x), A(y)\},$

(ii) $A(xy) \geq \max\{A(x), A(y)\}$, for all $x, y \in R$.

2.1.3 Example:

Consider the fuzzy subset $A$ of $R$ defined by $A(x) = 0.8$, if $x \in \mathbb{Z}$ and $A(x) = 0.4$ otherwise. Then $A$ is a **fuzzy ideal** of $R$.

2.1.4 Definition:

Let $R$ be a ring. A fuzzy ideal $A$ of $R$ is said to be a **fuzzy normal ideal** of $R$ if $A(xy) = A(yx)$, for all $x, y \in R$. 
2.1.4 Example:

The set $\mathbb{R}$ of real numbers under ordinary addition and multiplication is a ring.

Define $A: \mathbb{R} \to [0, 1]$ by

$$A(x) = \begin{cases} 0.7, & \text{if } x \text{ is even} \\ 0.3, & \text{if } x \text{ is odd.} \end{cases}$$

Then $A$ is a fuzzy normal ideal of $\mathbb{R}$.

2.1.5 Definition:

Let $\mathbb{R}$ be a ring. A fuzzy subset $A$ of $\mathbb{R}$ is called an anti-fuzzy subring of $\mathbb{R}$ if the following conditions are satisfied:

(i) $A(x - y) \leq \max \{ A(x), A(y) \}$,

(ii) $A(xy) \leq \max \{ A(x), A(y) \}$, for all $x, y \in \mathbb{R}$.

2.1.6 Definition:

Let $\mathbb{R}$ be a ring. An anti-fuzzy subring $A$ of $\mathbb{R}$ is said to be an anti-fuzzy normal subring of $\mathbb{R}$ if $A(xy) = A(yx)$, for all $x, y \in \mathbb{R}$.

2.1.7 Definition:

Let $\mathbb{R}$ be a ring. A fuzzy subset $A$ of $\mathbb{R}$ is called an anti-fuzzy ideal of $\mathbb{R}$ if the following conditions are satisfied:

(i) $A(x - y) \leq \max \{ A(x), A(y) \}$,

(ii) $A(xy) \leq \min \{ A(x), A(y) \}$, for all $x, y \in \mathbb{R}$. 
2.1.8 Definition:

Let $R$ be a ring. An anti-fuzzy ideal $A$ of $R$ is said to be an anti-fuzzy normal ideal of $R$ if $A(xy) = A(yx)$, for all $x$ and $y \in R$.

2.1.9 Definition:

Let $R$ and $R'$ be any two rings. Let $f: R \rightarrow R'$ be any function and let $A$ be a fuzzy ideal in $R$, $V$ be a fuzzy ideal in $f(R) = R'$, defined by

$$V(y) = \sup_{x \in f^{-1}(y)} A(x),$$

for all $x \in R$ and $y \in R'$. $A$ is called a pre-image of $V$ under $f$ and is denoted by $f^{-1}(V)$.

2.1.10 Definition:

Let $R$ and $R'$ be any two rings. Let $f: R \rightarrow R'$ be any function and let $A$ be an anti-fuzzy ideal in $R$, $V$ be an anti-fuzzy ideal in $f(R) = R'$, defined by

$$V(y) = \inf_{x \in f^{-1}(y)} A(x),$$

for all $x \in R$ and $y \in R'$. $A$ is called a pre-image of $V$ under $f$ and is denoted by $f^{-1}(V)$.

2.1.11 Definition:

Let $A$ be an anti-fuzzy ideal, the subset $x+A$ defined by

$$(x+A)(y) = A(y-x)$$

is called a coset of the anti-fuzzy ideal $A$.

2.1.12 Definition:

Let $A$ be a fuzzy subset of $X$. For $t \in [0, 1]$, the lower level subset of $A$ is the set

$$\bar{A}_t = \{ x \in X : A(x) \leq t \}.$$
2.1.13 Definition:

Let $A$ be an anti-fuzzy ideal of a ring $R$. The ideal $A_t$, $t \in [0,1]$ and $t \geq A(0)$ is called the lower level ideal of $A$.

2.1.1 Theorem:

Let $R$ and $R'$ be any two rings with identity. Let $f : R \rightarrow R'$ be a homomorphism.

Then,

(i) $f(0) = 0'$, $f(1) = 1'$, where $0$, $1$ and $0'$, $1'$ are the identities of $R$ and $R'$ respectively.

(ii) $f(-a) = -f(a)$, for all $a \in R$.

Proof:

It is trivial.

2.1.2 Theorem:

Let $R$ and $R'$ be any two rings with identity. Let $f : R \rightarrow R'$ be an anti-homomorphism. Then

(i) $f(0) = 0'$, $f(1) = 1'$, where $0$, $1$ and $0'$, $1'$ are the identities of $R$ and $R'$ respectively, and

(ii) $f(-a) = -f(a)$, for all $a \in R$.

Proof:

Let $a \in R$. Then,

$f(a) = f(a + 0) = f(0) + f(a)$

and $f(a) = f(a \cdot 1) = f(1) \cdot f(a)$, as $f$ is an anti-homomorphism.
As \( f(a) \in R^l \), then
\[
0^l + f(a) = f(a)
\]
and  
\[
1^l \cdot f(a) = f(a)
\]
which implies that  
\[
0^l + f(a) = f(0^l) + f(a)
\]
and  
\[
1^l \cdot f(a) = f(1^l) \cdot f(a), \quad \text{by cancellation law}
\]
which implies that  
\[
f(0^l) = 0^l \quad \text{and} \quad f(1^l) = 1^l.
\]
Thus (i) is proved.

Now,  
\[
f(a) + f(-a) = f(-a + a)
\]
\[
= f(0) = 0^l.
\]
Hence  
\[
f(-a) = -f(a), \quad \text{for all} \ a \in R.
\]
Thus (ii) is proved.

