Chapter 4

Generalized synchrony in chaotically driven flows

In the previous Chapter, we have studied the scenarios of generalized synchronization in quasiperiodically and chaotically driven mappings and observed that there are transitions to weak GS which arises through distinct bifurcation scenarios [100, 112]. Here we study chaotically driven flows [129] to examine the generality of the dynamical transitions from strong to weak GS, via intermittency as well as via a blowout bifurcation.

These dynamical transitions can be quantitatively characterized by adapting tools that were originally introduced in the study of strange nonchaotic dynamics such as parameter sensitivity exponents [91], distribution of local Lyapunov exponents [102] and the variance [100]. Similar measures are useful in characterizing the transitions between strong and weak generalized synchronization, as well as the disappearance of weak GS, namely the transition from weak GS to a state of no generalized synchrony.

As drive we take a Rössler oscillator [111] and as response we consider two model nonlinear dynamical systems that have experimental realizations as well [27, 155]. We examine the driven Duffing oscillator [27] and study the transition to weak GS from strong GS through intermittency. We further study a different driven nonlinear oscillator [155] that makes this transition through a blowout bifurcation. The transition from weak synchrony to asynchrony in driven Duffing oscillator is discussed and we also characterized the dynamics and the geometry of the various limit sets across the transition. The onset of weak GS in a forced excitable system [134] is also presented in this Chapter.
4.1 Transitions to weak GS

We exploit the parallels that were noted earlier (see Chapter 3) between quasiperiodic and chaotic driving to observe that when a situation of strong generalized synchronization arises, the existence of a smooth implicit function \( \Phi \) corresponds to the case of a nonfractal nonchaotic attractor. Similarly, when there is weak generalized synchronization, \( \Phi \) is non-smooth, corresponding to a strange or fractal attractor.

The chaotic drive that we use in the present work is the Rössler oscillator

\[
\begin{align*}
\frac{du}{dt} &= (-v - w)f, \\
\frac{dv}{dt} &= (u + av)f, \\
\frac{dw}{dt} &= [b + w(u - c)]f,
\end{align*}
\]

with parameters \( a = b = 0.2 \) and the factor \( f \) is a scaling parameter that is introduced in order to have some flexibility in adjusting the natural frequency of the chaotic oscillator.

Specifically, we use the output signal \( u(t) \) to drive the response.

4.1.1 Intermittency

The response system that we study first is a forced Duffing oscillator which is given by the following dynamical equations,

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= -hy + x - x^3 + Ax \cos \theta, \\
\frac{d\theta}{dt} &= 1.
\end{align*}
\]

Note that for \( A = 0 \), this is passive Duffing oscillator, but for nonzero \( A \), the system is known to display a variety of dynamics including limit cycles and chaotic behaviors.

We subject this oscillator to external forcing by the Rössler oscillator (with time scale parameter in Eq. (4.1) is taken to be \( f = 1.49 \)); the above equation for \( dy/dt \) is thus modified as

\[
\frac{dy}{dt} = -hy + x - x^3 + Ax \cos \theta + Aa \sin x.
\]

The natural frequency of the drive and the amplitude of the signal \( u \) can be adjusted by varying the parameters \( f \) and \( c \). Recall that the case of a sinusoidal \( u(t) \) has been studied earlier in the context of transitions to SNAs \([46, 99, 145]\).
Figure 4.1: Schematic phase diagram of driven Duffing oscillator Eq. (4.3) in parameter space: White corresponds to regions of GS where dynamics are stabilized, and the shaded region corresponds to asynchrony, where the dynamics is chaotic.

Figure 4.1 shows the phase diagram of the driven Duffing oscillator Eq. (4.3) as a function of the parameters $c$ in the drive equations of motion and $h$ in the response, keeping $A = 0.15$ and $a_r = 0.125$. Regions in white (shaded) represent the dynamical state of generalized synchrony (asynchrony), namely the response conditional Lyapunov exponent is negative (positive).

