CHAPTER-V

INTUITIONISTIC FUZZY LEVEL SUBSETS

5.1 Introduction:

In this chapter, the basic definitions and properties of the intuitionistic fuzzy level subsets of an intuitionistic fuzzy subset are discussed. Using these concepts, some results are established.

5.1.1 Definition:

Let $A$ be an intuitionistic fuzzy subset of $X$. For $t \in [0, 1]$, the level subset of $A$ is the set, $A_t = \{ x \in X : \mu_A(x) \geq t \text{ and } \nu_A(x) \leq t \}$. This is called an intuitionistic fuzzy level subset of $A$.

5.2 – PROPERTIES OF INTUITIONISTIC FUZZY LEVEL SUBSETS:

5.2.1 Theorem:

Let $A$ be an intuitionistic fuzzy subgroup of a group $G$. Then for $t \in [0, 1]$ such that $t \leq \mu_A(e)$ and $t \geq \mu_A(e)$, $A_t$ is a subgroup of $G$.

Proof:

For all $x$ and $y$ in $A_t$,

we have $\mu_A(x) \geq t$ and $\nu_A(x) \leq t$ and $\mu_A(y) \geq t$ and $\nu_A(y) \leq t$.

Now, $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$, as $A$ is an IFSG of a group $G$

$\geq \min \{ t, t \} = t$.

Therefore, $\mu_A(xy^{-1}) \geq t$. 

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Also, \( v_A(xy^{-1}) \leq \max\{v_A(x), v_A(y)\} \), as \( A \) is an IFSG of a group \( G \)
\[ \leq \max\{t, t\} = t, \]
which implies that \( v_A(xy^{-1}) \leq t \).

That is, \( \mu_A(xy^{-1}) \geq t \) and \( v_A(xy^{-1}) \leq t \).

Therefore, \( xy^{-1} \in A_t \).

Hence \( A_t \) is a subgroup of a group \( G \).

5.2.1 Definition:

Let \( A \) be an intuitionistic fuzzy subgroup of a group \( G \). The subgroup \( A_t \) for \( t \in [0,1] \) such that \( t \leq \mu_A(e) \) and \( t \geq v_A(e) \) is called a level subgroup of \( A \).

5.2.2 Theorem:

Let \( A \) be an intuitionistic fuzzy subgroup of a group \( G \). If two level subgroups \( A_{t_1}, A_{t_2} \) such that \( t_1, t_2 \in [0,1] \), \( t_1, t_2 \leq \mu_A(e) \) and \( t_1, t_2 \geq v_A(e) \) with \( t_2 < t_1 \) of \( A \) are equal iff there is no \( x \) in \( G \) such that \( t_1 > \mu_A(x) > t_2 \) and \( t_2 < v_A(x) < t_1 \).

Proof:

Assume that \( A_{t_1} = A_{t_2} \).

Suppose there exists \( x \in G \) such that \( t_1 > \mu_A(x) > t_2 \) and \( t_2 < v_A(x) < t_1 \).

Then \( A_{t_1} \subseteq A_{t_2} \).

This implies that \( x \in A_{t_1} \), but not in \( A_{t_2} \), which is a contradiction to \( A_{t_1} = A_{t_2} \).

Therefore there is no \( x \in G \) such that \( t_1 > \mu_A(x) > t_2 \) and \( t_2 < v_A(x) < t_1 \).
Conversely,

if there is no $x \in G$ such that $t_1 > \mu_A(x) > t_2$ and $t_2 < \nu_A(x) < t_1$,

then $A_{t_1} = A_{t_2}$ (by the definition of level subset).

5.2.3 Theorem:

Let $G$ be a group and $A$ be an intuitionistic fuzzy subset of $G$. If $A_t$ is a subgroup of $G$, $t \in [0, 1]$ such that $t \leq \mu_A(e)$ and $t \geq \nu_A(e)$, then $A$ is an intuitionistic fuzzy subgroup of $G$.

Proof:

Let $G$ be a group and $x, y$ in $G$.

Let $\mu_A(x) = t_1$ and $\mu_A(y) = t_2$. $\nu_A(x) = t_3$ and $\nu_A(y) = t_4$.

Suppose $t_1 < t_2$, then $x, y \in A_{t_1}$.

As $A_{t_1}$ is a subgroup of $G$, then $xy^{-1} \in A_{t_1}$.

