CHAPTER 4

*ON TRANSFINITE CONVEX DIMENSION

4.1 INTRODUCTION

In [VAD₁] Van de vel introduced the notion of convex dimension cind for a topological convex structure. In this chapter we introduce the notion of transfinite convex dimension trcind. In section 4.2 we compare the transfinite topological dimension and transfinite convex dimension (Prop 4.2.3). A characterization of trcind in terms of hyperplanes (Cor 4.2.5) is obtained. In section 4.3 we obtain a characterization of trcind in terms of mappings to cubes (Prop 4.3.1). Throughout this chapter we assume that the convex structure is $S_1$ and closure stable.

4.2 TRANSFINITE CONVEX DIMENSION

4.2.1. Definition

Let $X$ be a topological convex structure. Then:

1. $\text{trcind} (X) = -1$ if and only if $X = \emptyset$

2. $\text{trcind} (X) \leq \alpha$, where $\alpha$ is an ordinal if and only if to each pair consisting of a convex closed set $C$ and a point $p \in X \setminus C$, there exists a convex closed screening $(A, B)$ and an ordinal $\beta < \alpha$ such that $\text{trcind} (A \cap B) \leq \beta$.

We say that $\text{trcind} (X) = \alpha$ if and only if $\text{trcind} (X) \leq \alpha$ but $\text{trcind} (X) \neq \beta$ for any $\beta < \alpha$.

* Some of the results in this chapter are presented in the international conference on Transform Techniques and their applications at St. Joseph's College, Irinjalkuda, 2000 Dec.
4.2.2 Proposition

\[ \text{trcind (C)} \leq \text{trcind (X)} \text{ for each convex subset C of a topological convex structure X.} \]

Proof

Let \( \text{trcind (X)} = \alpha \), an ordinal. Assume that the result is true for spaces with dimension less than \( \alpha \). Let \( p \in C \) and \( D \subseteq C \), where \( D \) is a convex closed subset of \( C \) such that \( p \notin D \). Then \( \text{cl} (D) \), the closure of \( D \) is a convex closed set in \( X \) and \( D = \text{cl}(D) \cap C \) with \( p \notin \text{cl}(D) \). Since \( \text{trcind (X)} = \alpha \), there exists a \( \beta < \alpha \) and a screening \( (A, B) \) of convex closed sets in \( X \) such that \( \text{trcind (A \cap B)} \leq \beta \).

Take \( A' = A \cap C \) and \( B' = B \cap C \). Then \( (A', B') \) is a pair of convex closed sets in \( C \) which screen \( p \) and \( D \). Also \( A' \cap B' \subseteq A \cap B \), then by assumption \( \text{trcind (A' \cap B')} \leq \beta \).

4.2.3 Proposition

Let \( X \) be a topological convex structure of which the weak topology is separable and metrizable. Then \( \text{trind (X_w)} \leq \text{trcind (X)} + k \), for some integer \( k \).

(Here \( \text{trind} \) denotes the transfinite small inductive dimension).

Proof

Let \( \alpha \) be an infinite ordinal and let \( \text{trcind (X)} \leq \alpha \). Assume that the result is true if \( \text{trcind} < \alpha \). Let \( A \subseteq X \) be a closed set and \( p \notin A \). Since we are considering the weak topology, there exists convex closed sets \( C_1, C_2, \ldots, C_m \) such...
that \( A \subseteq \bigcup_{i=1}^{m} C_i \) and \( p \notin \bigcup_{i=1}^{m} C_i \). By the definition of \( \text{trcind} \), for each \( i = 1, 2, \ldots, m \), there is a convex closed screening \((D_i, E_i)\) of \( p \) and \( C_i \) such that \( \text{trcind}(D_i \cap E_i) < \beta_i \) where \( \beta_i < \alpha \). Take \( D = \bigcup_{i=1}^{m} D_i \). Then \( D \) is a closed neighbourhood of \( p \) disjoint from \( A \) and \( \text{Bd}(D) \subseteq (D_i \cap E_i) \). This is because

\[
\text{Bd}(D) \subseteq \bigcup_{i=1}^{m} \text{Bd}(D_i)
\]

\[
= \bigcup_{i=1}^{m} \text{cl}(D_i) \cap \text{cl}(X \setminus D_i)
\]

\[
= \bigcup_{i=1}^{m} (\text{cl}(D_i) \cap \text{cl}(E_i \setminus D_i)) \subseteq \bigcup_{i=1}^{m} (D_i \cap E_i).
\]