To study the dynamical behavior of the drive as well as the response, we focus on the $h = 0.2$ line in this phase diagram. Figure 4.2 shows the variation of largest Lyapunov exponents (LEs) of the drive Eq. (4.1) and response Eq. (4.3) evaluated from an ensemble of different initial conditions as a function of the parameter $c$. The Rössler system shows a variety of dynamical regimes as the parameter $c$ is varied: Chaotic dynamics for parameters where LEs are positive and regular dynamics where LEs are zero. $P_1$, $P_2$, $P_3$, and $P_4$ are some of the largest periodic windows in the dynamics of the drive. The modulation induced in Eq. (4.3) by varying $c$ takes the dynamics of the response into different dynamical regimes of GS and at sufficiently high values of $c$, regions of asynchrony is reached. There is some evidence for multistability around the periodic window $P_4$ where states of GS and asynchrony coexist: this region is marked MS; see Fig. 4.2.

To study the dynamical transitions from weak to strong GS, we focus in the parameter
Figure 4.2: Variation of Lyapunov exponents of the drive Eq. (4.1) (black) and response Eq. (4.3) (Red) along the line $h = 0.2$. Coexistence of GS and asynchrony states are observed in the region MS.

range $c = 5.18$ to 5.19 around the region $P_1$. Figure 4.3(a) shows the variation of the largest Lyapunov exponent $\lambda_d$ of the drive as a function of this parameter. The dynamics is chaotic for $c \leq c_f \approx 5.18505$ (marked by the arrow), with an abrupt transition to a limit cycle for $c > c_f$. The response dynamics of the Duffing oscillator Eq. (4.3) is nonchaotic throughout this range, see Fig. 4.3(b). However at $c = c_f$, the Lyapunov exponent $\lambda_r$ decreases sharply while beyond this point it decreases slowly.

The intermittency transition in the drive induces intermittency in the response as well. Shown in Fig. 4.4(a) is the Poincaré section for an orbit in the $(x, \theta)$ plane [46]: the limit set is nonsmooth and dynamics on it is both intermittent and nonchaotic. A regime of weak generalized synchronization is manifest. Below $c_f$ the dynamics is characterized by intermittency. Upon increasing $c$ above $c_f$, a regime of strong GS results; the limit set becomes smooth as shown in Fig. 4.4(b). Similar dynamical transitions of GS are also observed inside $P_2$ window in Fig. 4.2.

The dynamical transitions from weak GS to strong GS can also be detected in the behavior of variance of the finite-time Lyapunov exponents. Shown in Fig. 4.3(c) is the variance of the response finite–time Lyapunov exponents. In general at this intermittency transition both the Lyapunov exponents and the variance show abrupt changes, with power–law variation [46, 99, 145].
Figure 4.3: The largest Lyapunov exponent (a) $\lambda_d$ of the driving Rössler oscillator Eq. (4.1) and (b) $\lambda_r$ of response Duffing oscillator Eq. (4.3). (c) Variance of $\lambda_r$ evaluated from ensembles of initial conditions. In (c) wGS and sGS corresponds to regime of weak and strong generalized synchronization, and $c_I$ indicates the bifurcation point in the drive and response systems respectively.
Figure 4.4: Projection of the dynamics of the response Duffing oscillator Eq. (4.3) on Poincaré section of \((x, \theta \mod 2\pi)\) plane. (a) Intermittent limit set at \(c = 5.185\) in the regime of weak GS. (b) Smooth limit set \(c = 5.1851\) in the regime of strong GS.
4.1.2 Blowout

In systems possessing a symmetric invariant subspace, the destabilization of this subspace by variation of a system parameter results in the so-called "on-off intermittency" at a blowout bifurcation. A well-studied model in this context is the nonlinear oscillator [155, 148]

\[ \ddot{x} + \kappa \dot{x} + \eta x^3 + (\mu + f_1(t) + f_2(t)) \sin(2\pi x) = 0, \]  

(4.4)

where \( f_i(t), i = 1, 2 \) are arbitrary time-dependent functions, and the invariant subspace is the point \( x = 0, \dot{x} = 0. \)

Figure 4.5: Variation of (a) Lyapunov exponent \( \lambda_r \) of response Eq. (4.5) (b) Variance of Lyapunov exponents \( \lambda_r \) with \( \kappa \) (\( \kappa_b \) is the point of blowout bifurcation). (c) Projection of trajectory on the stroboscopic section at \( \kappa = 4.5 \) showing a nonsmooth limit set. (d) Trajectory of \( y_n \) on the nonsmooth limit set showing on-off intermittency.
We take one forcing function to be harmonic, and the second to be the Rössler drive, rewriting the system as follows,

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\kappa y - \eta x^3 + (\mu + \sin \theta + u) \sin(2\pi x), \\
\dot{\theta} &= 1.
\end{align*}
\]  \tag{4.5}