### 2.2 - FUZZY AND ANTI-FUZZY IDEALS OF A RING R UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM:

#### 2.2.1 Theorem:

Let \( R \) and \( R^l \) be any two rings. The homomorphic image of a fuzzy ideal of \( R \) is a fuzzy ideal of \( R^l \).

**Proof:**

Let \( R \) and \( R^l \) be any two rings.

Let \( f: R \to R^l \) be a homomorphism.
That is \( f( x \pm y ) = f( x ) \pm f( y ) \) and \( f( xy ) = f( x )f( y ) \), for all \( x \) and \( y \in \mathbb{R} \).

Let \( V = f( A ) \), where \( A \) is a fuzzy ideal of \( \mathbb{R} \).

We have to prove that \( V \) is a fuzzy ideal of \( \mathbb{R}' \).

For \( f( x ) \) and \( f( y ) \in \mathbb{R}' \), we have

\[
V( f(x) - f(y) ) = V( f(x - y) ), \quad \text{as } f \text{ is a homomorphism}
\]

\[
\geq A( x - y ),
\]

\[
\geq \min\{ A( x ), A( y ) \} \quad \text{as } A \text{ is a fuzzy ideal of } \mathbb{R},
\]

which implies that

\[
V( f(x) - f(y) ) \geq \min\{ V( f(x) ), V( f(y) ) \}.
\]

And,

\[
V( f(x)f(y) ) = V( f(xy) ), \quad \text{as } f \text{ is a homomorphism}
\]

\[
\geq A( xy ),
\]

\[
\geq \max\{ A( x ), A( y ) \}, \quad \text{as } A \text{ is a fuzzy ideal of } \mathbb{R},
\]

which implies that

\[
V( f(x)f(y) ) \geq \max\{ V( f(x) ), V( f(y) ) \}.
\]

Hence \( V \) is a fuzzy ideal of a ring \( \mathbb{R}' \).

2.2.2 Theorem:

Let \( R \) and \( \mathbb{R}' \) be any two rings. The homomorphic pre-image of a fuzzy ideal of \( \mathbb{R}' \) is a fuzzy ideal of \( R \).

Proof:

Let \( R \) and \( \mathbb{R}' \) be any two rings.

Let \( f : R \to \mathbb{R}' \) be a homomorphism.

That is \( f( x \pm y ) = f( x ) \pm f( y ) \) and \( f( xy ) = f( x )f( y ) \), for all \( x \) and \( y \in \mathbb{R} \).

Let \( V = f( A ) \), where \( V \) is a fuzzy ideal of \( \mathbb{R}' \).

We have to prove that \( A \) is a fuzzy ideal of \( R \).
For \(x\) and \(y \in R\), we have
\[
A(x - y) = V(f(x - y)), \quad \text{since } A(x) = V(f(x))
\]
\[
= V(f(x) - f(y)), \quad \text{as } f \text{ is a homomorphism}
\]
\[
\geq \min\{V(f(x)), V(f(y))\}, \quad \text{as } V \text{ is a fuzzy ideal of } R'
\]
\[
= \min\{A(x), A(y)\},
\]
which implies that \(A(x - y) \geq \min\{A(x), A(y)\}\).

And,
\[
A(xy) = V(f(xy)), \quad \text{since } A(x) = V(f(x))
\]
\[
= V(f(x)f(y)), \quad \text{as } f \text{ is a homomorphism}
\]
\[
\geq \max\{V(f(x)), V(f(y))\}, \quad \text{as } V \text{ is a fuzzy ideal of } R'
\]
\[
= \max\{A(x), A(y)\},
\]
which implies that \(A(xy) \geq \max\{A(x), A(y)\}\).

Hence \(A\) is a fuzzy ideal of a ring \(R\).

2.2.3 Theorem:

Let \(R\) and \(R'\) be any two rings. The homomorphic image of an anti-fuzzy ideal of \(R\) is an anti-fuzzy ideal of \(R'\).

Proof:

Let \(R\) and \(R'\) be any two rings.

Let \(f : R \rightarrow R'\) be a homomorphism.

That is \(f(x + y) = f(x) + f(y)\) and \(f(xy) = f(x)f(y)\), for all \(x\) and \(y \in R\).

Let \(V = f(A)\), where \(A\) is an anti-fuzzy ideal of \(R\).

We have to prove that \(V\) is an anti-fuzzy ideal of \(R'\).

For \(f(x)\) and \(f(y) \in R'\), we have
\[ V( f(x) - f(y) ) = V( f(x - y) ), \] as \( f \) is a homomorphism

\[ \leq A(x - y), \]

\[ \leq \max\{ A(x), A(y) \}, \] as \( A \) is an anti-fuzzy ideal of \( R \),

which implies that \( V( f(x) - f(y) ) \leq \max\{ V(f(x)), V(f(y)) \} \).

And, \( V( f(x) f(y) ) = V( f(xy) ), \) as \( f \) is a homomorphism

\[ \leq A(xy), \]

\[ \leq \min\{ A(x), A(y) \}, \] as \( A \) is an anti-fuzzy ideal of \( R \),

which implies that \( V( f(x) f(y) ) \leq \min\{ V( f(x) ), V( f(y) ) \} \).

Hence \( V \) is an anti-fuzzy ideal of a ring \( R' \).

2.2.4 Theorem:

Let \( R \) and \( R' \) be any two rings. The homomorphic pre-image of an anti-fuzzy ideal of \( R' \) is an anti-fuzzy ideal of \( R \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f: R \to R' \) be a homomorphism.

That is \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( V \) is an anti-fuzzy ideal of \( R' \).

We have to prove that \( A \) is an anti-fuzzy ideal of \( R \).