Now, $\mu_A(xy^{-1}) \geq t_1 = \min \{ t_1, t_2 \}$

$= \min \{ \mu_A(x), \mu_A(y) \}$

which implies that $\mu_A(xy^{-1}) \geq \min \{ \mu_A(x), \mu_A(y) \}$.

And, suppose $t_3 < t_4$, then $x$ and $y \in A_{t_3}$.

As $A_{t_3}$ is a subgroup of $G$, then $xy^{-1} \in A_{t_3}$.

$\nu_A(xy^{-1}) \leq t_4 = \max \{ t_3, t_4 \}$

$= \max \{ \nu_A(x), \nu_A(y) \}$

which implies that $\nu_A(xy^{-1}) \leq \max \{ \nu_A(x), \nu_A(y) \}$.

Hence $A$ is an intuitionistic fuzzy subgroup of a group $G$. 

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5.2.4 Theorem:

Let \( A \) be an intuitionistic fuzzy subgroup of a group \( G \). For \( t_1, t_2 \in [0,1] \) such that \( t_1, t_2 \leq \mu_A(e) \) and \( t_1, t_2 \geq \nu_A(e) \). If \( A_{t_1}, A_{t_2} \) are two level subgroups of \( A \), then \( A_{t_1} \cap A_{t_2} \) is a level subgroup of \( A \).

Proof:

Let \( t_1, t_2 \in [0,1] \).

Case (i):

If \( t_1 < t_2 \), then \( A_{t_2} \subseteq A_{t_1} \).
Therefore, \( A_{t_1} \cap A_{t_2} = A_{t_1} \), but \( A_{t_1} \) is a level subgroup of \( A \).

Case (ii):

If \( t_2 < t_1 \), then \( A_{t_2} \subseteq A_{t_1} \).
Therefore, \( A_{t_1} \cap A_{t_2} = A_{t_2} \), but \( A_{t_1} \) is a level subgroup of \( A \).

Case (iii):

If \( t_2 = t_1 \), then \( A_{t_1} = A_{t_2} \).

In all cases \( A_{t_1} \cap A_{t_2} \) is a level subgroup of an intuitionistic fuzzy subgroup \( A \) of a group \( G \).

5.2.5 Theorem:

Let \( A \) be an intuitionistic fuzzy subgroup of a group \( G \). For \( t \in [0,1] \) such that \( t \leq \mu_A(e) \) and \( t \geq \nu_A(e) \), if \( A_i, \ i \in I \) are level subgroups of \( A \), then \( \bigcap_{i \in I} A_i \) is also a level subgroup of \( A \).
Proof:

It is trivial.

5.2.6 Theorem:

Let $A$ be an intuitionistic fuzzy subgroup of a group $G$. For $t_1, t_2 \in [0, 1]$ such that $t_1, t_2 \leq \mu_A(e)$ and $t_1, t_2 \geq \nu_A(e)$, if $A_{t_1}, A_{t_2}$ are level subgroups of $A$, then $A_{t_1} \cup A_{t_2}$ is a level subgroup of $A$.

Proof:

Let $t_1, t_2 \in [0, 1]$.

Case (i):

If $t_1 < t_2$, then $A_{t_1} \subseteq A_{t_2}$.

Therefore, $A_{t_1} \cup A_{t_2} = A_{t_2}$, but $A_{t_1}$ is a level subgroup of $A$.

Case (ii):

If $t_2 < t_1$, then $A_{t_2} \subseteq A_{t_1}$.

Therefore, $A_{t_1} \cup A_{t_2} = A_{t_1}$, but $A_{t_2}$ is a level subgroup of $A$.

Case (iii):

If $t_2 = t_1$, then $A_{t_1} = A_{t_2}$.

In all cases $A_{t_1} \cup A_{t_2}$ is a level subgroup of $A$.

5.2.7 Theorem:

Let $A$ be an intuitionistic fuzzy subgroup of a group $G$. For $t_i \in [0, 1]$ such that $t_i \leq \mu_A(e)$ and $t_i \geq \nu_A(e)$, if $A_{t_i}, i \in I$ are level subgroups of $A$, then $\bigcup_{i \in I} A_{t_i}$ is a level subgroup of $A$.  

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5.2.7 Theorem:

Let $A$ be an intuitionistic fuzzy subgroup of a group $G$. For $t_i \in [0, 1]$ such that $t_i \leq \mu_A(e)$ and $t_i \geq \nu_A(e)$, if $A_i$, $i \in I$ are level subgroups of $A$, then

\[ \bigcup_{i \in I} A_i \] is a level subgroup of $A$.