Here \(\kappa, \eta\) and \(\mu\) are parameters of the response, and the chaotic signal \(u(t)\) is the output of the Rössler oscillator with \(a = b = 0.2, c = 5.7,\) and \(f = 1.\)

The largest Lyapunov exponent of the response, \(\lambda_r\) of Eq. (4.5) from an ensemble of initial conditions with \(\eta = 2\) and \(\mu = -1.1\) is computed as a function of \(\kappa\). As can be seen in Fig. 4.5(a), when \(\kappa\) decreases, \(\lambda_r\) increases sharply to zero at \(\kappa_b \approx 4.76\), and then decreases, showing the blowout bifurcation in the response dynamics. Thus in the entire range, the response system is in generalized synchrony with the drive: strong GS for \(\kappa > \kappa_b\) and weak GS for \(\kappa \leq \kappa_b\). This abrupt change in the nature of the synchronization is evident in the behavior of the variance of finite-time Lyapunov exponents. In Fig. 4.5(b), with decrease in \(\kappa\), a sudden increase in the variance is observed at the bifurcation, signifying the destabilization of the invariant subspace, and this results in a nonsmooth limit set as shown in Fig. 4.5(c) (this is a stroboscopic section of a trajectory at \(\theta = 2\pi n, n = 1, 2\ldots\)). A typical trajectory on the invariant set is transversely unstable, and exhibits on–off intermittency [135]; see Fig. 4.5(d) for the case of \(\kappa = 4.5.\)

4.2 From weak GS to asynchrony

When system parameters are varied the subsystem Lyapunov exponents in the coupled systems need not remain nonpositive. There is a loss of generalized synchronization when the largest of the subsystem Lyapunov exponents becomes positive: the dynamics becomes asynchronous. We examine the behavior of the largest conditional Lyapunov exponent at the transition from weak GS to asynchrony since it is known that in unforced systems, along any route to chaos, the maximal Lyapunov exponent has a characteristic signature at the transition [31, 40, 97].

The transition from nonchaotic to chaotic dynamics in forced systems has also been analyzed earlier [67], and it appears that the arguments and reasoning that was used in the study of this transition in quasiperiodically forced systems may be more generally applicable. Following the reasoning outlined in [67], we can argue that a linear variation of
Lyapunov exponents will be seen in the case of chaotic forcing. Suppose a transition from weak GS to asynchrony takes place at a critical value of drive parameter $\beta = \beta_c$ without any abrupt changes in the geometrical structure of the limit sets. A trajectory can visit both the expanding and contracting region of the phase space. Let $\lambda_e(\beta)$ and $\lambda_c(\beta)$ be the average expansion and contraction rate of a trajectory visiting the regions of synchrony and asynchrony with frequencies $f_e(\beta)$ and $f_c(\beta)$ respectively. The terms $f_e(\beta)\lambda_e(\beta)$ and $f_c(\beta)\lambda_c(\beta)$ gives the average expansion rate and contraction rate respectively. So, the Lyapunov exponent $\lambda_r$ of the response which determine the average rate of expansion or contraction can be written as

$$\lambda_r(\beta) = f_e(\beta)\lambda_e(\beta) - f_c(\beta)\lambda_c(\beta).$$ (4.6)
Figure 4.6(a) and 4.6(b) shows the essentially indistinguishable limit sets of forced
Duffing Eq. (4.3) in regions of synchrony and asynchrony respectively. Although, morpho­
logically both limit sets are strange or nonsmooth without any distinctive changes across
the transition, but due to global stability in weak GS regime, trajectories in the same basin
of attraction synchronize whereas asynchronous motion takes place when global stability
is lost. Figure 4.6(c) and 4.6(d) show trajectories contrasting the case of weak generalized
synchrony and asynchrony phase.

In such scenarios where phase-space structure does not change drastically, one can
assume that the dynamical quantities defined in this region, i.e., \( \lambda_c(\beta) \), \( \lambda_c(\beta) \), \( f_c(\beta) \), and
\( f_c(\beta) \) are smooth functions of \( \beta \). Taylor expansion around the transition point \( \beta = \beta_c \) gives

\[
\lambda_r(\beta) \approx A(\beta_c)(\beta - \beta_c) + B(\beta_c)
\]

where \( A(\beta_c) \) and \( B(\beta_c) \) are slope and intercept of the line passing approximately linear
from weak GS to regions of synchrony. Although there are small fluctuations in the behavior
of Lyapunov exponents (which arise due to initial condition dependence in numerical
computation of LE over finite interval of time), as in the case of SNAs [67], the Lyapunov
exponent \( \lambda_r \) crosses zero approximately linearly, and with slope \( \approx 0.025 \); this can be seen
in Fig. 4.2 (dotted line).