For \( x \) and \( y \in R \), we have

\[ A(x - y) = V( f(x - y) ), \] since \( A(x) = V(f(x)) \)

\[ = V( f(x) - f(y) ), \] as \( f \) is a homomorphism

\[ \leq \max\{ V( f(x) ), V( f(x) ) \}, \] as \( V \) is an anti-fuzzy ideal of \( R' \)

\[ = \max\{ A(x), A(y) \}, \]
which implies that $A(x - y) \leq \max\{ A(x), A(y) \}$.

And, $A(xy) = V(f(xy))$, since $A(x) = V(f(x))$

\[ = V(f(x)f(y)), \text{ as } f \text{ is a homomorphism} \]

\[ \leq \min\{ V(f(x)), V(f(y)) \}, \text{ as } V \text{ is an anti-fuzzy ideal of } R' \]

\[ = \min\{ A(x), A(y) \}, \]

which implies that $A(xy) \leq \min\{ A(x), A(y) \}$.

Hence $A$ is an anti-fuzzy ideal of a ring $R$.

**2.2.5 Theorem**:

Let $R$ and $R'$ be any two rings. The anti-homomorphic image of a fuzzy ideal of $R$ is a fuzzy ideal of $R'$.

**Proof**:

Let $R$ and $R'$ be any two rings.

Let $f: R \rightarrow R'$ be an anti-homomorphism.

That is $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all $x$ and $y \in R$.

Let $V = f(A)$, where $A$ is a fuzzy ideal of $R$.

We have to prove that $V$ is a fuzzy ideal of $R'$.

For $f(x)$ and $f(y) \in R'$, we have

\[ V(f(x) - f(y)) = V(f(y) - f(x)), \text{ as } f \text{ is an anti-homomorphism} \]

\[ \geq A(y - x), \]

\[ \geq \min\{ A(x), A(y) \}, \text{ as } A \text{ is a fuzzy ideal of } R, \]

which implies that $V(f(x) - f(y)) \geq \min\{ V(f(x)), V(f(y)) \}$. 

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And,
\[ V( f(x)f(y) ) = V( f(xy) ) , \text{ as } f \text{ is an anti-homomorphism} \]
\[ \geq A(yx), \]
\[ \geq \max\{ A(x), A(y) \}, \text{ as } A \text{ is a fuzzy ideal of } R, \]
which implies that
\[ V( f(x)f(y) ) \geq \max\{ V( f(x) ), V( f(y) ) \}. \]
Hence \( V \) is a fuzzy ideal of a ring \( R' \).

2.2.6 Theorem :
Let \( R \) and \( R' \) be any two rings. The anti-homomorphic pre-image of a fuzzy ideal of \( R' \) is a fuzzy ideal of \( R \).

Proof:

Let \( R \) and \( R' \) be any two rings.
Let \( f : R \rightarrow R' \) be an anti-homomorphism.
That is \( f( x + y ) = f(y) + f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x \) and \( y \in R \).
Let \( V = f( A ) \), where \( V \) is a fuzzy ideal of \( R' \).
We have to prove that \( A \) is a fuzzy ideal of \( R \).
Now, let \( x \) and \( y \in R \). Then,
\[ A( x - y ) = V( f( x - y ) ) , \text{ since } A(x) = V( f(x) ) \]
\[ = V( f(y) - f(x) ) , \text{ as } f \text{ is an anti-homomorphism} \]
\[ \geq \min\{ V( f(x) ), V( f(y) ) \} , \text{ as } V \text{ is a fuzzy ideal of } R' \]
\[ = \min\{ A(x), A(y) \} , \]
which implies that
\[ A( x - y ) \geq \min\{ A(x), A(y) \} . \]
And,
\[ A( xy ) = V( f( xy ) ) , \text{ since } A(x) = V( f(x) ) \]
\[ = V( f(y)f(x) ) , \text{ as } f \text{ is an anti-homomorphism} \]
\[ \text{as } V \text{ is a fuzzy ideal of } R' \]
\[ = \max \{ A(x), A(y) \}, \]
which implies that \[ A(xy) \geq \max \{ A(x), A(y) \}. \]
Hence \( A \) is a fuzzy ideal of a ring \( R \).

\textbf{2.2.7 Theorem :}

Let \( R \) and \( R' \) be any two rings. The anti-homomorphic image of an anti-fuzzy ideal of \( R \) is an anti-fuzzy ideal of \( R' \).

\textbf{Proof :}

Let \( R \) and \( R' \) be any two rings.

Let \( f : R \to R' \) be an anti-homomorphism.

That is \( f(x + y) = f(y) \pm f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x \) and \( y \in R \).

Let \( V = f(A) \), where \( A \) is an anti-fuzzy ideal of \( R \).

We have to prove that \( V \) is an anti-fuzzy ideal of \( R' \).

Now, let \( f(x) \) and \( f(y) \in R' \). Then,

\[ V( f(x) - f(y) ) = V( f( y - x ) ), \text{ as } f \text{ is an anti-homomorphism} \]
\[ \leq A( y - x ), \]
\[ \leq \max \{ A(x), A(y) \}, \text{ as } A \text{ is an anti-fuzzy ideal of } R, \]
which implies that \( V( f(x) - f(y) ) \leq \max \{ V( f(x) ), V( f(y) ) \} \).

And, \( V( f(x)f(y) ) = V( f(yx) ), \text{ as } f \text{ is an anti-homomorphism} \)
\[ \leq A(yx), \]
\[ \leq \min \{A(x), A(y) \}, \text{ as } A \text{ is an anti-fuzzy ideal of } R, \]
which implies that \( V( f(x)f(y) ) \leq \min \{ V(f(x)), V(f(y)) \} \).

Hence \( V \) is an anti-fuzzy ideal of a ring \( R' \).
2.2.8 Theorem:

Let $R$ and $R'$ be any two rings. The anti-homomorphic pre-image of an anti-fuzzy ideal of $R'$ is an anti-fuzzy ideal of $R$.