Proof:

It is trivial.

5.2.8 Theorem:

Let $A$ be an intuitionistic fuzzy subgroup of a group $G$. If $A$ is an intuitionistic fuzzy characteristic subgroup of a group $G$, then each level subgroup of $A$ is a characteristic subgroup of a group $G$.

Proof:

Let $A$ be an intuitionistic fuzzy characteristic subgroup of a group $G$.

Let $x$ and $y \in G$ and $t \in \text{Im } A$; $f \in \text{Aut } G$ and $x \in A_i$.

Now, $\mu_A(f(x)) = \mu_A(x) \geq t$, since $A$ is an IFCSG of $G$.

Therefore, $\mu_A(f(x)) \geq t$.

And, $\nu_A(f(x)) = \nu_A(x) \leq t$, since $A$ is an IFCSG of $G$.

Therefore, $\nu_A(f(x)) \leq t$.

Therefore, $\mu_A(f(x)) \geq t$ and $\nu_A(f(x)) \leq t$.

So that $f(x) \in A_i$.

Hence $f(A_i) \subseteq A_i$ \hspace{1cm} (1).

For the reverse inclusion,

let $x \in f(A_i)$ and let $y$ in $G$ be such that $f(y) = x$.

Then, $\mu_A(y) = \mu_A(f(x)) = \mu_A(x) \geq t$, 

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Therefore, $\mu_A(y) \geq t$, $\nu_A(y) \leq t$.

Hence $y \in A_t$, whence $x \in f(A_t)$.

Hence $f(A_t) \subseteq A_t$-------------------------(2).

From (1) and (2), we get

$A_t$ is a characteristic subgroup of a group $G$.

5.2.9 Theorem:

Any subgroup $H$ of a group $G$ can be realized as a level subgroup of
some intuitionistic fuzzy subgroup of $G$.

Proof:

Let $A$ be the intuitionistic fuzzy subset of $G$ defined by

$$
\mu_A(x) = \begin{cases} 
    t_1 & \text{if } x \in H, \quad 0 < t_1 < 1 \\
    0 & \text{if } x \notin H.
\end{cases}
$$

and

$$
\nu_A(x) = \begin{cases} 
    t_2 & \text{if } x \in H, \quad 0 < t_2 < 1 \\
    0 & \text{if } x \notin H.
\end{cases}
$$

and $t_1 + t_2 \leq 1$,

where $H$ is a subgroup of a group $G$.

We claim that $A$ is an intuitionistic fuzzy subgroup of a group $G$.

Let $x$ and $y \in G$.

Case (i):

If $x$ and $y \in H$, then $xy^{-1} \in H$, since $H$ is a subgroup of $G$.

Therefore, $\mu_A(xy^{-1}) = t_1$, $\mu_A(x) = t_1$, $\mu_A(y) = t_1$. 

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So, $\mu_A(\, xy^{-1} ) \geq \min\{ \mu_A(x), \mu_A(y) \}$.

And, $v_A(\, xy^{-1} ) = t_2, v_A(x) = t_2, v_A(y) = t_2$.

So, $v_A(\, xy^{-1} ) \leq \max\{ v_A(x), v_A(y) \}$.

Case (ii):

If $x \in H, y \notin H$, then $xy^{-1} \notin H$.

Then $\mu_A(x) = t_1, \mu_A(y) = 0, \mu_A(x \ y^{-1}) = 0$.

Therefore, $\mu_A(\, xy^{-1} ) \geq \min\{ \mu_A(x), \mu_A(y) \}$.

And, $v_A(x) = t_2, v_A(y) = 0, v_A(x \ y^{-1}) = 0$.

Therefore, $v_A(\, xy^{-1} ) \leq \max\{ v_A(x), v_A(y) \}$.

Case (iii):

Suppose $x$ and $y \notin H$, then $xy^{-1}$ may or may not belong to $H$.

Clearly $\mu_A(\, xy^{-1} ) \geq \min\{ \mu_A(x), \mu_A(y) \}$

and $v_A(\, xy^{-1} ) \leq \max\{ v_A(x), v_A(y) \}$.

In any case, $\mu_A(\, xy^{-1} ) \geq \min\{ \mu_A(x), \mu_A(y) \}$ and

$v_A(\, xy^{-1} ) \leq \max\{ v_A(x), v_A(y) \}$.

Thus in all the cases $A$ is an intuitionistic fuzzy subgroup of $G$.