By induction hypothesis, \( \text{trcind}(D_i \cap E_i) \leq \beta_i + m_i \) for every \( i \), where each \( m_i \) is an integer. Now by the sum theorem for \( \text{trcind} \) ([CH2], See prop 0.3.10),

\[
\text{trcind \text{Bd}}(D) \leq \text{trcind}(\bigcup_{i=1}^{m} (D_i \cap E_i)).
\]

\[
\leq \max(\beta_i + m_i) + m
\]

\[
= \max\{\alpha + m_i\} + m
\]

\[
= \alpha + k \text{ for some integer } k.
\]

4.2.4 Proposition

Let \( X \) be a non empty FS3 space with connected convex sets. If \( H \subseteq X \) is a half space, then \( \text{trcind}(\text{cl}(H) \setminus H) < \text{trcind}(X) \).

Proof

Assume that \( \text{trcind}(X) \leq \alpha \), an ordinal and that the result is true if \( \text{trcind} < \alpha \). Let \( C \subseteq \text{cl}(H) \setminus H \) be a convex closed set and \( p \notin C \). Consider \( \text{cl}(H) \).
Since $X$ is closure stable, $\text{cl} \ (H)$ is convex in $X$. Then $\text{trcind} \ (\text{cl} \ (H)) \leq \text{trcind} \ (X) \leq \alpha$.

Also $C = \text{cl} \ (C) \cap \text{cl} \ (H) \setminus H$, where $\text{cl} \ (C)$ is convex closed in $\text{cl} \ (H)$ and $p \notin \text{cl} \ (C)$.

Then there exists a convex closed screening $(A, B)$ of $\text{cl} \ (C)$ and $p$ in $\text{cl} \ (H)$ such that $\text{trcind} \ (A \cap B) \leq \beta < \alpha$. Without loss of generality we can assume that $(A, B)$ is a minimal screening pair. Also since $H$ is dense in $\text{cl} \ (H)$, we can conclude that $H \cap A \cap B$ is dense in $A \cap B$. Therefore,

$$(\text{cl} \ (H) \setminus H) \cap A \cap B$$

$$= \text{cl} \ (H) \cap (A \cap B) \setminus H$$

$$= \text{cl} \ (H) \cap (\text{cl} \ (A \cap B)) \setminus H$$

$$= (A \cap B) \setminus H = (A \cap B) \setminus (H \cap A \cap B)$$

Since $(A \cap B)$ is a relative half space of $A \cap B$, by inductive hypothesis, $\text{trcind} \ ((\text{cl} \ (H) \setminus H) \cap A \cap B) \leq \gamma$, where $\gamma < \beta$. This shows that each relatively convex closed set $C$ of $\text{cl} \ (H) \setminus H$ and each point $p \notin C$ of $\text{cl} \ (H) \setminus H$, can be screened by convex closed sets of the form

$(\text{cl} \ (H) \setminus H) \cap A$ and $(\text{cl} \ (H) \setminus H) \cap B$ and $\text{trcind} \ ((\text{cl} \ (H) \setminus H) \cap A \cap B) \leq \gamma < \beta$.

Thus $\text{trcind} \ (\text{cl} \ (H) \setminus H) \leq \beta < \alpha$.

A set of the type $\text{cl} \ (H) \setminus H$ where $H$ is an open half space of $X$ is called a hyperplane.
4.2.5 Corollary

Let \( X \) be an FS3 space with connected convex sets. The following statements are equivalent.

1. \( \text{trcind} (X) \leq \alpha \), where \( \alpha \) is an ordinal.
2. Corresponding to each hyper pane \( H \subseteq X \), there exists a \( \beta < \alpha \) such that \( \text{trcind} (H) \leq \beta \).

Proof

(1) \( \Rightarrow \) (2) by using prop (4.2.4) above. Now assume (2). Let \( C \) be a convex closed set in \( X \) and \( p \in C \). By FS3, there exists a continuous \( cp \) functional \( f : X \to \mathbb{R} \) separating \( p \) and \( C \). Let \( f(C) \subseteq (-\infty,0] \) and \( f(p) > 0 \).