4.3 Characterizing strangeness: parameter sensitivity

Although the behavior of the variance of the distribution of FTLEs can detect the dynamical
transition from strong to weak GS, it does not give much information about the geometry
of the limit sets. The nature of this geometry, in particular the degree of “strangeness”
can be determined by analyzing the sensitivity of the dynamics to perturbations of external
forcing [91] or of the system parameters [86].

Consider the “drive-response” system

\[
\begin{align*}
\frac{dx}{dt} &= F(x, u, \alpha), \\
\frac{du}{dt} &= G(u, \beta),
\end{align*}
\]

where \( x \in \mathbb{R}^m \) and \( u \in \mathbb{R}^d \) are the dynamical variables of the response and drive systems
respectively. The vector fields \( F \) and \( G \) are taken to be continuous and differentiable, and
\( \alpha \) and \( \beta \) represent control parameters.
The sensitivity of the response \( x \) to perturbations of the drive initial conditions, namely \( \partial x / \partial u_0 \) gives a good measure for quasiperiodically forced systems [91]. This does not appear to be useful for other types of forcing. To see this, for \( u \in R \), by differentiating Eq. (4.8) with respect to the drive variable, one gets

\[
\frac{d}{dt} \frac{\partial x}{\partial u_0} = \sum_{j=1}^{r} \frac{\partial F}{\partial x_j} \frac{\partial x_j}{\partial u_0} + \frac{\partial F}{\partial u} \frac{\partial u}{\partial u_0}.
\]

(4.10)

Note that in the last term, \( \partial u / \partial u_0 \) is determined by the stability characteristics of the drive: for quasiperiodic forcing, \( \partial u / \partial u_0 = 1 \) and thus \( \partial x / \partial u_0 \) which is obtained by integrating Eq. (4.10) is a good measure to quantify the strangeness of the limit sets [91]. The quantity \( |\partial x / \partial u_0| \) is bounded for smooth limit sets, whereas it is unbounded for nonsmooth limit sets. However, if \( \partial u / \partial u_0 \neq 1 \) the situation is less clear.

In such cases, the parameter sensitivity [56, 86] provides a good measure to characterize dynamical transitions associated with a change in the morphology of the invariant set. Adapting the parameter sensitivity analysis for the present case of time continuous systems, note that upon differentiating Eq. (4.8) with respect to system parameter \( \alpha \), one gets

\[
\frac{d}{dt} \frac{\partial x}{\partial \alpha} = \sum_{j=1}^{r} \left( \frac{\partial F}{\partial x_j} \right)_a \left( \frac{\partial x_j}{\partial \alpha} \right)_x + \left( \frac{\partial F}{\partial \alpha} \right)_x,
\]

(4.11)

which can be solved to obtain \( \partial x / \partial \alpha \), the sensitivity of response with respect to a system parameter. Depending on the dynamics of the response, the quantity

\[
\gamma = \max_{0 \leq k \leq t} \left| \frac{\partial x_k}{\partial \alpha} \right|
\]

(4.12)

namely the upper envelope of the signal \( \partial x_k / \partial \alpha \) has three typical behaviors.

When the largest of the subsystem Lyapunov exponents is positive, \( \gamma \sim \exp(\lambda_r t) \) because of exponential divergence of the orbits. This corresponds to a lack of generalized synchronization. In regions of generalized synchrony where the largest exponent is nonpositive, it can be shown that \( \gamma \sim t^\mu \) with exponents \( \mu = 0 \) for regular dynamics, namely for the case of strong GS, and \( \mu \neq 0 \) for irregular dynamics, namely for the case of weak GS.