5.2.10 Theorem:

Let $A$ be an intuitionistic fuzzy subset of a set $X$. Then

$\mu_A(x) = \sup\{ t \ / \ x \in A \}$ and $v_A(x) = \inf\{ t \ / \ x \in A \}$, where $x \in X$.

Proof:

Let $\alpha_1 = \sup\{ t \ / \ x \in A \}$ and $\varepsilon > 0$ be arbitrary.
Then $\alpha_1 - \varepsilon < \sup \{ t / t \in A_t \}$,

which implies that $\alpha_1 - \varepsilon < t$, for some $t$ such that $x \in A_t$.

That is, $\alpha_1 - \varepsilon < \mu_A(x)$, since $\mu_A(x) \geq t$.

Therefore, $\alpha_1 \leq \mu_A(x)$, since $\varepsilon > 0$ is arbitrary \textbf{---------- (1).}

Now, assume that $\mu_A(x) = s$.

Then $x \in A_s$ and so $s \in \{ t / x \in A_t \}$.

Hence $s \leq \sup \{ t / x \in A_t \}$, where $\mu_A(x) \leq \alpha_1 \textbf{---------- (2).}$

From (1) and (2), we get

$$\mu_A(x) = \alpha_1 = \sup \{ t / x \in A_t \}.$$ And, let $\alpha_2 = \inf \{ t / x \in A_t \}$ and $\varepsilon > 0$ be arbitrary.

Then $\alpha_2 + \varepsilon > \inf \{ t / t \in A_t \}$,

which implies that $\alpha_2 + \varepsilon > t$, for some $t$ such that $x \in A_t$.

That is, $\alpha_2 + \varepsilon > \nu_A(x)$ since $\nu_A(x) \leq t$.

Therefore, $\alpha_2 \geq \nu_A(x)$, since $\varepsilon > 0$ is arbitrary \textbf{---------- (3).}

Now, assume that $\nu_A(x) = s$.

Then $x \in A_s$ and so $s \in \{ t / x \in A_t \}$.

Hence $s \leq \inf \{ t / x \in A_t \}$, where $\nu_A(x) \geq \alpha_2 \textbf{---------- (4).}$

From (3) and (4), we get,

$$\nu_A(x) = \alpha_2 = \inf \{ t / x \in A_t \}.$$  

\textbf{5.2.11 Theorem :}

Two different intuitionistic fuzzy subgroups of a group may have identical family of level subgroups.
Proof:

We consider the following example:

Let \( G \) be Klein's four group:

\[
G = \{ e, a, b, ab \}, \text{ where } a^2 = e = b^2, \, ab = ba.
\]

Define intuitionistic fuzzy subsets \( A \) and \( B \) of \( G \) by

\[
A = \{ \langle e, 0.7, 0.1 \rangle, \langle a, 0.6, 0.2 \rangle, \langle b, 0.4, 0.3 \rangle, \langle ab, 0.4, 0.3 \rangle \} \text{ and }
\]

\[
B = \{ \langle e, 0.8, 0.2 \rangle, \langle a, 0.7, 0.3 \rangle, \langle b, 0.5, 0.4 \rangle, \langle ab, 0.5, 0.4 \rangle \}.
\]

Clearly \( A \) and \( B \) are intuitionistic fuzzy subgroups of \( G \).

And, \( \text{Im } \mu_A = \{ 0.7, 0.6, 0.4 \}, \text{ Im } \nu_A = \{ 0.1, 0.2, 0.3 \} \).

The level subgroups of \( A \) are \( A_{0.7} = \{ e \}, A_{0.6} = \{ e, a \}, A_{0.4} = \{ e, a, b, ab \} = G \).

And, \( \text{Im } \mu_B = \{ 0.8, 0.7, 0.5 \}, \text{ Im } \nu_B = \{ 0.2, 0.3, 0.4 \} \).

The level subgroups of \( B \) are \( B_{0.8} = \{ e \}, B_{0.7} = \{ e, a \}, B_{0.5} = \{ e, a, b, ab \} = G \).

Thus the two intuitionistic fuzzy subgroups \( A \) and \( B \) have the same family of level subgroups.

But \( A \neq B \), because \( \mu_A \neq \mu_B \).

5.2.12 Theorem:

Let \( G \) be a finite group and \( A \) be an intuitionistic fuzzy subgroup of \( G \).

If \( t_i, t_j \) are elements of the image set of \( A \) such that \( A_{t_i} = A_{t_j} \), then \( t_i, t_j \) need not be equal.

