Chapter 1

NONLINEAR PHENOMENA AND SOLITONS

1.1 Introduction

Progress in physics as in many other subjects follows the trend of diversifying and discovering as many new objects as possible and finally attempting to solve them. One of the important aspects is the evolution of physical systems, which is taking place as a result of forces acting on them. The evolution will result in the dynamics of the physical system. The nature of forces will determine whether it is a linear dynamical system or a nonlinear dynamical system. Linear systems are characterized by a class of linear differential equations. Most of the methods in mathematics can be used to solve linear differential equations. But nonlinear systems are characterized by nonlinear differential equations, ordinary differential equations (ODE) or partial differential equations (PDE). These equations cannot be solved by methods used to solve a linear differential equation. Since most of the natural phenomena are nonlinear in nature, nonlinear ODEs and PDEs play a central role in almost all physical systems.

A remarkable development in our understanding of certain classes of nonlinear PDEs known as evolution equations, has taken place in the past
two decades. The key to our present knowledge of these equations is the realization that they possess a special type of elementary solution. These special solutions take the form of localized disturbances or pulses that retain their shape even after interaction among themselves. Thus they act like particles. These localized disturbances have come to be known as a solitons. The concept of solitons and the inverse scattering technique (IST) for exact solutions of some nonlinear PDEs have had far-reaching influence and consequence in various branches of mathematics, physics and engineering [1-3]. The most interesting developments that have been made in the study of nonlinear PDEs were by means of the methods developed by a group of scholars centered around Martin Kruskal, in the 1960s [1]. The advent of new ideas and methods, results and applications, has led to the study of nonlinear PDEs in subjects such as fluid mechanics, nonlinear optics, nonlinear circuits, neural networks and spin systems [4-7].

The development of mathematical properties of a large class of solvable nonlinear evolution equations started with the first recorded observation of the ‘great solitary’ wave by Scott Russell in 1834. The other class of nonlinear evolution equations includes the Korteweg-de Vries (KdV), the sine-Gordan (sG) and the nonlinear Schrödinger equations (NLSE) [6].

Nonlinear differential equations fall into two broad categories, the so-called “integrable” and “nonintegrable” systems. As a consequence, in nonlinear dynamics, there emerged two prominent sub-disciplines, “Solitons” and “Chaos”, corresponding to the integrable and nonintegrable types, respectively. The “solitonic systems” are characterized by regular or
predictable behavior for all times of evolution. Contrary to this, “chaotic systems” exhibit completely irregular and unpredictable behavior with sensitive dependence on initial conditions.

1.2 The Soliton and Its History

In nature, not all waves disperse or spread, and hence diminish over distances. But there are many cases of fairly permanent and powerful waves that have the capacity to travel extraordinary distances without diminishing in size or shape, and which are described by nonlinear equations of dispersive type. This kind of a wave was first observed by the Scottish naval architect Scott Russel in 1834.

In 1830s Scott Russel carried out investigations on the shape of hulls of ships and observed the speed and forces needed to propel them. In 1834, riding on horseback, he observed the “Great Wave of Translation” in the union canal and reported his observations to the British Association [8]:

“I believe I shall best introduce the phenomenon by describing the circumstances of my own acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion, it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a
rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel”.

This rolling pile of water is a solitary wave which maintained its shape and speed much larger than the conventional wave. Scott-Russel also performed laboratory experiments (Fig 1.1) [8], generating solitary waves by dropping a weight at one end of water channel, and obtained the relation \( c^2 = g(h + a) \), where \( h \) is the undisturbed depth of water, \( a \) is the amplitude of the wave, \( g \) the acceleration due to gravity and \( c \) the speed of the wave.

![Fig 1.1 Scott Russell’s laboratory experiment: solitary waves in a tank](image)

The Dutch Physicists Korteweg and de Vries, in 1895, deduced a famous wave equation based on Scott Russell’s explanation and obtained
the “Korteweg-de Vries” (KdV) equation [9] which is a simple nonlinear dispersive wave equation whose modern version is given by

\[ U_t + 6UU_{xx} + U_{xxx} = 0 \]  \hspace{1cm} (1.1)

The solution is written as

\[ U(x,t) = \frac{c}{2} \text{sech}^2\left[ \frac{\sqrt{c(x-ct)}}{2} \right] \]  \hspace{1cm} (1.2)

which is shown in Fig 1.2 [8].

