CHAPTER 1

BENDING PROBLEMS ON PLATES
SECTION 1.1

THE AXI-SYMMETRIC BENDING OF CIRCULAR PLATE UNDER THE COMBINED ACTION OF LATERAL LOADS AND FORCES IN THE MIDDLE PLANE OF THE PLATE WITH VARIABLE FLEXURAL RIGIDITY AND VARIABLE INTENSITY *

INTRODUCTION

A lot of works has been done to study the bending problems of isotropic, anisotropic and orthotropic, circular, annular plates with constant and variable thickness. Bending problem of circular plate with variable thickness was first discussed by Holzer\(^{[25]}\)(1918). Since then bending problems of circular plates and discs have been discussed by many researchers. Conway\(^{[13]}\)(1958) solved the bending problem of symmetrically loaded circular plate of variable thickness. The effect of stresses due to concentrated couple at the centre of an elastic elliptic plate with fixed edge was investigated by Bose\(^{[4]}\)(1965). Gupta and Sharma\(^{[23]}\)(1982) had analysed asymmetric bending of circular sandwich plate including transverse shear in facing. Sinha and Basuli\(^{[48]}\)(1986) had solved the bending problem of circular plate under the combined action of lateral load and forces in the middle.


The object of this section is to investigate theoretically the problem of axi-symmetric bending of a circular plate having a small initial curvature and under the combined action of lateral loads and forces in the middle plane of the plate with flexural rigidity $D$ varying as a biquadratic function of radial distance $r$ expressed in the form,

$$D = D_0 + D_1r + D_2r^2 + D_3r^3 + D_4r^4$$  \hspace{1cm} (1.1.1)

and the load intensity $q$ assumed in the following polynomial form as,

$$q = q_0 + q_1r + q_2r^2 + q_3r^3 + q_4r^4$$  \hspace{1cm} (1.1.2)

where $q_0$ and $D_0$ are the values of $q$ and $D$ at the centre of the plate i.e., at $r = 0$ and $D_i$ and $q_i$ ($i = 1, 2, 3, 4$) are real constants.

The corresponding results for linear variation and quadratic variation of flexural rigidity and load intensity follow as special cases and are in agreement with the previous results obtained by Sinha and Basuli\textsuperscript{[48]} (1986) and De and Banik\textsuperscript{[14]} (2010) respectively. Corresponding results for cubic variation of flexural rigidity and load intensity are obtained as a special case which is also a new contribution to the literature. A comparative study of the
initial deflection and lateral deflection are calculated numerically and shown graphically for both cubical and biquadratic variations. Bending moments for both cubical and biquadratic cases are also obtained numerically and shown graphically.

FORMULATION AND SOLUTION OF THE PROBLEM

The differential equation of bending of radially symmetric circular plate of variable flexural rigidity under the combined action of lateral loads and forces of compression in the middle plane of the plate is given by, (vide, Timoshenko and Woinowsky-Krieger[54 (1983)],

\[ M_r + \frac{dM_r}{dr} - M_r = -Q_r + N \frac{d^2w}{dr^2} \]  \hspace{1cm} (1.1.3)

where,

\[ Q_r = \frac{1}{r} \int_0^r rqdr \]  \hspace{1cm} (1.1.4)

and \( N \) is the force of compression.

We consider a thin circular plate with a small initial deflection and is under the combined action of lateral loads and forces in the middle plane of the plate. If \( w_0 \) is the initial deflection and \( w_i \) be the additional deflection due to supplied loads and forces, then the total deflection will be,

\[ w = w_0 + w_i \]  \hspace{1cm} (1.1.5)

The bending moments \( M_r \) and \( M_t \) depend only on the additional deflection. Hence, the bending moments are given by,

\[ M_r = -D \left( \frac{d^2w_i}{dr^2} + \frac{\nu}{r} \frac{dw_i}{dr} \right) \] and \[ M_t = -D \left( \frac{1}{r} \frac{d^2w_i}{dr^2} - \frac{\nu}{r} \frac{dw_i}{dr} \right) \]  \hspace{1cm} (1.1.6)
Using equations (1.1.4), (1.1.5) and (1.1.6) in the equation (1.1.3), the differential equation of equilibrium for an initially curved circular plate of varying flexural rigidity is obtained as,

\[ D \frac{d}{dr} \left( \frac{d^2 w_i}{dr^2} + \frac{1}{r} \frac{dw_i}{dr} \right) + \frac{dD}{dr} \left( \frac{d^2 w_i}{dr^2} + \frac{v}{r} \frac{dw_i}{dr} \right) = \frac{1}{r} \int_0^r q r \ dr - N \frac{d^2 (w_0 + w_i)}{dr^2} \]