Proof:

Let $R$ and $R'$ be any two rings.

Let $f : R \rightarrow R'$ be an anti-homomorphism.

That is $f( x \pm y ) = f(y) \pm f(x)$ and $f(xy) = f(y)f(x)$, for all $x$ and $y \in R$.

Let $V = f(A)$, where $V$ is an anti-fuzzy ideal of $R'$.

We have to prove that $A$ is an anti-fuzzy ideal of $R$.

Now, let $x$ and $y \in R$. Then,

$$A( x - y ) = V( f( x - y ) ),$$

since $A(x) = V( f(x) )$

$$= V( f(y) - f(x) ),$$

as $f$ is an anti-homomorphism

$$\leq \max\{ V( f(x) ), V( f(y) ) \},$$

as $V$ is an anti-fuzzy ideal of $R'$

$$= \max\{ A(x), A(y) \},$$

which implies that $A( x - y ) \leq \max\{ A(x), A(y) \}$.

And, $A(xy) = V( f(xy) )$, since $A(x) = V( f(x) )$

$$= V( f(y)f(x) ),$$

as $f$ is an anti-homomorphism

$$\leq \min\{ V( f(x) ), V( f(y) ) \},$$

as $V$ is an anti-fuzzy ideal of $R'$

$$= \min\{ A(x), A(y) \},$$

which implies that $A( xy ) \leq \min\{ A(x), A(y) \}$.

Hence $A$ is an anti-fuzzy ideal of a ring $R$. 

2.3 - FUZZY AND ANTI-FUZZY NORMAL IDEALS OF A RING R

UNDER HOMOMORPHISM AND ANTI-HOMOMORPHISM:

2.3.1 Theorem:

Let R and R' be any two rings. The homomorphic image of a fuzzy normal ideal of R is a fuzzy normal ideal of R'.

Proof:

Let R and R' be any two rings.

Let \( f : R \to R' \) be a homomorphism.

That is \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( A \) is a fuzzy normal ideal of R.

We have to prove that \( V \) is a fuzzy normal ideal of R'.

For \( f(x) \) and \( f(y) \in R' \), then clearly \( V \) is a fuzzy ideal,

since \( A \) is a fuzzy ideal.

Now, \( V(f(x)f(y)) = V(f(xy)) \), as \( f \) is a homomorphism

\[ \geq A(xy), \]

\[ = A(yx), \text{ as } A \text{ is a fuzzy normal ideal of } R. \]
\[ V(f(yx)) = V(f(y)f(x)), \text{ as } f \text{ is a homomorphism,} \]

which implies that \[ V(f(x)f(y)) = V(f(y)f(x)). \]

Hence \( V \) is a fuzzy normal ideal of a ring \( R' \).

2.3.2 Theorem:

Let \( R \) and \( R' \) be any two rings. The homomorphic pre-image of a fuzzy normal ideal of \( R' \) is a fuzzy normal ideal of \( R \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f: R \to R' \) be a homomorphism.

That is \( f(x \pm y) = f(x) \pm f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( V \) is a fuzzy normal ideal of \( R' \).

We have to prove that \( A \) is a fuzzy normal ideal of \( R \).

Let \( x \) and \( y \in R \). Then clearly \( A \) is a fuzzy ideal,

since \( V \) is a fuzzy ideal.

Now, \( A(xy) = V(f(xy)) \), since \( A(x) = V(f(x)) \)

\[ = V(f(x)f(y)), \text{ as } f \text{ is a homomorphism} \]
= \mathcal{V}( f(y)f(x) ), \text{ as } \mathcal{V} \text{ is a fuzzy normal ideal of } R' \\
= \mathcal{V}( f(y) ), \text{ as } f \text{ is a homomorphism} \\
= \mathcal{A}(yx), \text{ since } \mathcal{A}(x) = \mathcal{V}( f(x) ),

which implies that \( \mathcal{A}(xy) = \mathcal{A}(yx) \).

Hence \( \mathcal{A} \) is a fuzzy normal ideal of a ring \( R \).

2.3.3 Theorem :

Let \( R \) and \( R' \) be any two rings. The homomorphic image of an anti-fuzzy normal ideal of \( R \) is an anti-fuzzy normal ideal of \( R' \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f: R \to R' \) be a homomorphism.

That is \( f(x + y) = f(x) + f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x, y \in R \).

Let \( \mathcal{V} = f(\mathcal{A}) \), where \( \mathcal{A} \) is an anti-fuzzy normal ideal of \( R \).

We have to prove that \( \mathcal{V} \) is an anti-fuzzy normal ideal of \( R' \).

Let \( f(x) \) and \( f(y) \in R' \). Then clearly \( \mathcal{V} \) is an anti-fuzzy ideal, since \( \mathcal{A} \) is an anti-fuzzy ideal.
Now, \( V(f(x)f(y)) = V(f(xy)) \), as \( f \) is a homomorphism

\[
\leq A(xy)
\]

\[= A(yx), \text{ as } A \text{ is an anti-fuzzy normal ideal of } R\]

\[\geq V(f(yx))
\]

\[= V(f(y)f(x)), \text{ as } f \text{ is a homomorphism,} \]

which implies that \( V(f(x)f(y)) = V(f(y)f(x)) \).

Hence \( V \) is an anti-fuzzy normal ideal of a ring \( R' \).

2.3.4 Theorem:

Let \( R \) and \( R' \) be any two rings. The homomorphic pre-image of an anti-fuzzy normal ideal of \( R' \) is an anti-fuzzy normal ideal of \( R \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f : R \rightarrow R' \) be a homomorphism.

That is \( f(x \pm y) = f(x) \pm f(y) \) and \( f(xy) = f(x)f(y) \), for all \( x \) and \( y \in R \).