![Fig: 1.2. Solitary wave solution of the KdV equation](image)

Modern developments in the theory and application of KdV solitary wave began with the famous work of E. Fermi, J. Pasta and S. Ulam for the dynamics of a chain of weakly coupled oscillators in the early 1950s [3], as shown in figure (1.3) [8]. Their investigations identified that in presence of nonlinearity the energy assigned to the lowest mode is returned back to that mode after exciting only the first few modes in a characteristic time called recurrence time. This phenomenon is called the FPU recurrence phenomenon [2].
Their observation was contradictory to the concept of equipartition of energy, where the total energy of a molecule is equally distributed among all degrees of freedom of molecule and the average kinetic energy per degree of freedom at temperature $T$ is $\frac{kT}{2}$, where $k$ is the Boltzmann constant. For a decade, the FPU problem remained as one unrelated to solitary waves. Norman Zabuski and Martin Kruskal studied the KdV equation as a model FPU problem and reconfirmed the recurrence phenomenon. Zabuski and Kruskal [10] in 1965 carried out numerical experiments on the KdV equation and an explanation was given by them for the above anomaly. They had observed that the delicate balance between the nonlinear term $(UU_x)$ and dispersion term $(U_{xxx})$ in equation (1.1) results in a solitary wave pulse which moves with uniform velocity proportional to the amplitude. Another important observation is that, after interaction, such solitary waves emerge unaffected in their amplitude and velocity, except for phase shifts. This made Zabuski and Kruskal to coin the term ‘Soliton’ to characterize it, in analogy with particles.

The new concepts that emerge from Zabuski-Kruskal experiments are:

i) A solitary wave can arise when the nonlinearity balances linear dispersion.
ii) In appropriate nonlinear systems, these solitary waves can interact elastically like particles without changing their shape and velocities.

At the time of the discovery of solitons, there was no mathematical tool for solving the initial value problem of integrable nonlinear PDEs. Gardner, Greene, Kruskal and Miura (GGKM) [11,12] subsequently developed an elegant technique known as the inverse scattering transform (IST) to solve nonlinear PDEs. They formulated the initial value problem for the KdV equation, resorting to the ideas of direct and inverse scattering technique. This method is one of the basic tools used to study Nonlinear Evolution Equations (NLEE). It provides a procedure for explicitly obtaining the soliton solutions and qualitative information concerning the general solutions. A rigorous mathematical approach suitable for dealing with nonlinear problems was set up by Lax [13]. Accordingly, an integrable PDE can be brought to a standard form involving what is called “Lax pair” of operators. Following this, Zakharov and Shabat [14] generalized this to a (2x2) linear matrix eigenvalue problem and have shown that the nonlinear Schrödinger equation (NLSE) given by

$$iq_t + q_{xx} + |q|^2 q = 0$$  \hspace{1cm} (1.3)$$

is solved by IST and obtained the soliton solution for it. The so-called AKNS formalism was developed by Ablowitz, Kaup, Newell and Segur [15] in order to identify a class of soliton possessing nonlinear evolution equations. Besides IST, there exist several methods like Hirota bilinearization method [16, 17] and Backlund transformation [18] that are used to solve integrable nonlinear PDEs.
1.3 Examples of Nonlinear Evolution Equations

There exist many other important nonlinear dynamical systems which are also soliton-possessing nonlinear evolution equations of dispersive type, having the same general properties as the KdV equation. They are also completely integrable through IST procedure, and expressible in bilinear form. They possess Lax pairs, infinite conservation laws, and Backlund transformations which are discussed in the Chapter II. Some standard examples are the modified KdV equation, the sG equation, the NLSE, the continuous Heisenberg ferromagnetic spin equation and the Toda lattice equation.