Take \( H = f^{-1} (-\infty, f(p)/2) \). Then \( \text{cl} (H) \) and \( \text{cl} (\overline{X\setminus H}) \) is a convex closed screening of \( p \) and \( C \) and \( \text{cl} (H) \cap \text{cl} (\overline{X\setminus H}) = \text{Bd} (H) \) and \( \text{trcind} (\text{Bd} (H)) \leq \beta < \alpha \).

4.2.6 Proposition

Let \( X \) be an FS3 space with connected convex sets. If \( C \) is a non-empty convex subset of \( X \) of dimension \( \alpha > 0 \), an ordinal, then the intersection of all relatively dense convex subsets of \( C \) is relatively dense in \( C \).

Proof

Assume that the result is true for all convex sets with dimension less than \( \alpha \). Let \( D \) be a convex closed set in \( X \) and \( p \in X \setminus D \). By FS3, there exists an \( \alpha \)-pseudo dense convex subset of \( D \) induces a dense convex subset of \( D \). Therefore the sets \( A_i \cap D \) are all dense in \( D \). Therefore \( \bigcap (A_i \cap D) = E \cap D \), which is a contradiction.

Let \( \text{trcind} (C) = \alpha \). Assume that the result is true for all convex sets with dimension less than \( \alpha \). Let \( E = \bigcap A_i \), where each \( A_i \) is a relatively dense convex subset of \( C \). To show that \( \text{cl} (E) = C \). Let \( p \in C \setminus \text{cl} (E) \). Then \( \text{cl} (E) \cap C \) is
a convex closed set in $\mathcal{C}$ and $p \not\in \text{cl} (E) \cap \mathcal{C}$. Then there exists a minimal convex closed screening of $\text{cl} (E) \cap \mathcal{C}$ and $p$ whose intersection $D$ satisfies $\text{trcind} (D) \leq \beta$, where $\beta < \alpha$. Also each dense convex subset of $\mathcal{C}$ induces a dense convex subset of $D$. Therefore the sets $A_i \cap D$ are all dense in $D$. Therefore $\bigcap_i \{ A_i \cap D \} = E \cap D$ is relatively dense $D$. Thus $E \cap D \neq \emptyset$, which is a contradiction.

4.2.7 Proposition

In an FS$_3$ space with connected convex sets, a convex set and its closure have the same convex dimension.

Proof

Let $X$ be the space and $C \subseteq X$ be convex in $X$. Without loss of generality assume that $C$ is dense in $X$. We will show that $\text{trcind} (C) = \text{trcind} (X)$. We have $\text{trcind} (C) \leq \text{trcind} (X)$. To show that $\text{trcind} (X) \leq \text{trcind} (C)$. Let $\text{trcind} (C) \leq \alpha$, an ordinal. We prove the result by transfinite induction.

Assume that the result is true for all convex sets with dimension less than $\alpha$. Let $D$ be a convex closed set in $X$ and $p \in X \setminus D$. By FS$_3$, there exists an open half space $O \subseteq X$ such that $D \subseteq \text{cl} (O)$ and $p \not\in \text{cl} (O)$. Now consider a minimal convex closed screening $D_1, D_2$ of $D$ and $p$ with $D_1 \subseteq \text{cl} (O)$ and $D_2 \subseteq X \setminus O$. Now $D_1 \cap D_2 \subseteq \text{cl} (O) \cap X \setminus O \subseteq \text{Bd} (O)$. Since $C$ is dense in $X$, $\text{cl}_C (O \cap C) = \text{cl} (O) \cap C$. Similarly $\text{cl}_C (X \setminus O \cap C) = X \setminus O \cap C$. Therefore $\text{Bd} (O) \cap C$ is the relative boundary of $O \cap C$ in $C$. Then, $\text{trcind} (D_1 \cap D_2 \cap C) \leq \text{trcind} (\text{Bd} (O) \cap C) \leq \beta < \alpha$. (By corollary (4.2.5)). Since $C$ is dense and convex,
D₁ ∩ D₂ ∩ C is a dense subset of D₁ ∩ D₂. By induction hypothesis
trcind (D₁ ∩ D₂ ∩ C) = trcind (D₁ ∩ D₂).

Thus trcind (X) ≤ α.

4.2.8 Proposition

In an FS₃ space with connected convex sets and of dimension α, an
ordinal, each dense half space has a non-empty interior. In fact, its interior meets
every non-empty convex open set of the space.