Let \( V = f(A) \), where \( V \) is an anti-fuzzy normal ideal of \( R' \).

We have to prove that \( A \) is an anti-fuzzy normal ideal of \( R \).

Let \( x \) and \( y \in R \). Then clearly \( A \) is an anti-fuzzy ideal,

since \( V \) is an anti-fuzzy ideal.
Now, \( A(xy) = V(f(xy)) \), since \( A(x) = V(f(x)) \)

\[ = V(f(x)f(y)), \text{ as } f \text{ is a homomorphism} \]

\[ = V(f(y)f(x)), \text{ as } V \text{ is an anti-fuzzy normal ideal of } R' \]

\[ = V(f(yx)), \text{ as } f \text{ is a homomorphism} \]

\[ = A(yx), \text{ since } A(x) = V(f(x)), \]

which implies that \( A(xy) = A(yx) \).

Hence \( A \) is an anti-fuzzy normal ideal of a ring \( R \).

2.3.5 Theorem:

Let \( R \) and \( R' \) be any two rings. The anti-homomorphic image of a fuzzy normal ideal of \( R \) is a fuzzy normal ideal of \( R' \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f : R \to R' \) be an anti-homomorphism.

That is \( f(x + y) = f(y) + f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x \) and \( y \in R \).

Let \( V = f(A) \), where \( A \) is a fuzzy normal ideal of \( R \).

We have to prove that \( V \) is a fuzzy normal ideal of \( R' \).

Let \( f(x), f(y) \in R' \). Then clearly \( V \) is a fuzzy ideal of \( R' \), since \( A \) is a fuzzy ideal of \( R \).
Now,  \( V(f(x)f(y)) = V(f(yx)) \), as \( f \) is an anti-homomorphism
\[
\geq A(yx)
\]
\[
= A(xy), \quad \text{as } A \text{ is a fuzzy normal ideal of } R
\]
\[
\leq V(f(xy))
\]
\[
= V(f(y)f(x)), \quad \text{as } f \text{ is an anti-homomorphism},
\]
which implies that  \( V(f(x)f(y)) = V(f(y)f(x)) \).

Hence \( V \) is a fuzzy normal ideal of a ring \( R' \).

2.3.6 Theorem:

Let \( R' \) and \( R' \) be any two rings. The anti-homomorphic pre-image of a fuzzy normal ideal of \( R' \) is a fuzzy normal ideal of \( R \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f : R \rightarrow R' \) be an anti-homomorphism.

That is \( f(x + y) = f(y) + f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( V \) is a fuzzy normal ideal of \( R' \).

We have to prove that \( A \) is a fuzzy normal ideal of \( R \).

Let \( x, y \in R \). Then clearly \( A \) is a fuzzy ideal of \( R \),
since \( V \) is a fuzzy ideal of \( R' \).
Now, \( A(xy) = V(f(xy)) \), since \( A(x) = V(f(x)) \)
\[ = V(f(y)f(x)), \text{ as } f \text{ is an anti-homomorphism} \]
\[ = V(f(x)f(y)), \text{ as } V \text{ is a fuzzy normal ideal of } R' \]
\[ = V(f(yx)), \text{ as } f \text{ is an anti-homomorphism} \]
\[ = A(yx), \text{ since } A(x) = V(f(x)) \],
which implies that \( A(xy) = A(yx) \).

Hence \( A \) is a fuzzy normal ideal of a ring \( R \).

2.3.7 Theorem:

Let \( R \) and \( R' \) be any two rings. The anti-homomorphic image of an anti-fuzzy normal ideal of \( R \) is an anti-fuzzy normal ideal of \( R' \).

Proof:

Let \( R \) and \( R' \) be any two rings.

Let \( f : R \rightarrow R' \) be an anti-homomorphism.

That is \( f(x + y) = f(y) + f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( A \) is an anti-fuzzy normal ideal of \( R \).

We have to prove that \( V \) is an anti-fuzzy normal ideal of \( R' \).

Let \( f(x), f(y) \in R' \). Then clearly \( V \) is an anti-fuzzy ideal of \( R' \), since \( A \) is an anti-fuzzy ideal of \( R \).
Now,

\[ V( f(x)f(y) ) = V( f(yx) ), \text{ as } f \text{ is an anti-homomorphism} \]

\[ \leq A(xy) \]

\[ = A(xy), \text{ as } A \text{ is an anti-fuzzy normal ideal of } R \]

\[ \geq V(f(xy)) \]

\[ = V( f(y)f(x) ), \text{ as } f \text{ is an anti-homomorphism}, \]

which implies that \[ V( f(x)f(y) ) = V( f(y)f(x) ). \]

Hence \( V \) is an anti-fuzzy normal ideal of a ring \( R' \).

2.3.8 Theorem:

Let \( R \) and \( R^1 \) be any two rings. The anti-homomorphic pre-image of an anti-fuzzy normal ideal of \( R^1 \) is an anti-fuzzy normal ideal of \( R \).

Proof:

Let \( R \) and \( R^1 \) be any two rings.

Let \( f : R \to R^1 \) be an anti-homomorphism.

That is \( f(x \pm y) = f(y) \pm f(x) \) and \( f(xy) = f(y)f(x) \), for all \( x, y \in R \).

Let \( V = f(A) \), where \( V \) is an anti-fuzzy normal ideal of \( R^1 \).

We have to prove that \( A \) is an anti-fuzzy normal ideal of \( R \).