Proof:

We consider the following example:

Let \( G \) be Klein's four group.
\[ G = \{ e, a, b, ab \}, \text{ where } a^2 = e = b^2, ab = ba. \]

Define intuitionistic fuzzy subgroup \( A \) by
\[ A = \{ (e, 0.5, 0.4), (a, 0.4, 0.5), (b, 0.3, 0.6), (ab, 0.3, 0.6) \}. \]

Case(i):

If \( t_i = 0.5 \text{ and } t_j = 0.4 \) are in \( \text{Im } \mu_A \), then \( A_{0.5} = \{ e \}, A_{0.4} = \{ e \} \) are
the level subgroups of \( G \).
Clearly \( t_i, t_j \) are different.

Case(ii):

If \( t_i = 0.5 \text{ and } t_j = 0.4 \) are in \( \text{Im } \nu_A \), then \( A_{0.5} = \{ e \}, A_{0.4} = \{ e \} \) are the
level subgroups of \( G \).
Clearly \( t_i, t_j \) are different.

Result: In a fuzzy group, if \( t_i, t_j \) are elements of the image set of \( A \) such that
\[ A_{t_i} = A_{t_j}, \text{ then } t_i = t_j. \]

5.2.13 Theorem:

Let \( I \) be the subset of \([0, 1]\) and let \( G \) be a group with subgroups \( \{ H_i \}, i \in I \) such that \( \cup H_i = G \) and \( i < j \) implies that \( H_i \subset H_j \). Then an intuitionistic fuzzy subset \( A \) of \( G \) defined by \( \mu_A(x) = \sup \{ i / x \in H_i \} \) and \( \nu_A(x) = \inf \{ i / x \in H_i \} \) is an intuitionistic fuzzy subgroup of \( G \).

Proof:

Let \( A \) be an intuitionistic fuzzy subset of \( G \) defined by
\[ \mu_A(x) = \sup \{ i / x \in H_i \} \text{ and } \nu_A(x) = \inf \{ i / x \in H_i \}, \text{ where } i \in I \subset [0, 1]. \]

Let \( x \text{ and } y \in G \) and \( \mu_A(x) = m_1 \text{ and } \mu_A(y) = n_1 \).

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If $\mu_A(xy) = \sup \{ i / xy \in H_j \} < \min\{ m_1, n_1 \}$, then there exists $j$ such that $x$ and $y$ are elements of $H_j$, but $xy$ is not an element of $H_j$, since $H_j$ is a subgroup of $G$.

This is a contradiction.

Therefore, $\mu_A(xy) \geq \min\{ m_1, n_1 \}$, which implies that $\mu_A(xy) \geq \min\{ \mu_A(x), \mu_A(y) \}$.

Clearly $\mu_A(x^{-1}) = \mu_A(x)$.

Also, $v_A(x) = m_2$ and $v_A(y) = n_2$.

If $v_A(xy) = \inf\{ i / xy \in H_j \} > \max\{ m_2, n_2 \}$, then there exists $j$ such that $x$ and $y$ are elements of $H_j$, but $xy$ is not an element of $H_j$, since $H_j$ is a subgroup of $G$.

This is a contradiction.

Therefore, $v_A(xy) \leq \max\{ m_2, n_2 \}$, which implies that $v_A(xy) \leq \max\{ v_A(x), v_A(y) \}$.

Clearly $v_A(x^{-1}) = v_A(x)$.

Hence $A$ is an intuitionistic fuzzy subgroup of $G$.

5.2.14 Theorem:

If $A$ is an intuitionistic fuzzy normal subgroup of a group $G$, then for each level subgroup $A_t$, $t \in [0,1]$, $t \leq \mu_A(e)$ and $t \geq v_A(e)$ is a normal subgroup of $G$.

Proof:

Let $A$ be an intuitionistic fuzzy normal subgroup of a group $G$.

Let $A_t$ be any level subgroup of $A$. 

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To prove that $A_t$ is normal in $G$.

Let $x \in A_t$ and $g \in G$.

Then, $\mu_A(x) \geq t$ and $\nu_A(x) \leq t$.

Now, $\mu_A(g^{-1}xg) = \mu_A(xgg^{-1})$, since $A$ is an IFNSG of $G$
$$= \mu_A(x) \geq t.$$

And, $\nu_A(g^{-1}xg) = \nu_A(xgg^{-1})$, since $A$ is an IFNSG of $G$
$$= \nu_A(x) \leq t.$$

Hence $\mu_A(g^{-1}xg) \geq t$ and $\nu_A(g^{-1}xg) \leq t$.