Another suitably averaged quantity to consider is the lower envelope of different \( \gamma \) of different initial conditions,

\[
\Gamma_t = \min_{x_0, u_0} \{ \gamma \}
\]

(4.13)
and this proves to be more appropriate to use for characterization of the limit sets. In the region of nonchaotic dynamics, again

\[ \Gamma_t \sim t^\mu \]  

(4.14)

and the exponent \( \mu \) is a good measure for characterization of the dynamics and the "degree of strangeness". Figure 4.7(a) shows parameter sensitivity of the dynamics across the intermittent transition from weak to strong GS in the forced Duffing system. The sensitivity exponent (black line) for intermittent dynamics in regime of weak GS at \( c = 5.185 \) grows with \( \mu \approx 0.166 \) whereas it [red (dark gray) line] saturates for dynamics in regime of strong GS at \( c = 5.1851 \). This behavior can be contrasted with the distributions of finite-time Lyapunov exponents (FTLEs) [1] which is defined as

\[ P(\lambda_r,t) d\lambda_r = \text{Probability that } \lambda_r(t) \text{ takes a value} \]

between \( \lambda_r(t) \) and \( \lambda_r(t) + d\lambda_r(t) \).

(4.15)

Figure 4.7(b) shows the distributions of FTLEs of the forced Duffing oscillator evaluated for trajectory length \( t = 10 \). At \( c = 5.1851 \) for the case of strong GS, the distribution of FTLEs (solid line) is largely confined to the negative axis, whereas FTLEs for the intermittent weak GS at \( c = 5.185 \) (dashed line) extends into the positive axis [102, 24]. Figure 4.7(c) and 4.7(d) shows the dynamical difference between the regime of weak GS and asynchrony. The behavior of parameter sensitivity across the transition from weak GS to asynchrony is shown in Fig. 4.7(c). At the regime of weak GS at \( c = 9.2 \), there is a power law growth (black line) with exponent \( \mu \approx 6.782 \) but the behavior changes to exponential (red line) in regime of asynchrony at \( c = 9.79 \). In Fig. 4.7(d), the distribution of FTLEs across the transitions is shown: The mean of distribution shifts toward the positive components as one approach regions of asynchrony (red line) from weak GS (black line).

Shown in Fig. 4.7(e) and 4.7(f) are the corresponding quantities at the blowout transition in the forced systems, Eq. (4.5). The parameter sensitivity of the dynamics for weak GS in this system at \( \kappa = 4.5 \) has power law growth with exponent \( \mu \approx 2.636 \), while the distribution of FTLEs for trajectory length \( t = 300 \) is Gaussian; see Fig. 4.7(f).
Figure 4.7: Parameter sensitivity exponents (left panel) and corresponding distribution of finite time Lyapunov exponents (FTLEs) (right panel) of response dynamics for Eq. (4.3) in (a)-(d) and Eq. (4.5) in (e)-(f): (a) Strong GS (red line) and weak GS (black line); (b) Strong GS (black line) and weak GS (red line); (c) Asynchrony (red line) and weak GS (black line); (d) Asynchrony (red line) and weak GS (black line); (e) Weak GS due to blowout bifurcation and its (f) FTLEs distribution.
4.4 Weak generalized synchronization in excitable system

Excitability is common in many natural systems and examples can be drawn from chemical systems [142], neural systems [54], physiological processes [38], laser systems [37] and so on. Dynamical systems having a stable fixed point and a saddle, in response to external stimuli, give rise to excitability although otherwise they are intrinsically quiescent. Different types of forcing—harmonic [77], stochastic [33] and quasiperiodic [98]—are known to generate excitability.

Studies in quasiperiodically forced excitable systems [98] have shown that the systems can be driven into a regime of stable spiking dynamical behavior and strange nonchaotic attractors (SNAs) [39] are generated. SNAs on which aperiodic nonchaotic dynamics are realized became a subject of interest. Most biological systems exhibit stable aperiodic excitable behavior and thus SNAs in excitable systems are one means achieving this dynamics of objective.

Laser systems are a paradigm within which we can explore a variety of dynamical phenomena. Low frequency fluctuations (LFFs) [37] which are characterized by the existence of irregular pulses at large time intervals, and questions concerning their origin and modeling have been a subject of interest. The physical and dynamical mechanisms responsible for LFFs are based on Lang-Kobayashi equation. LFFs can be interpreted as a deterministic processes manifesting high-dimensional chaos [116] while experimental results of Giudici [37] suggest that laser systems can be model as an excitable systems and LFFs are induced by noise.

