CHAPTER 2
RELATIONS BETWEEN INFINITE
CONVEX INVARIANTS

2.1 INTRODUCTION

This chapter deals with relations among various invariants of a convex structure. Levi [LE] proved that for a finite subset of a convex structure, Radon dependence implies Helly dependence. In [HAR] Hammer proved that if $X$ is a join hull commutative space and has ramification property, then Radon dependence is equivalent to Helly dependence. In this chapter we first introduce the infinite Caratheodory number, infinite Radon number and infinite exchange number. We obtain relations between Radon, Caratheodory, Helly and exchange dependence for arbitrary subsets of a convex structure (prop 2.3.2). The inequalities of Levi [LE] and Sierksma [SI1] are discussed in the infinite context in (Prop 2.3.4). In (Prop 2.3.5), we investigate the behavior of convex invariants under convexity preserving images. We extend the Eckhoff-Jamison inequality [SI1] in (Prop 2.3.7).

2.2 INFINITE CONVEX INVARIANTS

In this section we introduce various infinite invariants.

2.2.1 Definition

Let $X$ be a convex structure and $F$ be any non empty subset of $X$. Then,

1. $F$ is said to be Caratheodory dependent if $\text{co}(F) \subseteq \bigcup_{a \in F} \text{co}(F \setminus \{a\})$. $F$ is said to be Caratheodory independent if it is not Caratheodory dependent.
(2) $F$ is Radon dependent if there is a partition $\{F_1, F_2\}$ of $F$ such that $\text{co}(F_1) \cap \text{co}(F_2) \neq \emptyset$. $F$ is said to be Radon independent if it is not Radon dependent.

(3) $F$ is called exchange dependent if for each $p \in F$, $\text{co}(F \setminus \{p\}) \subseteq \bigcup \{\text{co}(F \setminus \{a\}) | a \in F, a \neq p\}$. $F$ is said to be exchange independent otherwise.

2.2.2 Definition

Let $X$ be a convex structure. Then we say that the Caratheodory number $c(X) \leq \aleph_0$ if and only if each $F \subseteq X$ with $|F| > \aleph_0$ is Caratheodory dependent. In addition to this, if for any finite cardinal $\alpha$, there is a Caratheodory independent subset $F$ with $|F| \geq \alpha$, we say $c(X) = \aleph_0$. Generally if $\alpha$ is an infinite cardinal, then we say that $c(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is Caratheodory dependent. In addition if for any cardinal $\beta$ less than $\alpha$, there is a Caratheodory independent subset $F$ with $|F| \geq \beta$, then we say $c(X)$ is equal to $\alpha$.

2.2.3 Definition

The Radon number $r(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is Radon dependent, where $\alpha$ is any infinite cardinal. In addition if for any cardinal $\beta$ less than $\alpha$, there is a Radon independent subset $F$ with $|F| \geq \beta$, then we say $r(X)$ is equal to $\alpha$. 

Assume that the inclusion is not true. That is there exists
2.2.4 Definition

The exchange number $e(X) \leq \alpha$ if and only if each $F \subseteq X$ with $|F| > \alpha$ is exchange dependent, where $\alpha$ is any infinite cardinal. In addition if for any cardinal $\beta$ less than $\alpha$, there is an exchange independent subset $F$ with $|F| \geq \beta$, then we say $e(X)$ is equal to $\alpha$.

2.3 RELATIONS BETWEEN INFINITE CONVEX INVARIANTS

The properties given below are available for finite subsets of a join hull commutative space $X$. Here we prove them for arbitrary subsets of $X$.

2.3.1 Proposition

Let $X$ be a JHC space and $F \subseteq X$ be any set.

1. If $X$ has ramification property and if $F$ is Radon independent, then for each pair of subsets $F_1, F_2$ of $F$, $\text{co}(F_1) \cap \text{co}(F_2) = \text{co}(F_1 \cap F_2)$

2. If $X$ has decomposable segments and $F$ has at least two points, then for all $x \in \text{co}(F)$, $\text{co}(F) = \bigcup_{a \in F} \text{co}\{\{x\} \cup F \setminus \{a\}\}$.

