Chapter 3

INTEGRABILITY OF NONLINEAR SCHRÖDINGER EQUATIONS

3.1 Introduction

Among the important class of nonlinear systems, the well known nonlinear Schrödinger equation (NLSE) plays a significant role in the theory of envelope of wave trains in which no dissipation occurs [1,2]. The wide range of applicability of NLSE is due to the presence of the special type of stable solitary wave solutions, called envelope solutions. Solitons have been actively studied as nonlinear excitations in various systems, including liquid crystals, conducting polymers and optical fibers. These solitons are solitary waves and possess the remarkable property of preserving their shape and speed except for a phase shift during collision. It is this property which accounts for its name soliton, in analogy with particle-like behavior.

Solitonic systems are believed to be integrable in the sense that they possess infinite number of conservation laws. Integrability as a mathematical property can be studied independently in terms of the existence of solitons. Once this property is established, the existence of solitonic behavior is predictable. As explained in Chapter II, the Painlevé test and the Lax method
are standard techniques that reveal the integrability of the equation. In this chapter, we apply these methods to the Nonlinear Cubic-Quintic Schrödinger Equation (NLCQSE) and Nonlinear Septic Schrödinger Equation (NLSSE).

NLSE is an ubiquitous nonlinear wave equation of dispersive type [3], making its appearance in a variety of physical problems ranging from the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly dispersive medium [4] to Bose-Einstein condensates in condensed matter physics. It also describes water waves [3] at the free surface of an ideal fluid, and represents intense optical pulse propagation in optical fibers and photorefractive media (Kerr media) [5, 6]. It possesses close connection with the dynamics of Heisenberg ferromagnetic spin systems [7].

### 3.2 Specific Nonlinear Schrödinger Equations

NLSEs can be indexed by the degree of nonlinearity as follows.

NLSEs of the form

\[
i\partial_t u + \Delta u = \pm |u|^2 u
\]  

are the nonlinear cubic Schrödinger equation (NLCSE) which is a completely integrable system.

NLSEs of the form

\[
i\partial_t u + \Delta u = \pm |u|^4 u
\]  

are nonlinear quintic Schrödinger equations (NLQSE).

NLSEs of the form

\[
i\partial_t u + \Delta u = \pm |u|^6 u
\]  

are nonlinear septic Schrödinger equations (NLSSE).
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The NLSE or more precisely, the nonlinear cubic Schrödinger equation (NLCSE) reads as

\[ iU_t + \beta U_{xx} + \gamma |U|^2 U = 0, \]  

where \( \beta \) and \( \gamma \) are constants. Here \( U \) represents the potential which has the effect of trapping the wave energy, which otherwise tends to spread due to dispersion. At some values of the pulse width, the nonlinearity balances dispersion and a stationary pulse can be formed.

Zakharov and Shabat [8] solved the NLSE by IST. NLSE has two types of solution, namely bright soliton solutions which arise when the dispersion and cubic nonlinear coefficient have identical signs, and dark soliton solutions which occur when the two coefficients have opposite signs. Grimshaw has investigated the slowly varying solitary wave solution of the variable coefficient NLSE [9]. The NLSE describes phenomena with weak nonlinearity and strong dispersion such as waves in deep water, self-focusing of laser in dielectric, propagation of signals in optical fibers etc. This equation is obtained by neglecting optical higher order nonlinearity terms except the third order one. NLSE plays an important role both in experimental and theoretical studies in optical fiber communication.

In some cases, the effect of the fifth order nonlinear term dominates, while the third order effect may be absent. Then the above equation can be modified to be the nonlinear quintic Schrödinger equation (NLQSE). This equation can be expressed in terms of complex amplitude \( U(x,t) \) as
where $\gamma$ and $\delta$ are constants. Here the second term originates from group velocity dispersion (GVD). The third term in the equation originates from the nonlinear effect, due to the fact that the wavelength depends on the intensity of the wave. It is assumed that the refractive index $n_2$ is absent and only $n_4$ is present. NLQSE is obtained by modifying NLCSE by considering different order nonlinearity term in the electrical susceptibility of the medium in developing modified NLSE. The first step towards this was taken by Pushkarov et al [10] and later by Cowan et al [11]. They considered the effect of the quintic term appearing along with the conventional NLCSE. They have obtained the solutions of this nonlinear cubic-quintic Schrödinger equation (NLCQSE). NLQSE, NLCQSE, NLSSE and similar equations with higher order terms arise in the study of nonlinearities which might come into play at high intensities. We have studied the integrability of NLCQSE and extended the approach to higher orders.