1.3.1 The Korteweg-de Vries (KdV) Equation

The KdV equation, as described earlier, was deduced by the Dutch physicists Korteweg and de Vries as an ultimate explanation of the Scott-Russel phenomena. It is a simple nonlinear differential equation of the form given in equation (1.1). It is a completely integrable, infinite-dimensional nonlinear dynamical system, which possesses soliton solutions, exhibiting particle-like collision properties. The initial value problem of the KdV equation is completely solvable through the IST, making use of Lax operators. Using these linear operators, the KdV equation can be written as the consistency condition between two linear differential equations. This system not only appears in the description of shallow water, but also in the wave propagation in anharmonic lattices, in many other physical systems, in unidirectional wave propagation such as ion-acoustic plasma waves, collision-free hydro magnetic waves, stratified internal waves and so on [4,5].
1.3.2 The Modified KdV (MKdV) Equation

The wave propagation in the FPU lattice with quadratic nonlinear force may be described in a nontrivial way by the KdV equation. The lattice with cubic nonlinear force may be described by the MKdV equation:

\[ U_t + 6U_x^2 U_x + U_{xxx} = 0 \]  \hspace{1cm} (1.4)

1.3.3 The sine-Gordan (sG) Equation

The sG equation is a nonlinear hyperbolic equation in (1 + 1) dimensions, involving the D'Alembert operator and the sine of the unknown function [19]. It was originally considered in the nineteenth century for the study of surfaces of constant negative curvature. This equation attracted a lot of attention in the 1970s due to the presence of soliton solutions. There are two equivalent forms of the sG equation. In the (real) space-time coordinates, the equation is given by

\[ \psi_{tt} - \psi_{xx} + \sin \psi = 0 \]  \hspace{1cm} (1.5)

The sine–Gordon equation has the following 1-soliton solutions:

\[ \psi(x,t) = 4 \arctan e^{\gamma(x-vt) + \delta} \]  \hspace{1cm} (1.6)

where

\[ \gamma^2 = \frac{1}{1-v^2} \]  \hspace{1cm} (1.7)

The 1-soliton solution for which we have chosen the positive root for \( \gamma \), is called a kink, and represents a twist in the variable \( \psi \) which takes the system from one solution \( \psi = 0 \) to an adjacent one with \( \psi = 2\pi \). The 1-
soliton solution in which we take the negative root for $\gamma$ is called an antikink. The sG equation possesses multisoliton solutions as well. It can be used as a model equation for the propagation of transverse electromagnetic waves in superconducting systems.

1.3.4 The Toda Lattice Equation

A model equation introduced by M. Toda in 1967 to study wave propagation in nonlinear lattice admitting soliton solutions is the Toda Lattice equation which is given by

$$U_n = \exp[-(U_n - U_{n-1})] - \exp[-(U_{n+1} - U_n)]$$

(1.8)

where $n = 1, 2, 3, \ldots, n$.

1.3.5 The Nonlinear Schrödinger Equation (NLSE)

In theoretical physics, the Nonlinear Schrödinger Equation (NLSE) is a nonlinear version of Schrödinger equation in quantum mechanics. NLSE is an example of a universal nonlinear model that describes many physical nonlinear systems. It is a classical field equation with applications in hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids and various other nonlinear instability phenomena. Unlike the Schrödinger equation, it never describes the time evolution of a quantum state. It is an example of an integrable model and can be solved with the IST. The equation can be applied to modulation-instability of plane waves and the appearance of solitary waves will be analyzed [19]. In optics, the NLSE occurs in the Manakov system, a model of wave propagation in fiber optics. The equation models many nonlinearity effects in a fiber including, self-phase
modulation, four-wave mixing, second-harmonic generation, stimulated Raman scattering, etc.

The NLSE or more precisely, the nonlinear cubic Schrödinger equation (NLCSE) reads as

\[ iU_t + \beta U_{xx} + |U|^2 U = 0 \]  \hspace{1cm} (1.9)

where \( \beta \) and \( \gamma \) are constants. Here \( U \) represents the potential which has the effect of trapping the wave energy, which otherwise tends to spread due to dispersion. The different types of NLSE and its integrability aspects are considered in detail in Chapter III. The soliton propagation using NLSE is as shown in (Fig 1.4) [19].