Therefore, $g^{-1}xg \in A_t$ and hence $A_t$ is a normal subgroup of $G$.

5.2.15 Theorem:

Let $A$ and $B$ be intuitionistic fuzzy subsets of the sets $G$ and $H$, respectively, and let $t \in [0, 1]$. Then $(A \times B)_t = A_t \times B_t$.

Proof:

Let $(x,y) \in (A \times B)_t$
$$\Leftrightarrow \mu_{A \times B}(x,y) \geq t \text{ and } \nu_{A \times B}(x,y) \leq t$$
$$\Leftrightarrow \min \{ \mu_A(x), \mu_B(y) \} \geq t \text{ and } \max \{ \nu_A(x), \nu_B(y) \} \leq t$$
$$\Leftrightarrow \mu_A(x) \geq t \text{ and } \mu_B(y) \geq t \text{ and } \nu_A(x) \leq t \text{ and } \nu_B(y) \leq t$$
$$\Leftrightarrow \mu_A(x) \geq t \text{ and } \nu_A(x) \leq t \text{ and } \mu_B(y) \geq t \text{ and } \nu_B(y) \leq t$$
$$\Leftrightarrow x \in A_t \text{ and } y \in B_t$$
$$\Leftrightarrow (x,y) \in A_t \times B_t.$$

Therefore, $(A \times B)_t = A_t \times B_t$. 

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5.3 - HOMOMORPHISM AND ANTI-HOMOMORPHISM OF LEVEL
SUBGROUPS OF INTUITIONISTIC FUZZY SUBGROUPS

5.3.1 Theorem:

The homomorphic image of a level subgroup of an intuitionistic fuzzy
subgroup of a group G is a level subgroup of an intuitionistic fuzzy subgroup
of a group G'.

Proof:

Let G and G' be any two groups.

Let \( f: G \rightarrow G' \) be a homomorphism.

That is, \( f(xy) = f(x)f(y) \) for all \( x \) and \( y \in G \).

Let \( V = f(A) \), where A is an intuitionistic fuzzy subgroup of a group G.

Clearly V is an intuitionistic fuzzy subgroup of a group G'.

Let \( x \) and \( y \in G \), implies \( f(x) \) and \( f(y) \) in \( G' \).

Let \( A_t \) is a level subgroup of A.

That is, \( \mu_A(x) \geq t \) and \( \nu_A(x) \leq t \);

\[ \mu_A(y) \geq t \text{ and } \nu_A(y) \leq t; \]

\[ \mu_A(xy^{-1}) \geq t \text{ and } \nu_A(xy^{-1}) \leq t. \]

We have to prove that \( f(A_t) \) is a level subgroup of V.

Now,

\[ \mu_V( f(x) ) \geq \mu_A(x) \geq t, \text{ implies that } \mu_V( f(x) ) \geq t ; \]

\[ \mu_V( f(y) ) \geq \mu_A(y) \geq t, \text{ implies that } \mu_V( f(y) ) \geq t ; \text{ and} \]

\[ \mu_V( f(x)f(y))^{-1} = \mu_V( f(x)f(y) )^{-1}, \text{ as } f \text{ is a homomorphism} \]

\[ = \mu_V( f(xy^{-1}) ) , \text{ as } f \text{ is a homomorphism} \]

\[ \geq \mu_A( xy^{-1} ) \geq t. \]
which implies that $\mu_V(f(x)(f(y))^{-1}) \geq t$.

And,

$$v_V(f(x)) \leq v_A(x) \leq t,$$

implies that $v_V(f(x)) \leq t$ ; and

$$v_V(f(y)) \leq v_A(y) \leq t,$$

implies that $v_V(f(y)) \leq t$ ; and

$$v_V(f(x)(f(y))^{-1}) = v_V(f(x)f(y^{-1})), \text{ as } f \text{ is a homomorphism}$$

$$= v_V(f(xy^{-1})), \text{ as } f \text{ is a homomorphism}$$

$$\leq v_A(xy^{-1}) \leq t,$$

which implies that $v_V(f(x)(f(y))^{-1}) \leq t$.

Therefore, $\mu_V(f(x)(f(y))^{-1}) \geq t \text{ and } v_V(f(x)(f(y))^{-1}) \leq t$.

Hence $f(A_t)$ is a level subgroup of an intuitionistic fuzzy subgroup $V$ of a group $G'$.