A dynamical system which models low-frequency fluctuations in a semiconductor laser with optical feedback as an excitable system has been extensively studied. The system is defined by the following equations

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= x - y - x^3 + xy + \varepsilon_1 + \varepsilon_2 x^2,
\end{align*}
\]  (4.16)

where \(\varepsilon_1\) and \(\varepsilon_2\) are the system parameters. When perturbed with noise, this system exhibits excitable behavior and this has been studied both numerically and analytically [29, 28]. With quasiperiodic drive, the dynamics goes from being non-spiking to spiking and there is a transition from smooth limit cycle to SNAs.

We externally modulate the parameter \(\varepsilon_1\) of the response Eq. (4.16) as follows,

\[
\varepsilon_1 \rightarrow \varepsilon_1[1 + \varepsilon(R\cos t + a_d u)].
\]  (4.17)
The modulating signal is thus a superposition of a sinusoidal wave and a signal from a Rössler oscillator which is defined as follows

\[
\frac{du}{dt} = (-v - w)f, \\
\frac{dv}{dt} = (u + \alpha v)f, \\
\frac{dw}{dt} = [\beta + w(u - \gamma)]f,
\]

(4.18)

with $\alpha = \beta = 0.2$. The parameter $f = 1.49$ control the frequency of Rössler oscillators. Since no irrational frequencies are involved, the two external frequencies are not incommen-

Figure 4.8: Variation of (a) largest Lyapunov exponents $\lambda_{Res}$ of the response Eq. (4.16); the inset shows that Lyapunov exponents crosses zero linearly (b) largest Lyapunov exponent $\lambda_{Ross}$ of the Rössler system Eq. (4.18); $P_1$ and $P_2$ are the two largest periodic window [a sudden disappearance of chaotic attractors are characterized by intermittency($I$ is one such region)] (c) variance of $\lambda_{Res}$ as function of $\gamma$; inset shows the for low values of $\gamma$. 

76
surate and hence, no quasiperiodicity is introduced in the forcing. In our numerical simulations, the equations are integrated numerically using a fourth-order Runge-Kutta method with step size $h = 2\pi/200$ and the systems parameters are set as follows; $\varepsilon_1 = 0.08$, $\varepsilon_2 = 1$, $\varepsilon = 2$, $a_d = 0.15$ and $R = 0.6\gamma$ in Eq. (4.17). Figure 4.8(a) shows the largest Lyapunov exponent $\lambda_{Res}$ of the response Eq. (4.16) vs. the nonlinearity parameter $\gamma$ of the Rössler oscillator. In this parameter range, the dynamics of the Rössler changes from periodic to chaotic regions where there are many periodic windows inside it; see Fig. 4.8(b). As the parameter $\gamma$ of Rössler changes, the modulation Eq. (4.17) changes from stable periodic to unstable aperiodic signals or vice-versa.

With increasing forcing amplitude, the dynamics of the response goes from stable dynamics to chaos. In this transition to chaos, Lyapunov exponents passes through zero approximately linear with a slope $\approx 0.587$ in a similar manner to that of quasiperiodically forced systems [67]; see inset Fig. 4.8(a). In the region of stable dynamics, trajectories starting from different initial conditions coalesce and synchronize irrespective of different dynamical behavior and one can speak of generalized synchronization (GS) [114]. Inside this regime dynamical transitions can be detected in the behavior of the variance of $\lambda_{Res}$

![Figure 4.9: Poincaré section at different values of $\gamma$: (a) $\gamma=4.19682$ (smooth), (b) $\gamma=4.21194$ (nonsmooth), (c) $\gamma=4.694$ (intermittent), and (d) $\gamma = 4.70679$ (smooth).](image-url)
evaluated from an ensemble of initial conditions; see Fig. 4.8(c). Corresponding to regions where the dynamics in Eq. (4.18) is periodic or regular, the variance saturates and is characterized by smooth dynamical behavior in the response, while in regions where the drive dynamics is chaotic, variance increases indicating a nonsmooth dynamical behavior: see inset Fig. 4.8(c).

The dynamics across the above transition is shown in Fig. 4.9(a) and 4.9(b). In the nonspiking regime, a smooth limit cycle transforms into nonsmooth limit cycle i.e., strong GS transforms to weak GS. The limit cycles are smooth, corresponding to the values of $\gamma$ where the dynamics in Eq. (4.18) is periodic but it is nonsmooth when drive is chaotic. For strong GS, behavior of variance saturates whereas there are relatively large fluctuations in case of weak GS.