Proof

Let X be the space and let H ⊆ X be a dense half space. Let O ≠ ∅ be
a convex open set in X. Then H ∩ O is a relatively dense half space of O. By
corollary (4.2.4), trcind (O \ H) < trcind (O). Now by prop (4.2.7), O \ H is not
dense in O. Then Φ ≠ int O (O ∩ H) ⊆ int (H).

4.3 TRANSFINITE CONVEX DIMENSION AND CONVEXITY PRESERVING MAPS

4.3.1 Proposition

Let X be an FS₃ space with connected convex sets and let [0,1]^{N_0} be
equipped with the standard median convexity. If C ⊆ X is a convex set with
trcind(C) ≥ N₀, then there exists a continuous convexity preserving function
f = (fₙ) : X → [0,1]^{N₀}, where for each n, fₙ is a continuous convexity preserving
function from X → [0,1]ⁿ such that fₙ (C) = [0,1]ⁿ.
Proof

Since \( \text{trcind}(C) \geq \aleph_0 \), \( \text{trcind}(C) > n \) for all \( n \). Then for each \( n \), there
exists a continuous convexity preserving function \( f_n : X \rightarrow [0,1]^n \) satisfying
\( f_n(C) = [0,1]^n \) (See theorem[0.3.16 ]). Then the function \( f = (f_n) \) is a continuous
convexity preserving function from \( X \) to \( [0,1]^\aleph_0 \). For, let \( C \) be any subbasic
convex set in \( [0,1]^\aleph_0 \). Then \( C = \pi_i^{-1}(C_i) \), where \( C_i \) is convex in \( [0,1]^i \).

Therefore \( f^{-1}(\pi_i^{-1}(C_i)) = \{ x \in X : f_n(x) \in \pi_n(\pi_i^{-1}(C_i)) \text{ for all } n \} \)
\( = \cap_n f_n^{-1}(\pi_n(\pi_i^{-1}(C_i))) \), which is convex in \( X \).

4.3.2 Proposition

Let \( X \) and \( Y \) be FS3 spaces with connected convex sets and let \( f : X \rightarrow Y \) be a closed, continuous and convexity preserving function of \( X \) onto \( Y \).

Then \( \text{trcind}(X) \geq \text{trcind}(Y) \).

Proof

We will show that \( \text{trcind}(Y) \geq \alpha \) implies that \( \text{trcind}(X) \geq \alpha \), where \( \alpha \)
is any ordinal. Assume that the statement is valid for all \( \beta < \alpha \). Now if
\( \text{trcind}(Y) \geq \alpha \), then by corollary (4.2.5), there is an open half space \( O \) of \( Y \) such
that for any ordinal \( \beta < \alpha \), \( \text{trcind}(\text{Bd}(O)) \geq \beta \). Then the set \( P = f^{-1}(O) \) is an open
half space of \( X \) and since \( f \) is closed and surjective \( f(\text{Bd}(P)) = \text{Bd}(O) \). Hence \( f \)
duces a closed convexity preserving map from \( \text{Bd}(P) \) to \( \text{Bd}(O) \) which is onto
and by inductive assumption, \( \text{trcind}(\text{Bd}(P)) \geq \beta \). This implies that \( \text{trcind}(X) \geq \alpha \).
4.3.3 Corollary

Let $X$ be an FS$_3$ space with connected convex sets and with compact polytopes. Then $\text{trcind}(X) \geq \aleph_0$ if and only if for each $n$, there exists a polytope $P_n$ such that $\text{trcind}(P_n) \geq n$.

Proof

If for each $n$, there exists a polytope $P_n$ such that $\text{trcind}(P_n) \geq n$, then $\text{trcind}(X) \geq n$ for all $n$, and hence $\text{trcind}(X) \geq \aleph_0$. On the other hand if $\text{trcind}(X) \geq \aleph_0$, then there exists a continuous convexity preserving function $f = (f_n) : X \to [0,1]^\aleph_0$, where each $f_n : X \to [0,1]^n$ is continuous, convexity preserving and onto. For each $f_n$, take one pre-image of each corner point. Let $F_n$ be the resulting set. Then $f_n$ maps $\text{co}(F_n)$ onto $[0,1]^n$. Thus $\text{trcind}(\text{co}(F_n)) \geq n$. 

References