### 3.3 Higher Order Nonlinear Schrödinger Equations

The response of any dielectric to light becomes nonlinear for intense electromagnetic fields. At a fundamental level, the origin of nonlinear responses is related to anharmonic motion of bound electrons under the influence of an applied field [5]. As a result, the total polarization $P$ induced by electric dipoles is not linear in the electric field $E$, but satisfies the more general relation

$$P = \varepsilon_0 (\chi^{(1)} + \chi^{(2)} : EE + \chi^{(3)} : EEE + \ldots)$$

(3.6)
where $\varepsilon_0$ is the vacuum permittivity and $\chi^{(j)}(j = 1, 2, \ldots)$ is the $j^\text{th}$ order susceptibility. In general, $\chi^{(j)}$ is a tensor of rank $j + 1$. The linear susceptibility $\chi^{(1)}$ represents the dominant contribution to $P$. Its effect is included through the refractive index and attenuation constant. The second order susceptibility $\chi^{(2)}$ is responsible for nonlinear effects such as second harmonic generation and sum frequency generation. However, it is nonzero only for media that lack inversion symmetry at a molecular level. The lowest order nonlinear effect in optical fibers originates from third order susceptibility $\chi^{(3)}$, which is responsible for phenomena such as third harmonic generation, four-wave mixing and nonlinear refraction. Most of the nonlinear effects in optical fibers, therefore, originate from nonlinear refraction, a phenomenon referring to the intensity dependence of refractive index. In its simplest form, the refractive index can be written as

$$\tilde{n}(w) = n(w) + n_2 |E|^2 + \ldots$$ (3.7)

$n(w)$ is the linear index coefficient, $|E|^2$ is the optical intensity inside the fiber and $n_2$ is the nonlinear index coefficient related to $\chi^{(3)}$. These results can be used in the wave equation (5.31) to obtain different nonlinear equations.

### 3.3.1 Nonlinear Quintic Schrödinger Equation (NLQSE)

Mohanachandran et al [11] developed the nonlinear Schrödinger equation with quintic nonlinearity, and hence the equation is called nonlinear quintic Schrödinger equation (NLQSE), given by
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\[ iU_t + \gamma U_{xx} + \delta |U|^4 U = 0 \]  

(3.8)

A solitary wave solution has been obtained as

\[ U(x,t) = \frac{C_1 \exp(ikt)}{\sqrt{(C_2 \cosh \sqrt{2} C_1 t + 1)}} \]  

(3.9)

where \( C_1 = \sqrt{2k}, C_2 = \sqrt{1 + 4 \delta} \)

The solution to NLQSE is found to possess soliton behavior. They also studied the stability of the solution using variation method, and it is found to be advantageous over the solution to the conventional NLQSE. Since the pulse-width of the soliton solution of NLQSE is less than that of NLCSE, the solution of NLQSE is more stable. Cowan et al [12] have tested the stability of this solution using numerical methods.

3.3.2 Nonlinear Cubic-Quintic Schrödinger Equation (NLCQSE)

The NLCQSE considered is

\[ iU_t + \gamma U_{xx} + \delta |U|^2 U + \delta |U|^4 U = 0 \]  

(3.10)

In this equation, we consider the quintic-term along with the cubic term. The Painlevé analysis and Lax method are used to test the integrability of this equation.

3.3.2.1 Painlevé Analysis

In order to check the integrability of the higher order NLSEs, we use the Painlevé P-test for PDEs. As explained Chapter II, this method consists of determining the presence or absence of movable, critical singularity manifolds. When the system is free from movable critical manifold the P-
property holds, suggesting P-integrability. As already discussed in Chapter II, the three main steps involved in the P-analysis of PDE are (1) determination of leading order behavior; (2) identification of powers called resonances at which arbitrary functions can enter into Laurent series; (3) verifying that at the resonant values, sufficient number of arbitrary functions exist without introduction of movable critical manifolds. In the following section, this method is applied to higher order NLSEs such as NLQCSE and NLSSE.