![Image](image.png)

**Fig. 1.4** NLSE in one transverse dimension

Multi-component nonlinear evolution equations may exhibit shape-changing collision properties of solitons, in contrast to the elastic (shape-preserving) collision of solitons. For example, the coupled NLSE (CNLSE)
or the Manakov system, describing two modes (birefringence) propagation in optical fibers [20], admits shape-changing collisions [21, 22].

The Coupled Nonlinear Cubic Schrödinger Equation (CNLCSE) is of the form

\[
iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 = 0 \tag{1.9a}
\]

\[
iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 = 0 \tag{1.9b}
\]

Thus solitons in general, manifest in a large variety of wave/particle systems in nature, practically in any system that possesses both dispersion (in time or space) and nonlinearity. Solitons have been identified in optics, plasmas, fluids, condensed matter, particle physics and astrophysics. Yet over the past decade, the forefront of soliton research is shifted to optics. Notable among them is the concept of ‘optical soliton’ pioneered by Hasegawa [7] of Japan.

### 1.4 Nonlinear Optics & Optical Solitons

Nonlinear optics is the study of phenomena that occur as a consequence of the modification of the optical properties of a material system by the presence of light. The beginning of the field of nonlinear optics is often taken to be the discovery of second harmonic generation by Franken et al. in 1961, shortly after the demonstration of the first working of laser by Maiman in 1960. Nonlinear optical phenomena are nonlinear in the sense that they occur when the response of a material system to an applied optical field depends in a nonlinear manner upon the strength of the optical field. This is due to the harmonic motion of bound electrons under the influence of an
applied field. In the case of linear optics the dipole moment per unit volume or
the polarization is directly proportional to the applied field. But in the case of
nonlinearity, second and third harmonics are generated in the polarization
which depends upon the characteristics of the crystal. That is, second harmonic
generation occurs as a result of the part of the atomic response which
depends quadratically on the strength of the applied field.

Following the invention of lasers, nonlinear optics has emerged as
one of the most sought-after subjects in all the frontiers of science by both
theoreticians and experimentalists. Indeed, nonlinear optics is one of the vital
cores of recent scientific advancements. Nonlinear optics may find applications
ranging from high data transmission in optical communication, switching
amplifiers, pulse reshaping, pulse compression, tunable lasers and encoded
message transmission. Optical solitons revolutionised the scope of
telecommunication world and they are now a days perceived to be carriers of
communication signals in the near future.

Optical solitons have their roots in two very important scientific
advances of 1960s: the mathematical theory of solitons, starting in 1965 by
Zabuski and Kruskal [10], Lax [13], Zakharov and Shabat [14], Miura et al [23]
and development of laser [24, 25]. Mollenauer and his group in 1980s first
observed solitons in optical fibers. They used mode-locked color-centre laser
capable of emitting short pulses near 1.5 μm. The pulses were propagated
through an optical fiber of length 700m, the core diameter of the fiber being
9.3 μm.

Optical solitons fall into two categories- 1) Spatial optical solitons and
2) Temporal optical solitons. The best known characteristic of wave
propagation is that the beams that are in finite space, tend to broaden due to diffraction effect. In fact, such diffraction effects are fully equivalent to the broadening of temporal pulses propagating in the media that possess chromatic dispersion. If a strong nonlinear interaction occurs between the wave and the medium through which it is propagating, this paradigm is broken. As a result, a self-trapped beam or a ‘spatial soliton’ can form [26, 27]. Spatial solitons are optical beams that can propagate in a nonlinear medium without diffraction, that is, their beam-diameter remains invariant during propagation. Intuitively, a spatial soliton represents an exact balance between diffraction and nonlinearity-induced self-lensing or self-defocusing effects. A ‘Temporal soliton’ is formed when group velocity dispersion (GVD) is totally counteracted by temporal self-focusing or self-phase modulation (SPM) effects [28, 29]. Therefore, optical spatial and temporal solitons have become a candidate for optical communication networks.