Let \( x, y \in R \). Then clearly \( A \) is an anti-fuzzy ideal of \( R \), since \( V \) is an anti-fuzzy ideal of \( R^1 \).
Now, \( A(xy) = V(f(xy)), \) since \( A(x) = V(f(x)) \)

\[ = V(f(y)f(x)), \text{ as } f \text{ is an anti-homomorphism} \]

\[ = V(f(x)f(y)), \text{ as } V \text{ is an anti-fuzzy normal ideal of } R' \]

\[ = V(f(yx)), \text{ as } f \text{ is an anti-homomorphism} \]

\[ = A(yx), \text{ since } A(x) = V(f(x)), \]

which implies that \( A(xy) = A(yx). \)

Hence \( A \) is an anti-fuzzy normal ideal of a ring \( R. \)

\[ \textbf{2.4 \quad PROPERTIES OF ANTI-FUZZY IDEALS :} \]

\[ \textbf{2.4.1 Theorem :} \]

Let \( R \) be a ring. \( A \) is a fuzzy ideal of \( R \) iff \( A^c \) is an anti-fuzzy ideal of \( R. \)

\[ \textbf{Proof :} \]

Suppose \( A \) is a fuzzy ideal of \( R. \)

For all \( x \) and \( y \in R, \) we have

\[ A(x - y) \geq \min \{ A(x), A(y) \}, \]

implies that \( 1 - A^c(x - y) \geq \min \{ 1 - A^c(x), 1 - A^c(y) \}, \)

implies that \( A^c(x - y) \leq 1 - \min \{ A^c(x), A^c(y) \}, \)

implies that \( A^c(x - y) \leq \max \{ A^c(x), A^c(y) \}. \)

Also, \( A(xy) \geq \max \{ A(x), A(y) \}, \)

implies that \( 1 - A^c(xy) \geq \max \{ 1 - A^c(x), 1 - A^c(y) \}, \)
implies that $A'(xy) \leq 1 - \max\{1 - A'(x), 1 - A'(y)\}$,
implies that $A^c(xy) \leq \min\{A'(x), A'(y)\}$.
Thus $A^c$ is an anti-fuzzy ideal of $R$.
Converse also can be proved similarly.

2.4.2 Theorem:

Let $R$ be a ring. The union of any two anti-fuzzy ideals of $R$ is always an anti-fuzzy ideal.

Proof:
Let $R$ be a ring.
Let $A$ and $B$ be any two anti-fuzzy ideals of $R$.
Then for $x$ and $y \in R$, we have

$$A(x - y) \leq \max\{A(x), A(y)\} \quad \text{and} \quad B(x - y) \leq \max\{B(x), B(y)\},$$
implies that $A(x - y) \leq A(x)$ or $A(x - y) \leq A(y),$$and$ B(x - y) \leq B(x)$ or $B(x - y) \leq B(y).$ Clearly $$(A \cup B)(x) = \max\{A(x), B(x)\}, \quad \text{and} \quad$$(A \cup B)(x - y) = \max\{A(x - y), B(x - y)\}.$$Now, $$(A \cup B)(x - y) \leq \max\{A(x), B(x)\}$$ or $$(A \cup B)(x - y) \leq \max\{A(y), B(y)\},$$implies that $(A \cup B)(x - y) \leq (A \cup B)(x)$ or $(A \cup B)(x - y) \leq (A \cup B)(y),$ implies that $(A \cup B)(x - y) \leq \max\{(A \cup B)(x), (A \cup B)(y)\}.$ And, $A(xy) \leq \min\{A(x), A(y)\}$ and $B(xy) \leq \min\{B(x), B(y)\}.$
That is \( A(xy) \leq A(x) \) and \( A(xy) \leq A(y) \) and
\[
B(xy) \leq B(x) \quad \text{and} \quad B(xy) \leq B(y).
\]
Clearly \( (A \cup B)(x) = \max\{ A(x), B(x) \} \), and
\[
(A \cup B)(xy) = \max\{ A(xy), B(xy) \}.
\]
Now, \( (A \cup B)(xy) \leq \max\{ A(x), B(x) \} \)
\[
\text{and} \quad (A \cup B)(xy) \leq \max\{ A(y), B(y) \},
\]
implies that \( (A \cup B)(xy) \leq (A \cup B)(x) \)
\[
\text{and} \quad (A \cup B)(xy) \leq (A \cup B)(y),
\]
implies that \( (A \cup B)(xy) \leq \min\{ (A \cup B)(x), (A \cup B)(y) \} \).
Hence \( A \cup B \) is an anti-fuzzy ideal of a ring \( R \).

2.4.3 Theorem:

Let \( R \) be a ring. The intersection of any two anti-fuzzy ideals of \( R \) is always an anti-fuzzy ideal.

Proof:

Let \( R \) be a ring.

Let \( A \) and \( B \) be any two anti-fuzzy ideals of \( R \).

Then for \( x \) and \( y \in R \). We have
\[
A(x - y) \leq \max\{ A(x), A(y) \}
\]
\[
\text{and} \quad B(x - y) \leq \max\{ B(x), B(y) \}.
\]
That is \( A(x - y) \leq A(x) \) or \( A(x - y) \leq A(y) \)
\[
\text{and} \quad B(x - y) \leq B(x) \) or \( B(x - y) \leq B(y).
\]
Clearly \( (A \cap B)(x) = \min\{ A(x), B(x) \} \), and
\[
(A \cap B)(x - y) = \min\{ A(x - y), B(x - y) \}.
\]
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Now, \((A \cap B)(x - y) \leq \min\{ A(x), B(x) \}\)

or \((A \cap B)(x - y) \leq \min\{ A(y), B(y) \}\),

implies that \((A \cap B)(x - y) \leq (A \cap B)(x)\)

or \((A \cap B)(x - y) \leq (A \cap B)(y)\),

implies that \((A \cap B)(x - y) \leq \max\{ (A \cap B)(x), (A \cap B)(y) \}\).

And, \(A(xy) \leq \min\{ A(x), A(y) \}\)

and \(B(xy) \leq \min\{ B(x), B(y) \}\),

implies that \(A(xy) \leq A(x)\) and \(A(xy) \leq A(y)\)

and \(B(xy) \leq B(x)\) and \(B(xy) \leq B(y)\).