Before taking the solution in the form of Laurent series, let us write the equation (3.10) in terms of two complex valued functions \( q \) and \( p \), defined by \( q = U \) and \( p = U^* \)

Then the equation becomes

\[
\begin{align*}
iq_t + \gamma q_{xx} + \delta q^2 p(1 + qp) &= 0 \\
jp_t + \gamma p_{xx} + \delta p^2 q(1 + pq) &= 0
\end{align*}
\]

The solution of the equation in the Laurent series form is

\[
\begin{align*}
q &= \phi^{\alpha} \sum_{j=0}^{\infty} q_j \phi^j \\
p &= \phi^{\beta} \sum_{j=0}^{\infty} p_j \phi^j
\end{align*}
\]

To simplify calculation, we use the Kruskal ansatz: \( \phi(x,t) = x + \psi(t) \), where \( \psi(t) \) is an arbitrary function, and \( q \) and \( p \) are analytic functions such
that \( q_0 = 0 \) and \( p_0 = 0 \) in the neighborhood of noncharacteristic movable singularity manifold.

1. To find the leading order terms, assume that the leading orders are of the form \( q = q_0 \phi^\alpha \) and \( p = p_0 \phi^\beta \). Substituting in (3.11) and (3.12) and balancing dominant terms, we get \( \alpha = \beta = -\frac{1}{2} \) and \( p_0^2 q_0^2 = -\frac{3\gamma}{4\delta} \).

2. To get resonances, that is powers at which the arbitrary functions enter into generalized Laurent series, equations (3.13) & (3.14) is substituted in equations (3.11) and (3.12). Equating the leading order terms, the resonance equation obtained is given below.

\[
(j^2 - 2j - 3)(j - 2) = 0
\]

The resonant values are \( j = -1, 0, 2, 3 \). The resonances at \( j = -1, 0 \) imply that \( \psi(t) \) is arbitrary, and that there is only one equation defining \( p_0 \) and \( q_0 \) respectively (that is \( p_0 \) and \( q_0 \) is arbitrary).

3. To probe the existence of a sufficient number of arbitrary functions, equations (3.13) & (3.14) are substituted (3.11) & (3.12) and the co-efficients of different powers of \( \phi \) are collected.

For

\[
\phi^{-\frac{5}{2}} \Rightarrow p_0^2 q_0^2 = -\frac{3\gamma}{4\delta} \tag{3.16}
\]

\[
\phi^{-\frac{3}{2}} \Rightarrow p_0 q_0 + q_0 p_1 = -\frac{3}{8} \tag{3.17}
\]
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\[
p_0 q_1 - q_0 p_1 = \left(\frac{1}{\gamma}\right) i p_0 q_0 \phi_i \tag{3.18}
\]

\[
\phi^{-\frac{1}{2}} \Rightarrow \frac{3}{2} p_0 q_0 (q_0 p_2 - p_0 q_2) = i \frac{d}{dt} (p_0 q_0 - \frac{3}{2} \phi) \tag{3.19}
\]

The above equation indicates the arbitrariness of \( p_2 \) and \( q_2 \).

Similarly equating the powers of \( \phi^{\frac{1}{2}} \) and \( \phi^{\frac{3}{2}} \) leads to the arbitrariness of \( q_3 \) or \( p_3 \) and \( q_4 \) or \( p_4 \) respectively.

Thus the solution \((q, p)\) of NLCQSE admits required number of arbitrary functions without the introduction of movable critical manifolds. Since the exponent \( \alpha \) is half integer, we consider that NLCQSE is integrable in the restricted Painlevé sense.

### 3.3.2.2 Lax Method

The general version of the Lax method requires that if a given nonlinear equation can be written in terms of two operators, \( L \) and \( M \) where \( L \) is the operator of the specific problem, and \( M \) the operator governing the associated time evolution of the eigenfunction, then the compatibility condition is

\[
L_t - M + [L, M] = 0, \tag{3.20}
\]

which leads to original nonlinear equation.