5.3.2 Theorem:

The homomorphic pre-image of a level subgroup of an intuitionistic fuzzy subgroup of a group $G'$ is a level subgroup of an intuitionistic fuzzy subgroup of a group $G$.

Proof:

Let $G$ and $G'$ be any two groups.

Let $f: G \to G'$ be a homomorphism.

That is, $f(xy) = f(x)f(y)$ for all $x$ and $y \in G$.

Let $V = f(A)$, where $V$ is an intuitionistic fuzzy subgroup of a group $G'$.

Clearly $A$ is an intuitionistic fuzzy subgroup of a group $G$.

Let $f(x)$ and $f(y) \in G'$, implies $x$ and $y$ in $G$.

Let $f(A_t)$ is a level subgroup of $V$. 

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That is, \( \mu_v(\text{f}(x)) \geq t \) and \( \nu_v(\text{f}(x)) \leq t \);

\[
\mu_v(\text{f}(y)) \geq t \quad \text{and} \quad \nu_v(\text{f}(y)) \leq t;
\]

\[
\mu_v(\text{f}(x)(\text{f}(y))^{-1}) \geq t \quad \text{and} \quad \nu_v(\text{f}(x)(\text{f}(y))^{-1}) \leq t.
\]

We have to prove that \( A_t \) is a level subgroup of \( A \).

Now,

\[
\mu_{A_t}(x) = \mu_v(\text{f}(x)) \geq t, \text{ implies that } \mu_{A_t}(x) \geq t;
\]

\[
\mu_{A_t}(y) = \mu_v(\text{f}(y)) \geq t, \text{ implies that } \mu_{A_t}(y) \geq t; \text{ and }
\]

\[
\mu_{A_t}(xy^{-1}) = \mu_v(\text{f}(xy^{-1})),
\]

\[
= \mu_v(\text{f}(x)(\text{f}(y))^{-1}) \quad \text{as } \text{f} \text{ is a homomorphism}
\]

\[
= \mu_v(\text{f}(x)(\text{f}(y))^{-1}) \quad \text{as } \text{f} \text{ is a homomorphism}
\]

\[
\geq t,
\]

which implies that \( \mu_{A_t}(xy^{-1}) \geq t \).

And,

\[
\nu_{A_t}(x) = \nu_v(\text{f}(x)) \leq t, \text{ implies that } \nu_{A_t}(x) \leq t;
\]

\[
\nu_{A_t}(y) = \nu_v(\text{f}(y)) \leq t, \text{ implies that } \nu_{A_t}(y) \leq t; \text{ and }
\]

\[
\nu_{A_t}(xy^{-1}) = \nu_v(\text{f}(xy^{-1})),
\]

\[
= \nu_v(\text{f}(x)(\text{f}(y))^{-1}) \quad \text{as } \text{f} \text{ is a homomorphism}
\]

\[
= \nu_v(\text{f}(x)(\text{f}(y))^{-1}) \quad \text{as } \text{f} \text{ is a homomorphism}
\]

\[
\leq t,
\]

which implies that \( \nu_{A_t}(xy^{-1}) \leq t \).

Therefore, \( \mu_{A_t}(xy^{-1}) \geq t \) and \( \nu_{A_t}(xy^{-1}) \leq t \).

Hence \( A_t \) is a level subgroup of an intuitionistic fuzzy subgroup \( A \) of \( G \).
5.3.3 Theorem:

The anti-homomorphic image of a level subgroup of an intuitionistic fuzzy subgroup of a group $G$ is a level subgroup of an intuitionistic fuzzy subgroup of a group $G'$.

Proof:

Let $G$ and $G'$ be any two groups.

Let $f : G \rightarrow G'$ be an anti-homomorphism.

That is, $f(xy) = f(y)f(x)$ for all $x$ and $y \in G$.

Let $V = f(A)$, where $A$ is an intuitionistic fuzzy subgroup of $G$.

Clearly $V$ is an intuitionistic fuzzy subgroup of $G'$.

Let $x$ and $y \in G$, implies $f(x)$ and $f(y)$ in $G'$.

Let $A_t$ is a level subgroup of $A$.

That is, $\mu_A(x) \geq t$ and $\nu_A(x) \leq t$;

$\mu_A(y) \geq t$ and $\nu_A(y) \leq t$;

$\mu_A(y^{-1}x) \geq t$ and $\nu_A(y^{-1}x) \leq t$.

We have to prove that $f(A_t)$ is a level subgroup of $V$.