(1.1.7)

The initial deflection is considered as (vide, Timoshenko and Woinowsky-Krieger\(^{[54]}\) (1983)),

\[ w_0 = a_0 \left( a^2 - r^2 \right)^2 \]

(1.1.8)

where, \( a \) is the radius of the plate and \( a_0 \) is a constant.

Substituting the values of \( D, q \) and \( w_0 \) from equations (1.1.1), (1.1.2) and (1.1.8) in the equation (1.1.7) the equation of equilibrium becomes,

\[
\left( D_0 + D_1 r + D_2 r^2 + D_3 r^3 + D_4 r^4 \right) \frac{d^3 w_i}{dr^3} \\
+ \left( \frac{D_0}{r} + (2 \nu - 1) D_2 \frac{D_1}{r} + (2 \nu - 1) D_2 + (3 \nu - 1) D_3 + (4 \nu - 1) D_4 \right) \frac{d^2 w_i}{dr^2} \\
+ \left( - D_0 + (\nu - 1) D_2 (2 \nu - 1) D_3 + (3 \nu - 1) D_4 \right) \frac{dw_i}{dr} \\
= Na_0 (a^2 - 3r^2) + q_0 \frac{r}{2} + q_1 \frac{r^2}{3} + q_2 \frac{r^3}{4} + q_3 \frac{r^4}{5} + q_4 \frac{r^5}{6}
\]

(1.1.9)

This is a third order non-homogeneous differential equation with variable coefficients.

Using \( \psi = \frac{dw_i}{dr} \) in the equation (1.1.9) we obtain,
\[(D_0 + D_1 r + D_2 r^2 + D_3 r^3 + D_4 r^4) \frac{d^2 \psi}{dr^2}\]

\[+ \left\{ \frac{D_0}{r} + (2D_1 + N) + 3D_2 r + 4D_3 r^2 + 5D_4 r^3 \right\} \frac{d \psi}{dr}\]

\[+ \{ - \frac{D_0}{r^2} + (\nu - 1) \frac{D_1}{r} + (2\nu - 1)D_2 + (3\nu - 1)D_3 r + (4\nu - 1)D_4 r^2 \}\psi\]

\[= 4Na_0(a^2 - 3r^2) + q_0 \frac{r}{2} + q_1 \frac{r^2}{3} + q_2 \frac{r^3}{4} + q_3 \frac{r^4}{5} + q_4 \frac{r^5}{6}\]

\[\text{(1.1.10)}\]

To find the complementary function of the equation (1.1.10) we consider,

\[(D_0 + D_1 r + D_2 r^2 + D_3 r^3 + D_4 r^4) \frac{d^2 \psi}{dr^2}\]

\[+ \left\{ \frac{D_0}{r} + (2D_1 + N) + 3D_2 r + 4D_3 r^2 + 5D_4 r^3 \right\} \frac{d \psi}{dr}\]

\[+ \{ - \frac{D_0}{r^2} + (\nu - 1) \frac{D_1}{r} + (2\nu - 1)D_2 + (3\nu - 1)D_3 r + (4\nu - 1)D_4 r^2 \}\psi\]

\[= 0\]

We apply the Frobenius method of series solution to find the complementary function. The trial solution of the equation (1.1.11) is considered as,

\[\psi = \sum_{i=0}^{\infty} (d_i r^{i+1})\]