Clearly \((A \cap B)(x) = \min\{ A(x), B(x) \}\), and

\((A \cap B)(xy) = \min\{ A(xy), B(xy) \}\).

Now, \((A \cap B)(xy) \leq \min\{ A(x), B(x) \}\)

and \((A \cap B)(xy) \leq \min\{ A(y), B(y) \}\),

implies that \((A \cap B)(xy) \leq (A \cap B)(x)\)

and \((A \cap B)(xy) \leq (A \cap B)(y)\),

implies that \((A \cap B)(xy) \leq \min\{ (A \cap B)(x), (A \cap B)(y) \}\).

Hence \(A \cap B\) is an anti-fuzzy ideal of a ring \(R\).

2.4.4 Theorem:

Let \(R\) be a ring. The union of a family of anti-fuzzy ideals of \(R\) is an anti-fuzzy ideal.

Proof:

Let \(\{A_i\}_{i \in I}\) be a family of anti-fuzzy ideals and let \(A = \bigcup_{i \in I} A_i\).
Then for $x$ and $y \in R$, we have

$$A(x - y) = \sup_{i \in I} A_i(x - y)$$

$$\leq \sup_{i \in I} \max\{ A_i(x), A_i(y) \}$$

$$\leq \max\{ \sup_{i \in I} A_i(x), \sup_{i \in I} A_i(y) \}$$

$$= \max\{ A(x), A(y) \},$$

which implies that $A(x - y) \leq \max\{ A(x), A(y) \}$.

And,

$$A(xy) = \sup_{i \in I} A_i(xy)$$

$$\leq \sup_{i \in I} \min\{ A_i(x), A_i(y) \}$$

$$\leq \min\{ \sup_{i \in I} A_i(x), \sup_{i \in I} A_i(y) \}$$

$$= \min\{ A(x), A(y) \},$$

which implies that $A(xy) \leq \min\{ A(x), A(y) \}$.

Hence $A$ is an anti-fuzzy ideal of a ring $R$.

2.4.5 Theorem:

Let $R$ be a ring. The intersection of a family of anti-fuzzy ideals of $R$ is an anti-fuzzy ideal.

Proof:

Let $\{A_i\}_{i \in I}$ be a family of anti-fuzzy ideals and let $A = \bigcap_{i \in I} A_i$.

Then for $x$ and $y \in R$, we have

$$A(x - y) = \inf_{i \in I} A_i(x - y)$$
\[ \leq \inf_{i \in I} \max\{ A_i(x), A_i(y) \} \]
\[ \leq \max\{ \inf_{i \in I} A_i(x), \inf_{i \in I} A_i(y) \} \]
\[ = \max\{ A(x), A(y) \}, \]

which implies that \( A(x - y) \leq \max\{ A(x), A(y) \} \).

And,
\[ A(xy) = \inf_{i \in I} A_i(xy) \]
\[ \leq \inf_{i \in I} \min\{ A_i(x), A_i(y) \} \]
\[ \leq \min\{ \inf_{i \in I} A_i(x), \inf_{i \in I} A_i(y) \} \]
\[ = \min\{ A(x), A(y) \}, \]

which implies that \( A(xy) \leq \min\{ A(x), A(y) \} \).

Hence \( A \) is an anti-fuzzy ideal of a ring \( R \).

2.4.6 Theorem:

Let \( R \) be a ring. If \( A \) is an anti-fuzzy ideal of \( R \), then \( x + A = y + A \iff A(x - y) = A(0) \). In that case \( A(x) = A(y) \).

Proof:

Given \( A \) is an anti-fuzzy ideal of \( R \).

That is \( A(x - y) \leq \max\{ A(x), A(y) \} \) and \( A(xy) \leq \min\{ A(x), A(y) \} \).

Suppose that \( x + A = y + A \), implies that \( (x + A)(x) = (y + A)(x) \), implies that \( A(x - x) = A(x - y) \), implies that \( A(0) = A(x - y) \).
Conversely, assume that \( A( x - y ) = A(0) \), then
\[
(x + A)(z) = A(z - x)
\]
\[
= A(z - x + y - y)
\]
\[
\leq \max\{ A(z - y), A(0) \}
\]
\[
= A(z - y)
\]
\[
= (y + A)(z),
\]
which implies that \( (x + A)(z) \leq (y + A)(z) \). \( \text{--------------------------(1)} \)

Now,
\[
(y + A)(z) = A(z - y)
\]
\[
= A(z - y + x - x)
\]
\[
\leq \max\{ A(z - x), A(0) \}
\]
\[
= A(z - x), \quad \text{Since } A(0) \leq A(z - x)
\]
\[
= (x + A)(z),
\]
which implies that \( (y + A)(z) \leq (x + A)(z) \). \( \text{--------------------------(2)} \)

From (1) and (2) we get, \( x + A = y + A \).

2.4.7 Theorem:

Let \( R \) be a ring. Let \( A \) be an anti-fuzzy ideal and \( x, y, u, \) and \( v \) be any elements in \( R \), if \( x + A = u + A \) and \( y + A = v + A \), then

(i) \( (x + y) + A = (u + v) + A \)

(ii) \( (xy) + A = (uv) + A. \)

Proof:

Given \( A \) is an anti-fuzzy ideal of \( R \).

That is \( A(x - y) \leq \max\{ A(x), A(y) \} \) and \( A(xy) \leq \min\{ A(x), A(y) \} \).
By Theorem 2.4.6,

\[ A(x - u) = A(y - v) = A(0). \]

We get, \( A(x + y - u - v) = A(x - u + y - v) \leq \max\{ A(x - u), A(y - v)\} = A(0). \)

Again, by Theorem 2.4.6,

\[ (x + y) + A = (u + v) + A. \]

Thus (i) is proved.