All optical solitons require a strong enough nonlinear interaction between themselves and the material in which they propagate. This interaction requires the so-called diffraction-length for the spatial case or dispersion-length for the temporal (fiber) case, which is comparable to a ‘nonlinear length’ that characterizes self-focusing in the medium. In fibers, low losses allow propagation distance of kilometers, and as a result, the very weak nonlinearity becomes cumulatively sufficient for soliton formation. In spatial case, however, the sample sizes are typically limited to centimeters, and thus, either the nonlinearities or the operator powers need to be larger. The higher dimensionalities of spatial solitons lead to interesting phenomena and processes such as full three dimensional (3D) interaction.
between solitons and soliton spiraling, vortex solitons, angular momentum
effects, rotating dipole vector solitons etc.

Solitons in a fiber exist as a result of Kerr nonlinearity. In pure Kerr
media, the refractive index change $\Delta n$ is linearly proportional to the optical
intensity $I$, that is $\Delta n(I) = n_2 I$, where $n_2$ is the Kerr co-efficient. In the spatial
case, however, optical solitons have been observed, based on several other
nonlinear processes. In addition to Kerr nonlinearity, various photorefractive
effects, parametric $\chi^{(2)}$ mixing phenomena, and nonlinearities in liquid crystals
and polymers were found to support spatial solitons.

1.5 Fiber Linearities

An optical pulse propagating through a fiber experiences the
following linear effects.

1.5.1 Pulse Dispersion

When a light-pulse travels through an optical fiber temporal broadening
of the pulse occurs. This is pulse dispersion. This results in the overlapping of
adjacent pulses, which increases errors in deciphering the coded information.

There are mainly two kinds of pulse dispersion, one due to
intramodal effect and the other due to inter-delay effects. These dispersions
cause the spreading of the pulse passing through a fiber, and the effects can
be quantified by considering group velocities of the guided modes.
1.5.2 Group Velocity Dispersion

As a light pulse travels through the fiber, its shape gets broadened due to dispersion. The group velocity, the velocity at which the energy in a pulse travels in a fiber, is given by

\[ V_g = \frac{d\omega}{dk} \]  

which can also be called the velocity of the envelope-wave. In a nondispersive medium, the component waves and envelopes will travel with the same velocity, that is the shape of the pulse (envelope) remains unchanged as pulse propagates through the medium. But in an optical medium, the velocity depends on refractive index, which in turn depends on frequency (wavelength) of the wave. The velocity of the wave is \( \frac{\omega}{k} = v = \frac{c}{n} \), where \( c \) is the velocity of the wave in vacuum, and \( n \) is the refractive index of the medium. As the signal propagates along the fiber, each spectral component can be assumed to travel independently and undergo a time delay per unit length. The propagation time over a distance \( L \) for a given group velocity \( V_g \) is defined as \( \tau = \frac{L}{V_g} \). If the spectral width of the pulse is not large, it may be approximated that the delay difference per unit wavelength along the propagation path is given by \( \frac{d\tau}{d\lambda} \). If the wavelength of the spectral components is spread over a wavelength range \( d\lambda \), the total delay difference \( d\tau \) over a distance \( L \) is given by

\[ \delta\lambda = \frac{d\tau}{d\lambda} \delta\lambda = -\frac{L}{c} \lambda \frac{d^2 n}{d\lambda^2} \delta\lambda \]  

(1.11)
Thus the spread in arrival times depends on $\frac{d^2 n}{d\lambda^2}$. Therefore, the pulse gets spread as it moves along the fiber, because different component waves which constitute the pulse have different velocities. This phenomenon is known as Group Velocity Dispersion (GVD). The two mechanisms responsible for GVD are:

A. Material Dispersion

Material Dispersion is also known as chromatic dispersion. This arises due to the variation of refractive index of the core of the fiber as a function of wavelength. Dopants like (GeO$_3$, P$_2$O$_3$) in the core are responsible for dispersion.

B. Wave-Guide Dispersion

Wave-guide dispersion occurs in single mode fibers, due to the fact that only 80% of the optical power is confined to the core, while 20% of the optical power is confined to the cladding, resulting in distortion of the pulse in the cladding and core.