Substituting \(\psi\) in the equation (1.1.11) we obtain,

\[\sum_{i=0}^{\infty} \left\{ (i + c)(i + c - 1) + 5(i + c) + (4\nu - 1) \right\} D_4 r^{(i+c+4)} d_i\]

\[+ \sum_{i=0}^{\infty} \left\{ (i + c)(i + c - 1) + 4(i + c) + (3\nu - 1) \right\} D_3 r^{(i+c+3)} d_i\]

\[+ \sum_{i=0}^{\infty} \left\{ (i + c)(i + c - 1) + 3(i + c) + (2\nu - 1) \right\} D_2 r^{(i+c+2)} d_i\]
\[ + \sum_{i=0}^{\infty} \left[(i + c)(i + c - 1)D_1 + (i + c)(2D_1 + N) + (v - 1)D_1 \right] r^{(i+c)}d_i \]

\[ + \sum_{i=0}^{\infty} (i + c + 1)(i + c - 1)D_0 r^{(i-c)}d_i = 0 \]

(1.1.12)

Equating the co-efficient of the smallest power of \( r \) from both sides, the indicial equation is obtained as,

\[ (c + 1)(c - 1) = 0 \]

which gives, \( c = -1, 1 \).

Comparing co-efficients of different powers of \( r \) from both sides of the equation (1.1.12), \( d_i \)'s are determined. For \( c = -1 \), the solution will be singular at the origin \( (r = 0) \) and hence the series solution corresponding to \( c = -1 \) is omitted.

Hence the complementary function is obtained as,

\[ C.F. = A_0 r(1 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 + \ldots) \]  

(1.1.13)

where, \( A_0 \) is an arbitrary constant and

\[ A_1 = -\frac{(1 + v)D_1 + N}{3D_0}, \quad A_2 = \frac{(5 + v)D_1 + 2N((1 + v)D_1 + N)}{24D_0^2} - \frac{2(1 + v)D_2}{8D_0}, \]

\[ A_3 = -\frac{3(1 + v)D_3}{15D_0} - \frac{(7 + 2v)D_2}{15D_0} A_1 - \frac{(11 + v)D_1 + 3N}{15D_0} A_2, \]

\[ A_4 = -\frac{4(1 + v)D_4}{24D_0} - \frac{3(3 + v)D_3}{24D_0} A_1 - \frac{2(7 + v)D_2}{24D_0} A_2 - \frac{(19 + v)D_1 + 4N}{24D_0} A_3, \]

\[ \ldots \]

(1.1.14)

The series solution so obtained is found to be convergent.
The particular integral of the equation (1.1.10) is found to be,

\[ P.I. = Ar + Br^2 + Cr^3 \]  \hspace{1cm} (1.1.15)

where,

\[
A = \frac{q_0}{6D_3(v + 1)(v + 5)} - \frac{(35 + 9v)(v + 5)D_2 + 2N}{240D_3D_2(v + 1)(v + 5)(4v + 11)}D_3q_0 + \frac{q_0}{4D_3(v + 1)},
\]

\[
B = \frac{(35 + 9v)D_3q_0}{120(v + 5)(4v + 11)D_0D_4}, \quad C = \frac{q_0}{24D_0(v + 5)}
\]

Hence the complete solution of the equation (1.1.10) is obtained as,

\[ \psi = A_0r(1 + A_1r + A_2r^2 + A_3r^3 + A_4r^4 + \ldots) + Ar + Br^2 + Cr^3 \]  \hspace{1cm} (1.1.16)

Integrating above equation deflection \( w_1 \) is found to be,

\[ w_1 = A_0r^2\left(\frac{1}{2} + A_1 \frac{r}{3} + A_2 \frac{r^2}{4} + A_3 \frac{r^3}{5} + A_4 \frac{r^4}{6} + \ldots\right) + \frac{A}{2}r^2 + \frac{B}{3}r^3 + \frac{C}{4}r^4 + K_0 \]  \hspace{1cm} (1.1.17)

where \( K_0 \) is an arbitrary constant.

Hence the total deflection is obtained as,

\[ w = a_0\left(a^2 - r^2\right)^2 \]

\[ + A_0r^2\left(\frac{1}{2} + A_1 \frac{r}{3} + A_2 \frac{r^2}{4} + A_3 \frac{r^3}{5} + A_4 \frac{r^4}{6} + \ldots\right) + \frac{A}{2}r^2 + \frac{B}{3}r^3 + \frac{C}{4}r^4 + K_0 \]  \hspace{1cm} (1.1.18)

To find the arbitrary constants, we consider that the plate is simply supported.