Now, \( A(uv - xy) = A(uv - uy + uy - xy) \leq \max\{ A(uv - uy), A(uy - xy)\} \]

\[ = \max\{ A(u(v - y)), A((u - x)y)\} \]

\[ = \max\{ \min[A(u), A(y - y)], \min[A(u - x), A(y)]\} \]

\[ = \max\{ \min[A(u), A(0)], \min[A(0), A(y)]\} \]

\[ = A(0), \]

which implies that \( A(uv - xy) = A(0). \)

By Theorem 2.4.6,

\[ xy + A = uv + A. \]

Thus (ii) is proved.

2.4.8 Theorem:

If \( A \) is any anti-fuzzy ideal of a ring \( R \) and \( A(x) < A(y) \), for some \( x \) and \( y \) in \( R \), then \( A(x - y) = A(y) = A(y - x) \).

Proof:

If \( A \) is an anti-fuzzy ideal of \( R \) and \( A(x) < A(y) \).

That is \( A(x - y) \leq \max\{ A(x), A(y)\} \).
Now, \( A(x - y) \leq \max\{ A(x), A(y) \} = A(y) \).

Therefore, \( A(x - y) \leq A(y) \) \( \ldots \)(1).

Now, \( A(y) = A(x - (x - y)) \leq \max\{ A(x), A(x - y) \} \)

\[ = A(x - y). \]

Therefore, \( A(y) \leq A(x - y) \) \( \ldots \)(2).

From (1) and (2), we get \( A(x - y) = A(y) \).

And, \( A(y - x) \leq \max\{ A(y), A(x) \} = A(y) \).

Therefore, \( A(y - x) \leq A(y) \) \( \ldots \)(3).

Now, \( A(y) = A(y - x + x) \)

\[ \leq \max\{ A(y - x), A(x) \} \]

\[ = A(y - x). \]

Therefore, \( A(y) \leq A(y - x) \) \( \ldots \)(4).

From (3) and (4), we get \( A(y) = A(y - x) \).

Hence \( A(x - y) = A(y) = A(y - x) \).

2.4.9 Theorem:

Let \( A \) be an anti-fuzzy ideal of a ring \( R \). Then for \( t \in [0, 1] \) such that

\[ t \geq A(0), \quad \tilde{A}_t \text{ is an ideal of } R. \]

Proof:

For \( x \) and \( y \) in \( \tilde{A}_t \), we have

\[ A(x) \leq t \text{ and } A(y) \leq t. \]

Now, \( A(x - y) \leq \max\{ A(x), A(y) \} \)

\[ \leq \max\{ t, t \} = t, \]

which implies that \( A(x - y) \leq t. \)
Hence \( x - y \in \tilde{A}_t \).

Now, \( \tilde{A}(xy) \leq \min\{ \tilde{A}(x), \tilde{A}(y) \} \)
\[ \leq \min\{ t, t \} = t, \]
which implies that \( \tilde{A}(xy) \leq t \).

Hence \( xy \in \tilde{A}_t \).

Hence \( \tilde{A}_t \) is an ideal of \( R \).

**2.4.10 Theorem:**

Let \( A \) be an anti-fuzzy ideal of a ring \( R \). If two lower level ideals \( \tilde{A}_{t_1}, \tilde{A}_{t_2} \subseteq [0, 1] \) and \( t_1, t_2 \geq A(0) \) with \( t_1 < t_2 \) of \( A \) are equal, then there is no \( x \) in \( R \) such that \( t_1 < A(x) < t_2 \).

**Proof:**

Let \( \tilde{A}_{t_1} = \tilde{A}_{t_2} \).

Suppose that there exists \( x \in R \) such that \( t_1 < A(x) < t_2 \).

Then \( \tilde{A}_{t_1} \subseteq \tilde{A}_{t_2} \).

That is \( x \in \tilde{A}_{t_1} \) does not imply that \( x \) in \( \tilde{A}_{t_2} \).

which is a contradiction to the fact that \( \tilde{A}_{t_1} = \tilde{A}_{t_2} \).

Therefore, there is no \( x \) in \( R \) such that \( t_1 < A(x) < t_2 \).

**2.4.11 Theorem:**

Let \( R \) be a ring. If \( A \) be a fuzzy subset of \( R \) such that \( \tilde{A}_t \) is an ideal of \( R \), for \( t \in [0, 1] \) and \( t \geq A(0) \), then \( A \) is an anti-fuzzy ideal of \( R \).
Proof:

Let \( x \) and \( y \in R \), \( A(x) = t_1 \) and \( A(y) = t_2 \).

Suppose that \( t_1 < t_2 \), then \( x \) and \( y \in \overline{A}_{t_1} \).

As \( \overline{A}_{t_1} \) is a subideal of \( R \),

\[ x - y \in \overline{A}_{t_1} \quad \text{and} \quad ax, xa \in \overline{A}_{t_1}. \]

Now, \( A(x - y) \leq t_2 = \max\{ t_1, t_2 \} \)

\[ \leq \max\{ A(x), A(y) \}, \]

which implies that \( A(x - y) \leq \max\{ A(x), A(y) \} \).

Now, \( A(ax) \leq t_2 = \max\{ t_3, t_2 \} \)

\[ \leq \max\{ A(a), A(x) \}, \]

which implies that \( A(ax) \leq A(x) \), since \( A(a) = t_3 < t_2 \).

And, \( A(xa) \leq t_2 \)

\[ = \max\{ t_2, t_3 \} \]

\[ \leq \max\{ A(x), A(a) \}, \]

which implies that \( A(xa) \leq A(x) \), since \( A(a) = t_3 < t_2 \).

Hence \( A \) is an anti-fuzzy ideal of a ring \( R \).