Consider the NLCQSEs in equations (3.10), (3.11) & (3.12). Here we shall confine ourselves to the case where \( L \) and \( M \) are 2 x 2 matrices and \( L \) is a linear function of the form,
where \( q \) and \( r \) are complex valued functions of \( x \) and \( t \),

and

\[
M = 2i\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2i\lambda \begin{bmatrix} 0 & q \\ r & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & q \\ -r & 0 \end{bmatrix} - i\frac{\delta}{2} \begin{bmatrix} rq & 0 \\ 0 & -rq \end{bmatrix}
\]

\[
-\frac{i}{2} \begin{bmatrix} (rq)^2 & 0 \\ 0 & (-rq)^2 \end{bmatrix}
\]

Therefore,

\[
L_i = i \begin{bmatrix} 0 & q_i \\ r_i & 0 \end{bmatrix}
\]

and

\[
M_x = 2i\lambda \begin{bmatrix} 0 & q_x \\ r_x & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & q_x \\ -r_x & 0 \end{bmatrix} - i\frac{\delta}{2} \begin{bmatrix} (rq)_x & 0 \\ 0 & -(rq)_x \end{bmatrix}
\]

\[
-\frac{i}{2} \begin{bmatrix} (rq)_x^2 & 0 \\ 0 & (-rq)_x^2 \end{bmatrix}
\]

Substituting the values of \( L_i, M_x, L \) and \( M \) in equation (3.20) and solving, it is found that this equation is equivalent to the system of equations:

\[
ir_t + yr_{xx} + \delta r^2 q + \delta r^3 q^2 = 0
\]

\[
-iq_t + \gamma q_{xx} + \delta q^2 + \delta q^3 = 0
\]
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Setting \( q = r^* \) & \( r = q^* \), we get

\[
ir_t + r_{xx} + \delta |r|^2 (1 + |r|^2) r = 0
\]

(3.27)

\[
-iq_t + r q_{xx} + \delta |q|^2 (1 + |q|^2) q = 0
\]

(3.28)

provided \( \frac{\delta}{2} = r \) (3.29)

and

\[
\frac{\partial f(x,t)}{\partial x}[|q|^4 - |q|^2] = 0
\]

(3.30)

The existence of Lax pairs indicates that the system given by the NLCQSE may be integrable. The last equation emerges as a restriction on Lax integrability. Therefore the NLCQSE is integrable in the restricted Lax sense as well.

3.3.3 Nonlinear Septic Schrödinger Equation (NLSSE)

3.3.3.1 Painlevé Analysis

The NLSSE is given by

\[
iU_t + \gamma U_{xx} + \delta |U|^6 U = 0
\]

(3.31)

Now as in the previous case, the above equation is rewritten in terms of two complex valued functions \( q \) and \( p \), defined by \( q = U \) and \( p = U^* \). Then the equation becomes,

\[
iq_t + r q_{xx} + \delta q^5 p^2 = 0
\]

(3.32)
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\[-ip_x + p_{xx} + \delta \phi^5 q^2 = 0\]

(3.33)

The Laurent series form is

\[q = \phi^\alpha \sum_{j=0}^{\infty} q_j \phi^j\]

(3.34)

\[p = \phi^\beta \sum_{j=0}^{\infty} p_j \phi^j\]

(3.35)

Here also Kruskal ansatz is used.

1. In order to find the leading order terms, the leading orders are assumed to be of the form \(q = q_0 \phi^\alpha\) and \(p = p_0 \phi^\beta\). Substituting in (3.32) and (3.33) and balancing dominant terms, the value of \(\alpha = \beta = -\frac{1}{3}\) and \(p_0^3 q_0^3 = \frac{4\gamma}{9\delta}\).

2. To find resonant values, equation (3.34) and (3.35) is expanded and substituted with \(\alpha = -\frac{1}{3}\) in (3.34) and (3.35). Equating leading order terms, the resonant values are found at \(j = \frac{15 \pm 25i}{18}, 0\) and \(\frac{5}{3}\).

One of the resonances is found to be imaginary. According to Painlevé, imaginary resonance indicates nonintegrability [13].

\[3.3.3.2\ \text{Lax Method}\]

From the failure of the Painlevé test, it follows that for the NLSSE, one may predict the breakdown of Lax test also. However, Lax pair is tried for this equation. But it seems that there do not exist two
linear operators which obey Lax condition for integrability. Hence NLSSE is not integrable in the Lax sense too.

3.4 Conclusion

We have analyzed the problem of integrability of the nonlinear cubic-quintic Schrödinger (NLCQSE) equation using the Painlevé and Lax methods. Even though the cubic case has long been known to be integrable, the complete cubic-quintic system is integrable only in the restricted sense, that is, subject to some conditions pinned on the modulus of the complex function $q$. The quintic system is also known to be integrable in the restricted sense [14]. The septic and all higher order cases are non-integrable. However, for modeling optical solitons of very narrow pulse width, the property of restricted integrability may be sufficient.
References