The total GVD is the sum of material and waveguide dispersion. A quantity called mode propagation constant $\beta = \frac{n(\omega)\omega}{c}$ is introduced and its Taylor series expansion is given by [32]

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \cdots,$$  \hspace{1cm} (1.12)

where $\beta_m = \left(\frac{d^m \beta}{d\omega^m}\right)$, $m = 0, 1, 2, \ldots$
A little bit of algebra will show that

\[ \beta_2 = \frac{d\beta_1}{d\omega} = \frac{\lambda^3}{2\pi^2} \frac{d^2 n}{d\lambda^2} \]  \hspace{1cm} (1.13)

where \( \beta_2 \) is responsible for pulse broadening and is called the GVD parameter. In the anomalous dispersion regime, the GVD parameter is negative and in normal dispersion regime, it is positive. In anomalous dispersion regime, the high frequency (blue shifted) components of the optical pulse travel faster than the low frequency components (red shifted). The opposite occurs in normal dispersion regime. The anomalous regime plays an important role in nonlinear fiber optics, in particular in the formation of highly stable and distortionless solitons.

1.6 Fiber Nonlinearities

As the intensity of the incident pulse is high (picoseconds/nanosecond), the response of the fiber becomes nonlinear, and, therefore, nonlinear effects also have to be taken into account to study the propagation of optical solitons.

1.6.1 Nonlinear Polarization

The presence of an applied optical field causes motion of internal electric charges within the system resulting in a field-induced electric dipole moment. It acts as a new source emitting a secondary electromagnetic wave. This polarization phenomenon is characterized by the electric polarization vector \( \vec{P} \). In a linear dielectric, the polarizability \( \vec{P} \) can be written as

\[ \vec{P} = \chi \vec{E} \]  \hspace{1cm} (1.14)
where $\chi$ is the linear susceptibility and $\bar{E}$ the applied electric field. But if the strength of the light beams is sufficiently high (for example, as in a laser beam), the polarizability of the medium depends on higher powers of electric field strength and the medium is said to be nonlinear. Thus we can write the polarizability of a nonlinear medium as

$$
P = \varepsilon_0 (\chi^{(1)}E + \chi^{(2)}EE + \chi^{(3)}EEE + \ldots)$$  \hspace{1cm} (1.15)

where $\varepsilon_0$ is called the vacuum permittivity and $\chi^{(j)} (j = 1,2,3,\ldots)$ is the $j^{th}$ order susceptibility tensor. The linear susceptibility $\chi^{(1)}$ contains the dominant contribution to $\bar{P}$, whose effect is induced through the refractive index $n$. The second order susceptibility $\chi^{(2)}$ is responsible for nonlinear effects such as second harmonic generation and sum-frequency generation.

The lowest order nonlinear effects in optical fibers originate from the third order polarization $\chi^{(3)}$, which is responsible for phenomena such as third-harmonic generation, four-wave mixing and nonlinear refraction. The refractive index and susceptibility of a nonlinear medium are related by

$$
n = \left(1 + \chi^{(1)} + \chi^{(2)}E + \chi^{(3)}EE + \ldots\right)^{\frac{1}{2}} \hspace{1cm} (1.16)
$$

For an optical fiber, this relation takes the form,

$$
n = \left(1 + \chi^{(1)} + \chi^{(2)}E + \chi^{(3)}EE\right)^{\frac{1}{2}}
= n_0 + \frac{1}{2} \chi^{(3)}EE
= n_0 + n_2 I \hspace{1cm} (1.17)
$$
where $n_0$ represents the usual, weak-field refractive index, $n_z$ the nonlinear refractive index and $I$ denotes the averaged intensity of the optical field, given by

$$I = \frac{n_0 c}{2\pi} |E|^2.$$  \hfill (1.18)

$n_z$ is related to $\chi^{(3)}$ by

$$n_z = \frac{12\pi^2}{n_0^2 c^2} \chi^{(3)}.$$  \hfill (1.19)

Equation (1.17) indicates that the refractive index of a medium depends on the intensity of light propagating through it.

The intensity-dependence of refractive index has interesting applications like Self-Phase Modulation (SPM) and Cross-Phase Modulation (XPM).