Hence the boundary conditions are given by,

\[ w = 0 \text{ at } r = a \]

\[ \text{and } \frac{d^2w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} = 0 \text{ at } r = a. \]  \hspace{1cm} (1.1.19)
Substituting $w_i$ from the equation (1.1.17) and $D$ from the equation (1.1.1) in the equation (1.1.6) bending moments are obtained as,

$$M_r = -D_0(A + A_0 + Av + A_0v) - \{D_1(A + A_0 + Av + A_0v)
+ D_0(2A_i + 2B + Bv + A_i)v\}r$$

$$-\{D_2(A + A_0 + Av + A_0v) + D_1(2A_i + 2B + Bv + A_i)v
+ D_0(3A_2 + 3B + Bv + A_2)v\}r^2$$

$$-\{D_3(A + A_0 + Av + A_0v) + D_2(2A_i + 2B + Bv + A_i)v
+ D_1(3A_2 + 3B + Bv + A_2)v + D_0(4 + v)A_3\}r^3$$

$$-\{D_4(A + A_0 + Av + A_0v) + D_3(2A_i + 2B + Bv + A_i)v
+ D_2(3A_2 + 3B + Bv + A_2)v + D_1(4 + v)A_3 + D_0(5 + v)A_4\}r^4$$

$$-\{D_4(2A_i + 2B + Bv + A_i)v
+ D_3(3A_2 + 3B + Bv + A_2)v + D_2(4 + v)A_3
+ D_1(5 + v)A_4 + D_0(6 + v)A_5\}r^5$$

(1.1.20)

and

$$M_r = -\left(\frac{A + A_0}{r}\right)D_0 - \{(2A_i + 2B)D_0 + (A + A_0)D_1\}
- \{(2A_i + 2B)D_1 + (A + A_0)D_2 + D_0(3A_2 + 3C - Av - A_0v)\}r$$

$$-\{(2A_i + 2B)D_2 + (A + A_0)D_3 + D_1(3A_2 + 3C - Av - A_0v)
+ D_0(4A_3 - A_i v - Bv)\}r^2$$

$$-\{(2A_i + 2B)D_3 + (A + A_0)D_4 + D_2(3A_2 + 3C - Av - A_0v)
+ D_1(4A_3 - A_i v - Bv) + D_0(5A_4 - A_2v - Cv)\}r^3$$
\[-\{(2A_1 + 2B)D_4 + D_3 (3A_2 + 3C - A_0 v - A_0 v) + D_2 (4A_3 - A_1 v - B v) + D_1 (5A_4 - A_2 v - C v) + D_0 (6A_5 - A_3 v)\} r^4 \]
\[-\{(3A_2 + 3C - A_0 v - A_0 v) + D_3 (4A_3 - A_1 v - B v) + D_2 (5A_4 - A_2 v - C v) + D_1 (6A_5 - A_3 v) + D_0 (7A_6 - A_4 v)\} r^5 \]

\textbf{Special Case 1:} Putting, \(D_2 = D_3 = D_4 = 0\), \(q_2 = q_3 = q_4 = 0\), in the equation (1.1.13) and calculating the corresponding particular integral separately, it is found that, the corresponding results obtained here for linearly varying flexural rigidity and load intensity, are in agreement with the existing results obtained by Sinha and Basuli\[48]\textbf{(1986)}.

\textbf{Special Case 2:} Putting \(D_3 = D_4 = 0\), \(q_3 = q_4 = 0\), in the equation (1.1.13) and calculating particular integral separately, it is observed that, the corresponding results obtained here are in agreement with the previous results obtained by De and Banik\[14]\textbf{(2010)}.

\textbf{Special Case 3:} Putting \(D_4 = q_4 = 0\) in the equation (1.1.9), the corresponding equation of equilibrium for cubic variation is obtained as,

\[
\left(D_0 + D_1 r + D_2 r^2 + D_3 r^3\right) \frac{d^3 w_i}{dr^3} + \left(\frac{D_0}{r} + (2D_1 + N) + 3D_2 r + 4D_3 r^2\right) \frac{d^2 w_i}{dr^2} + \left(\frac{-D_0}{r^2} + (3\nu - 1)\frac{D_1}{r} + (2\nu - 1)D_2 + (3\nu - 1)D_3 r\right) \frac{dw_i}{dr} = N a_0 \left(a^2 - 3r^2\right) + q_0 \frac{r}{2} + q_1 \frac{r^2}{3} + q_2 \frac{r^3}{4} + q_3 \frac{r^4}{5} \]