Proof

Case 1.

Let $F$ be any subset of a convex structure $X$. Then $F_1 \cap F_2 = \emptyset$, then the result follows from Radon independence. We obviously have $\text{co}(F_1 \cap F_2) \subseteq \text{co}(F_1) \cap \text{co}(F_2)$. Now to show that

$$\text{co}(F_1) \cap \text{co}(F_2) \subseteq \text{co}(F_1 \cap F_2).$$

Assume that the inclusion is not true. That is there exists
\[ x \in \text{co}(F_1) \cap \text{co}(F_2) \text{ but } x \notin \text{co}(F_1 \cap F_2). \]

Since \( x \in \text{co}(F_1) \), by domain finiteness of the hull operator, there exists a finite subset \( F_1' \subseteq F_1 \) such that \( x \in \text{co}(F_1') \). Similarly there is a finite subset \( F_2' \subseteq F_2 \) such that \( x \in \text{co}(F_2') \). Thus \( x \in \text{co}(F_1') \cap \text{co}(F_2') \) and since the inclusion is true in the finite case \( x \in \text{co}(F_1' \cap F_2') \). But \( F_1' \cap F_2' \subseteq F_1 \cap F_2 \) and this contradicts that \( x \notin \text{co}(F_1 \cap F_2) \).

**Case 2.**

Let \( F \subseteq X \) be any subset. Fix \( x \in \text{co}(F) \). By domain finiteness we can find a finite set \( F_1 \subseteq F \) with \( x \in \text{co}(F_1) \). But we have \( \text{co}(F_1) = \bigcup_{a \in F_1} \text{co} \{ z \cup F_1 \setminus \{a \} \} \) for every \( z \in \text{co}(F_1) \). Let \( y \in \text{co}(F) \). We will show that \( y \in \text{co} \{ x \cup F \setminus \{a \} \} \) for some \( a \in F \). If \( y \in \text{co}(F_1) \), then

\[ y \in \text{co}(F_1) = \bigcup_{a \in F_1} \text{co} \{ x \cup F_1 \setminus \{a \} \} \subseteq \bigcup_{a \in F} \text{co} \{ x \cup F \setminus \{a \} \}. \]

If \( y \notin \text{co}(F_1) \), then we can find a finite subset \( F_2 \subseteq F \) such that \( y \in \text{co}(F_2) \). Take \( F_3 = F_1 \cup F_2 \). Then \( x, y \in \text{co}(F_3) \) and the result follows as in the above case.

### 2.3.2 Proposition

Let \( F \) be any subset of a convex structure \( X \). Then

1. Radon dependence implies Helly dependence.
2. If \( X \) is JHC and has ramification property, then Radon dependence is equivalent to Helly dependence.
3. If \( X \) is JHC and has decomposable segments, then Helly dependence implies exchange dependence.
4. If $X$ has the cone-union property, then exchange dependence implies Caratheodory dependence.

Proof

1. Let $F$ be Radon dependent. Then there is a partition $\{F_1, F_2\}$ of $F$ such that $\text{co} (F_1) \cap \text{co} (F_2) \neq \emptyset$. Let $p \in \text{co}(F_1) \cap \text{co}(F_2)$. For each $a \in F$, either $F_1 \subseteq F \setminus \{a\}$ or $F_2 \subseteq F \setminus \{a\}$. Then $p \in \text{co}(F \setminus \{a\})$. That is $\bigcap_{a \in F} \text{co}(F \setminus \{a\}) \neq \emptyset$. Therefore $F$ is Helly dependent.

2. Suppose $X$ is JHC and has ramification property. Let $F$ be Radon independent. Then $\bigcap_{a \in F} \text{co} (F \setminus \{a\}) \neq \emptyset$.