In the vicinity of the sudden disappearance of chaotic attractors, for instance around the point $I$ in Fig. 4.9(b), a saddle-node bifurcation in drive induces intermittency in the response. Figure 4.8(c) shows the intermittent limit cycle at $\gamma = 4.694$ and with slight increase of value to $\gamma = 4.70679$ in the periodic region in Eq. (4.18), limit cycle becomes smooth again; see Fig. 4.9(d).

Excitable or spiking behavior starts around the point $E$, namely $\gamma \approx 4.999$, where the variance shows a sudden increase as shown in Fig. 4.8(c). Around this point, there is a transition from nonsmooth limit cycle to fractal attractor. Once the regime of excitability is reached by applying appropriate forcing, the limit set remains fractal or nonsmooth as

![Figure 4.10: Poincaré section at $\gamma = 5.21672$; the limit set is strange and nonsmooth.](image)
shown in Fig. 4.10 irrespective of the dynamics of the drive. In the regime of excitable behavior, the variance of the Lyapunov exponents has high fluctuations.

Clearly, there are two dynamical transitions that are operative in Eq. (4.16); (a) Transition from a smooth to nonsmooth limit cycles i.e. transition from strong GS to weak GS and (b) inside the regime of weak GS, nonsmooth limit cycle undergoes a crisis like transition to an enlarged strange limit set on which dynamics becomes excitable but remains nonchaotic.

4.5 Discussion

Here we have studied regimes of generalized synchronization in chaotically driven flows and investigate the role of chaos with an objective to generate stable aperiodic dynamics. These stable aperiodic motions are possible on limit sets obtained in regimes of weak generalized synchronization. Furthermore, there are dynamical transitions from strong generalized synchrony to weak generalized synchrony and to regions of asynchrony. These have parallels with scenarios of the dynamical transitions from torus to chaos via SNAs in quasiperiodically forced systems.

A major motivation that has governed our choice of the driven nonlinear dynamical system has been that the phenomena that are described should be experimentally realizable. Therefore, we have examined the Duffing oscillator, Eq. (4.3) as a model response system. An experimental system that is closely modeled by this oscillator is the magnetoelastic ribbon which has been earlier studied under the effect of quasiperiodic driving [27]. The second system we studied, Eq. (4.5), corresponds to a driven superconducting quantum interference device (SQUID) [155]. Thus we believe that the transitions discussed here can, in principle, be detected in laboratory experiments. We have also studied the effects of external forcing or stimulus with two frequencies which are not incommensurate in excitable systems and shown that strange nonchaotic limit set in excitable systems can be generated with chaos.

The phenomenon of generalized synchronization has a wide range of applicability. For instance, in the context of time-series analysis, the stabilization of recursive filters has been an important issue of practical importance [70, 137] to ensure that measurable quantities of the drive reconstructed from the filter output does not change the properties of the original drive [7]. From a technological point of view, generalized synchronization of aperiodic trajectories suggest applications in secure communications [153] and chaotic
masking [23]. The role of chaos in natural systems is of much importance and generation of such nonchaotic aperiodic dynamics could be of practical importance in technological [153] and medical applications [139]. Extensive studies of experimental data from electroencephalograms (EEGs) [76] suggest that dynamics are insensitive to initial conditions despite exhibiting complex irregular behavior. Our studies could provide a means to understand the stability of robust aperiodic dynamics and synchronization of physiological rhythms. However, questions concerning the characterization of the dynamical behavior and modeling of EEG are still open.

Chaotic motion—and therefore chaotic modulation—is widespread in natural systems. As a consequence, the occurrence of generalized synchrony may well be the most robust mechanism for the creation of temporal correlations in nature, given the fact that nonlinearity and coupling are both common features of natural systems. Indeed, stabilization of dynamics by quasiperiodic forcing has been suggested as a mechanism that neural systems could exploit [76, 98], although it is not easy to identify a reliable source of quasiperiodicity in nature. Given the ubiquity of stochasticity and chaos in biological systems over a wide range of time-scales [38, 58, 76, 104, 139], the stability of such dynamics may be the result of generalized synchronization to a chaotic environment. The entrainment of one system as a consequence of being driven by another can be a common method of dynamical control. The analysis of generalized synchrony is thus of considerable interest. Noise and chaos are ubiquitous in natural systems and understanding their roles in stabilizing dynamical systems, maintaining synchrony and rhythmic processes [38] in dynamical systems can open new frontiers of understanding.