**A. Self-Phase Modulation (SPM)**

SPM refers to the self-induced phase-shift experienced by an optical field during its propagation in optical fibers. Its magnitude can be obtained by noting that the phase of an optical field changes by

$$\phi = \tilde{n} k_0 L = (n + n_z |E|^2) k_0 L$$  \hfill (1.20)

where $k_0 = \frac{2\pi}{\lambda}$ and $L$, the fiber length.

The intensity-dependant phase-shift

$$\phi_{\text{int}} = n_z k_0 L |E|^2$$  \hfill (1.21)
is due to SPM. SPM is responsible for spectral broadening of ultra short pulses [30] and formation of optical soliton in anomalous dispersion regime of fibers [31, 32]. Due to SPM, the leading edge of pulse is downshifted in frequency and the trailing edge is uplifted in frequency leaving the central frequency of the pulse unchanged.

B. Cross-Phase Modulation (XPM)

XPM refers to the nonlinear phase-shift of an optical field induced by another field having a different wavelength, direction or state of polarization. Its origin can be understood by noting that the total electric field \( E \) in the equation (1.17) is given by

\[
\hat{E} = \frac{1}{2} \hat{x} [E_1 \exp(-i\omega_1 t) + E_2 \exp(-i\omega_2 t) + c.c]
\]

when the two optical fields are at frequencies \( \omega_1 \) and \( \omega_2 \), polarized along \( x \)-axis, propagate simultaneously inside the fiber. (c.c stands for the complex conjugate and \( \hat{x} \) is a unit vector along \( x \)-axis). The nonlinear phase-shift for the field, \( \omega_1 \) is given by

\[
\phi_{NL} = n_2 k_0 L (|E_1|^2 + |E_2|^2)
\]

Here we have neglected all terms that generate polarization at frequencies other than \( \omega_1 \) and \( \omega_2 \), because of their non-phase-matched character. The two terms on the RHS of equation (1.23) are due to SPM and XPM, respectively. XPM is responsible for asymmetric spectral broadening of co-propagating pulses.
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The nonlinear effects governed by the third order susceptibility \( \chi^{(3)} \) are elastic in the sense that no energy is exchanged between the electromagnetic field and the dielectric medium. A second class of nonlinear effect results from stimulated inelastic scattering in which the optical field transfers a part of its energy to the nonlinear medium. Two among them are Stimulated Brillouin Scattering (SBS) and Stimulated Raman Scattering (SRS) [33-35]. The main difference between the two is that optical phonons participate in SRS, while acoustic phonons participate in SBS.

1.7 NLSE in Optical Fibers

As mentioned in the above sections, the study of optical wave propagation in a nonlinear dispersive (dielectric) has been receiving considerable attention in recent times, because the fiber can support, under suitable circumstances, a stable pulse called optical soliton (Fig 1.6) [8,20, 36], which arises due to compensation of the effect of dispersion of the pulses by nonlinear response of the medium.

\[
\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathcal{P}}{\partial t^2}
\]

Fig 1.5 Wave propagation in an optical fiber

The analysis of such pulse propagation starts from Maxwell’s equation for the electromagnetic wave propagation in a dielectric medium:
where $\bar{P}$ is the induced polarization given in equation (1.15); $E$ represents the electric field; $c$ is the velocity of light; $\mu_0$ is the permeability free space, and $\chi^{(m)}$ is the $m^{th}$ order susceptibility factor.

In order to analyze equation (1.24), the following assumptions are taken into account:

- The nonlinear part of the induced polarization is treated as a small perturbation to the linear part.
- The optical field maintains its polarizability along the fiber.
- The fiber loss is small.
- The nonlinear response of the fiber is instantaneous.
- In a slowly varying envelope approximation for the pulse propagation along fiber, the electric field can be written as

$$
\hat{E}(r,t) = \frac{1}{2} \hat{e} [F(x,t)E(z,t)e^{(k_0z-\omega_0t)} + c.c]
$$

where $\hat{e}$ is the unit polarization vector of light to be linearly polarized, $E(z,t)$ is the slowly varying electric field, $F(x,y)$ is the mode distribution function in the $(x,y)$ plane, while $k_0$ and $\omega_0$ denote the propagation constant and central frequency of the optical pulse, respectively.