3. Let $F \subseteq X$ be Helly dependent. Then $\bigcap_{a \in F} \text{co} (F \setminus \{a\}) \neq \emptyset$.

4. Let $X$ satisfy the cone union property and let $F \subseteq X$ be exchange dependent. Fix a point $p \in F$. By exchange dependence we have $\text{co} (F \setminus \{p\}) \subseteq \bigcup \text{co}(F \setminus \{a\}/a \in F \setminus \{p\})$. Then by using cone union property, $\text{co} (F) = \text{co} (\{p\} \cup F \setminus \{p\}) \subseteq \bigcup \{\text{co}(\{p\} \cup \text{co}(F \setminus \{a\})/a \in F \setminus \{p\})\}$ $= \bigcup \text{co}(F \setminus \{a\}/a \in F \setminus \{p\})$. Therefore $F$ is Caratheodory dependent.
The following proposition can be used as an alternative definition for the Caratheodory number of a convex structure.

2.3.3 Proposition

Let $X$ be a convex structure, and $\alpha$ any infinite cardinal. Then $c(X) \leq \alpha$ if and only if for each $A \subseteq X$ and $p \in \text{co}(A)$, there is a subset $F$ of $A$ with $|F| \leq \alpha$ and $p \notin \text{co}(F)$.

Proof

Suppose that $c(X) \leq \alpha$ and $p \in \text{co}(A)$. By domain finiteness of the hull operator there is a finite set $F \subseteq A$ satisfying the condition. Now assume that the condition is true. Suppose $c(X) > \alpha$. Then there is a set $A \subseteq X$ with $|A| > \alpha$ which is Caratheodory independent. That is $\text{co}(A) \neq \bigcup_{a \in A} \text{co}(A \setminus \{a\})$. That is there is a point $x \in \text{co}(A)$ which is not in any of the sets $\text{co}(A \setminus \{a\})$ and in particular there is no subset $F$ of cardinality less than or equal to $\alpha$ containing $x$.

2.3.4 Proposition

Let $X$ and $Y$ be convex structures and $f: X \to Y$ be a convexity preserving surjection, then $h(X) \geq h(Y)$ and $r(X) \geq r(Y)$. If $f$ is also convex then $c(X) \geq c(Y)$ and $e(X) \geq e(Y)$. That is $c(X) \geq c(Y)$ and $e(X) \geq e(Y)$.

1. $h(X) \leq r(X)$

2. $e(X) \leq c(X) \leq \max\{h(X), e(X)\}$
Proof

1. Let \( r(X) = \alpha \), an infinite cardinal. We will show that \( h(X) \leq \alpha \).

Let \( F \subseteq X \) with \( |F| = \beta > \alpha \). Since \( F \) is Radon dependent there is a partition \( \{F_1, F_2\} \) of \( F \) such that \( \text{co}(F_1) \cap \text{co}(F_2) \neq \emptyset \). Let \( p \in \text{co}(F_1) \cap \text{co}(F_2) \). For each \( a \in F \), either \( F_1 \subseteq F \setminus \{a\} \) or \( F_2 \subseteq F \setminus \{a\} \). Then \( p \in \text{co}(F \setminus \{a\}) \). That is \( \cap_{a \in F} \text{co}(F \setminus \{a\}) \neq \emptyset \). Hence \( F \) is Helly dependent. Therefore \( h(X) \leq \alpha \).

2. Let \( c(X) = \alpha \), an infinite cardinal and let \( F \) be a subset of \( X \) with \( |F| > \alpha \). Take \( p \in F \). Then \( |F \setminus \{p\}| > \alpha \) and \( \text{co}(F \setminus \{p\}) \subseteq \cup_{a \in F} \text{co}(F \setminus \{a\}) \subseteq \cup_{a \in F} \text{co}(F \setminus \{a\}) \). This shows that \( F \) is exchange dependent. Therefore \( e(X) \leq c(X) \).