Now,

$\mu_V(f(x)) \geq \mu_A(x) \geq t$, implies that $\mu_V(f(x)) \geq t$;

$\mu_V(f(y)) \geq \mu_A(y) \geq t$, implies that $\mu_V(f(y)) \geq t$; and

$\mu_V(f(x)(f(y))^{-1}) = \mu_V(f(x)f(y^{-1}))$, as $f$ is an anti-homomorphism

$\geq \mu_A(y^{-1}x) \geq t$,

which implies that $\mu_V(f(x)(f(y))^{-1}) \geq t$. 

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And,

\[ v_{V}(f(x)) \leq v_{A}(x) \leq t, \text{ implies that } v_{V}(f(x)) \leq t; \]
\[ v_{V}(f(y)) \leq v_{A}(y) \leq t, \text{ implies that } v_{V}(f(y)) \leq t; \text{ and} \]
\[ v_{V}(f(x)(f(y))^{-1}) = v_{V}(f(x)f(y^{-1})), \text{ as } f \text{ is an anti-homomorphism} \]
\[ = v_{V}(f(y^{-1}x)), \text{ as } f \text{ is an anti-homomorphism} \]
\[ \leq v_{A}(y^{-1}x) \leq t, \]

which implies that \[ v_{V}(f(x)(f(y))^{-1}) \leq t. \]

Therefore, \[ \mu_{V}(f(x)(f(y))^{-1}) \geq t \text{ and } v_{V}(f(x)(f(y))^{-1}) \leq t. \]

Hence \( f(A_{A}) \) is a level subgroup of an intuitionistic fuzzy subgroup \( V \) of \( G'. \)

5.3.4 Theorem:

The anti-homomorphic pre-image of a level subgroup of an intuitionistic fuzzy subgroup of a group \( G' \) is a level subgroup of an intuitionistic fuzzy subgroup of a group \( G. \)

Proof:

Let \( G \) and \( G' \) be any two groups.

Let \( f : G \rightarrow G' \) be an anti-homomorphism.

That is, \( f(xy) = f(y)f(x) \), for all \( x \) and \( y \in G. \)

Let \( V = f(A) \), where \( V \) is an intuitionistic fuzzy subgroup of a group \( G'. \)

Clearly \( A \) is an intuitionistic fuzzy subgroup of a group \( G. \)

Let \( f(x) \) and \( f(y) \in G', \) implies \( x \) and \( y \) in \( G. \)

Let \( f(A_{A}) \) is a level subgroup of \( V. \)

That is, \[ \mu_{V}(f(x)) \geq t \text{ and } v_{V}(f(x)) \leq t; \]
\[ \mu_{V}(f(y)) \geq t \text{ and } v_{V}(f(y)) \leq t; \]
\[ \mu_{V}(f(y))^{-1}f(x)) \geq t \text{ and } v_{V}(f(y))^{-1}f(x)) \leq t. \]
We have to prove that $A_1$ is a level subgroup of $A$.

Now,

$$\mu_A(x) = \mu_V(f(x)) \geq t,$$

implies that $\mu_A(x) \geq t$.

$$\mu_A(y) = \mu_V(f(y)) \geq t,$$

implies that $\mu_A(y) \geq t$; and

$$\mu_A(xy^{-1}) = \mu_V(f(xy^{-1})),$$

$$= \mu_V(f(y^{-1})f(x)),$$

as $f$ is an anti-homomorphism

$$= \mu_V((f(y))^{-1}f(x)),$$

as $f$ is an anti-homomorphism

$$\geq t,$$

which implies that $\mu_A(xy^{-1}) \geq t$.

And,

$$\nu_A(x) = \nu_V(f(x)) \leq t,$$

implies that $\nu_A(x) \leq t$.

$$\nu_A(y) = \nu_V(f(y)) \leq t,$$

implies that $\nu_A(y) \leq t$; and

$$\nu_A(xy^{-1}) = \nu_V(f(xy^{-1})),$$

$$= \nu_V(f(y^{-1})f(x)),$$

as $f$ is an anti-homomorphism

$$= \nu_V((f(y))^{-1}f(x)),$$

as $f$ is an anti-homomorphism

$$\leq t,$$

which implies that $\nu_A(xy^{-1}) \leq t$.

Therefore, $\mu_A(xy^{-1}) \geq t$ and $\nu_A(xy^{-1}) \leq t$.

Hence $A_1$ is a level subgroup of an intuitionistic fuzzy subgroup $A$ of $G$. 

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