Rewriting Maxwell’s equation (1.24) by using the method of separation of variables and using the transformation, $T = t - \frac{z}{V_g}$, where $V_g$
is the group velocity given by \( V_g = \frac{\partial k}{\partial \omega} \), the wave equation for the evolution of \( \hat{E} \) is obtained as

\[
i \frac{\partial \hat{E}}{\partial k} - \frac{k'}{2} \frac{\partial^2 \hat{E}}{\partial T^2} + \gamma_0 |\hat{E}|^2 \hat{E} = 0
\] (1.26)

where \( \gamma_0 = \frac{n_2\omega_0}{cA_{eff}} \),

where \( A_{eff} \) denotes the effective core area of the single mode fiber, \( n_2 \), the nonlinear refractive index.

The parameter \( k'' = \frac{\partial^2 k}{\partial \omega_0} = -\frac{1}{V_g^2} (\frac{\partial V_g}{\partial \omega}) \) (at \( \omega = \omega_0 \)) accounts for GVD.

After normalizing equation (1.26) and using the transformations

\[
q = \left( \frac{\gamma_0 T_0^2}{|k''|} \right)^{\frac{1}{2}} E
\]

\[
\xi = \frac{2|k'|}{T_0^2}
\]

\[
\tau = \frac{T}{T_0}
\] (1.27)

Redefining \( \xi \) as \( z \) and \( \tau \) as \( t \), we get the NLSE,

\[
iq + \text{sgn}(k'')q_{xx} + 2|q|^2q = 0
\] (1.28)

where \( T_0 \) represents the width of the incident pulse, \( z \) and \( t \) are the normalized distance and time along the direction of propagation and \( q \), the normalized envelope. Interchanging \( t \) and \( x \), the standard form of NLSE is obtained as
\[ iq_t + q_{xx} \pm 2|q|^2 q = 0 \quad (q \in c). \] (1.29)

The solution of equation (1.29) has a bell-shaped form, as shown in Fig (1.7a) [37]. These solitons are called ‘bright solitons’. Those solitons are obtained in the anomalous dispersion regime.

![Bright solitary wave](image1.png)

**Fig 1.6a** Bright solitary wave

If we consider the normal dispersion regime (negative sign in equation (1.29), the solution takes ‘tanh’ form and the intensity profile associated with such solutions shows a dip in a uniform background, and hence, the solutions are called ‘dark solitons’ as shown in Fig (1.7b) [37].

![Dark solitary wave](image2.png)

**Fig 1.6 b** Dark solitary wave
1.8 Applications of Spatial Optical Solitons

Spatial optical solitons with various dimensionalities have been observed in various nonlinear media. One such application is the formation of spatial solitons in photorefractive media. Segev et al proposed a new kind of spatial soliton, the ‘Photorefractive Soliton’ (PR) in the year 1992 [38]. When illuminated, a space-charge field is formed in photorefractive materials, which induces nonlinear charges in the refractive index of the material, by the electro-optic (Pockels) effect. This change in refractive index can counter the effect of beam diffraction and form a PR soliton. PR solitons can be generated at low power levels of the order of several microwatts. The PR solitons have been investigated extensively by various groups, as it has potential application in all-optical switching, beam steering, optical interconnectors etc.

Under certain circumstances, two beams of light can interact in a photorefractive crystal in such a manner that energy is transferred from one beam to the other. This process, called two-beam coupling, can be used, for example, to amplify a weak, image-bearing signal beam by means of an intense pump beam. Exponential gains of 10 per centimeter are usually observed.

In this thesis the focus is on solitons, with specific emphasis on NLSEs, that is integrability studies of higher order NLSEs using the Painlevé Analysis & the Lax methods, solutions of coupled NLSE (Manakov model) using Hirota bilinear method and their applications in nonlinear optics, that is soliton propagation through photorefractive media (optical solitons).
1.9 Conclusion

In this chapter, we have reviewed the history of solitons and different nonlinear evolution equations. The application of solitons in optics, nonlinear optics and optical solitons is also considered. The importance of NLSE in optical fibers is highlighted in the context of fiber linearities and nonlinearities. Application of spatial solitons is also discussed.
References

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