To prove the other inequality, let \( \max \{h(X), c(X)\} = \alpha \) and \( F \subseteq X \) with \( |F| > \alpha \). Then \( F \) is Helly dependent. Then there is a point \( p \in \cap_{a \in F} \text{co}(F \setminus \{a\}) \). Consider \( F \cup \{p\} \). Now \( |F \cup \{p\}| > \alpha \) and is exchange dependent. Therefore \( \text{co}(F) \subseteq \cup_{a \in F} \text{co}(F \cup \{p\} \setminus \{a\}) = \cup_{a \in F} \text{co}(F \setminus \{a\}) \).

2.3.5 Proposition

Let \( X \) and \( Y \) be convex structures and \( f: X \to Y \) be a convexity preserving surjection, then \( h(X) \geq h(Y) \) and \( r(X) \geq r(Y) \). If \( f \) is also convex to convex then \( c(X) \geq c(Y) \) and \( e(X) \geq e(Y) \).

Proof

Let \( h(X) = \alpha \), an infinite cardinal and let \( G \subseteq Y \) with \( |G| = \beta > \alpha \). We will show that \( G \) is Helly dependent in \( Y \). For each \( b \in G \) there exists a \( e \in X \)
such that \( f(a) = b \). Denote \( F = f^{-1}(G) \). Since \( f \) is convexity preserving, 
\[
\{ \text{co}(F \setminus \{ a \}) \} \subseteq \text{co}(G \setminus \{ b \}) \neq \emptyset.
\]
But \( \bigcap_{a \in F} \text{co} F \setminus \{ a \} \neq \emptyset \). Therefore \( \bigcap_{b \in G} \text{co}(G \setminus \{ b \}) \neq \emptyset \).

Suppose \( r(x) = \alpha \). Let \( G \subseteq Y \) with \( |G| = \beta > \alpha \). For each \( b \in G \) there exists \( a \in X \) with \( f(a) = b \). Take \( F = f^{-1}(G) \). Since \( |F| > \alpha \), there is a partition \( \{ F_1, F_2 \} \) of \( F \) such that \( \text{co}(F_1) \cap \text{co}(F_2) \neq \emptyset \). Since \( f \) is convexity preserving, 
\[
f(\text{co}(F_1)) \subseteq \text{co}(f(F_1)) = \text{co}(G_1) \quad \text{and} \quad f(\text{co}(F_2)) \subseteq \text{co}(f(F_2)) = \text{co}(G_2).
\]
Since \( \text{co}(F_1) \cap \text{co}(F_2) \neq \emptyset \), 
\[
\text{co}(G_1) \cap \text{co}(G_2) \neq \emptyset.
\]

Now suppose that \( f \) is both convexity preserving, convex to convex and \( c(X) = \alpha \), an infinite cardinal. Let \( G \subseteq Y \) with \( |G| = \beta > \alpha \). For each \( b \in G \) there is an element \( a \in X \) such that \( f(a) = b \). Take \( F = f^{-1}(G) \). Since \( f \) is both convexity preserving and convex to convex, 
\[
f(\text{co}(F \setminus \{ a \})) = \text{co}(f(F \setminus \{ a \})) = \text{co}(G \setminus \{ b \}).
\]
This shows that 
\[
\text{co}(G) \subseteq \bigcup \text{co}(G \setminus \{ b \}).
\]

Therefore \( G \) is Caratheodory dependent and \( c(Y) \leq \alpha \). If we take \( e(X) = \alpha \) and \( G \subseteq Y \) with \( |G| = \beta > \alpha \), we can see that \( \text{co}(G \setminus \{ p \}) \subseteq \bigcup_{b \in G} \text{co}(G \setminus \{ b \}) \), \( b \neq p \) for each \( p \in G \) and hence \( G \) is exchange dependent. Therefore \( e(Y) \leq \alpha \).
2.3.6 Proposition

Let $X$ and $Y$ be convex structures and $f: X \rightarrow Y$ be an isomorphism. Let $h(X) = \alpha$, then $h(Y) = \alpha$.

Proof

Let $G \subseteq Y$ with $|G| = \beta > \alpha$. Since $f$ is a bijection, for each $b \in G$, there exists $a \in F \subseteq X$ such that $f(a) = b$. Since $f$ is both convexity preserving and convex to convex, $f(\operatorname{co} F\setminus \{a\}) = \operatorname{co} (G \setminus \{b\})$. Since $\cap_{a \in F} \operatorname{co}(F\setminus \{a\}) \neq \emptyset$, we have $f(\cap_{a \in F} \operatorname{co}(F\setminus \{a\})) = \cap \operatorname{co}(G\setminus \{b\}) \neq \emptyset$. Therefore $G$ is Helly dependent.

The following proposition is an extension of Eckhoff-Jamison inequality $[S1_1]$. See proposition (0.2.6).

2.3.7 Proposition

Let $X$ be a convex structure with the infinite star Helly number $h^*(X) = \alpha$ and Caratheodory number $c(X) = \beta$, both infinite cardinals, then the Radon number $r(X)$ satisfies $r(X) \leq \max \{\alpha, \beta\}$

Proof

Let $F \subseteq X$ with $|F| > \max \{\alpha, \beta\}$. We will show that $F$ is Radon dependent. Take $p \in F$. Then the sets $\operatorname{co}(F \setminus \{p\})$ and $\operatorname{co}(F \setminus A)$ for $p \not\in A \subseteq F$ and $|A| \leq \beta$ meet $\alpha$ by $\alpha$. Suppose $\operatorname{co}(F \setminus \{p\})$ belongs to the collection of $\alpha$ sets chosen. Among the remaining sets of the type $\operatorname{co}(F \setminus A_i)$, note that $|\cup A_i| \leq \max \{\alpha, \beta\}$. Then there exists a point $q \in F$, such that $q \neq p$ and $q \not\in \cup A_i$. 

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\( q \in \text{co}(F\{p\}) \cap \cap \text{co}(F\{A_i\}) | A_i | \leq \beta \). If \( \text{co}(F\{p\}) \) is not in the collection, then \( p \in \cap \{ \text{co}(F\setminus A_i) / | A_i | \leq \beta \} \). Since \( h^*(X) = \alpha \), each collection of convex sets meeting \( \alpha \) by \( \alpha \) has non empty intersection. Therefore there is a point \( x \in \text{co}(F\{p\}) \cap \cap \text{co}(F\setminus (A_i)) / p \notin A \subseteq F, | A | \leq \beta \). Also, since the Caratheodory number of \( X \) is \( \beta \), there is a set \( A \subseteq F\{p\} \) with \( | A | \leq \beta \) and \( x \in \text{co}(A) \) (By prop.2.3.3). Then \( \{A, F\setminus A\} \) is a partition of \( F \).

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This chapter considers the invariants rank and generating degree (both infinite) of a convex structure \( X \). The generating degree was defined using a generalisation of Dilworth's theorem for posets (Prop.3.2.2). For a non-coarse convex structure, rank is less than or equal to the generating degree (Prop.3.2.4). In section 3.3 we generalize Tverberg's theorem using (infinite) partition numbers.

Following closely the results on gated amalgams by Bandelt, Chepoi, and Van de Vel [VADs], we consider some infinite convex invariants for gated amalgams in section 3.4.

3.2 RANK AND GENERATING DEGREE

3.2.1 Definition

The rank of a convex structure \( X \) is defined to be the number \( d(X) \) as \( d(X) \leq \alpha \) (any cardinal finite or infinite) if and only if each subset of \( X \) with cardinality greater than \( \alpha \) is convexly dependent.

*Some of the results in this chapter are included in the paper "Relationship between rank and generating degree" presented in the national conference on Mathematical modeling conducted by Kerala Mathematical Association at Basellius College, Kottayam, 2002 Jan."