\textbf{(1.1.22)}
Hence putting \( D_4 = q_4 = 0 \) in the equation (1.1.13) and calculating the partial integral separately, deflection \( w_1 \) for cubic variation is found as,

\[
w_1 = A_0' r^2 \left( \frac{1}{2} + A_1' \frac{r^2}{3} + A_2' \frac{r^3}{4} + A_3' \frac{r^4}{5} + A_4' \frac{r^5}{6} + \ldots \right) + \frac{A'}{2} r^2 + \frac{B'}{3} r^3 + \frac{C'}{4} r^4 + K_0' \tag{1.1.23}
\]

where,

\( A_0', K_0' \) are arbitrary constants and

\[
A_1' = -\frac{(1 + v)D_1 + N}{3D_0}, \quad A_2' = \frac{((5 + v)D_1 + 2N)((1 + v)D_1 + N)}{24D_0^2} - \frac{2(1 + v)D_2}{8D_0}
\]

\[
A_3' = -\frac{3(1 + v)D_3}{15D_0} - \frac{(7 + 2v)D_2}{15D_0} A_1' - \frac{(11 + v)D_1 + 3N}{15D_0} A_2',
\]

\[
A_4' = -\frac{3(3 + v)D_1}{24D_0} A_1' - \frac{2(7 + v)D_2}{24D_0} A_2' - \frac{(19 + v)D_1 + 4N}{24D_0} A_3',
\]

\[...
\]

and

\[
A' = -\frac{4q_0}{5D_1(v + 1)(3v + 17)} - \frac{(29 + 7v)(v + 5)D_1 + 2N}{120D_0D_2D_3(v + 1)(v + 3)(3v + 17)} + \frac{q_0}{4D_2(v + 1)},
\]

\[
B' = \frac{(29 + 7v)D_2q_0}{60(v + 3)(3v + 17)D_0D_3},
\]

\[
C' = \frac{q_0}{5D_0(3v + 17)},
\]

The Particular integral become infinite for \( D_4 = 0 \), so particular integral for this case is calculated separately. Hence total deflection for cubical variation is obtained as,

\[
w = a_0 \left( a^2 - r^2 \right)^2 + A_0' r^2 \left( \frac{1}{2} + A_1' \frac{r^2}{3} + A_2' \frac{r^3}{4} + A_3' \frac{r^4}{5} + A_4' \frac{r^5}{6} + \ldots \right)
\]
\[ + \frac{A'}{2} r^2 + \frac{B'}{3} r^3 + \frac{C'}{4} r^4 + K'_0 \]  

(1.1.24)

Using boundary conditions (1.1.19) the arbitrary constants, \( A'_0, K'_0 \) are obtained.

The bending moments for cubic variation of flexural rigidity and load intensity are found to be,

\[
M_r = -D_0 (A' + A'_0 + A' \nu + A'_0 \nu) - \{D_1 (A' + A'_0 + A' \nu + A'_0 \nu) \\
+ D_0 (2A'_0 + 2B' + B' \nu + A'_0 \nu) \} r
\]

\[
- \{D_2 (A' + A'_0 + A' \nu + A'_0 \nu) + D_1 (2A'_0 + 2B' + B' \nu + A'_0 \nu) \\
+ D_0 (3A'_0 + 3B' + B' \nu + A'_0 \nu) \} r^2
\]

\[
- \{D_3 (A' + A'_0 + A' \nu + A'_0 \nu) + D_2 (2A'_0 + 2B' + B' \nu + A'_0 \nu) \\
+ D_1 (3A'_0 + 3B' + B' \nu + A'_0 \nu) + D_0 (4 + \nu) A'_3 \} r^3
\]

\[
- \{D_3 (2A'_0 + 2B' + B' \nu + A'_0 \nu) \\
+ D_2 (3A'_0 + 3B' + B' \nu + A'_0 \nu) + D_1 (4 + \nu) A'_3 + D_0 (5 + \nu) A'_3 \} r^4
\]

\[
- \{D_3 (3A'_0 + 3B' + B' \nu + A'_0 \nu) \\
+ D_2 (4 + \nu) A'_3 + D_1 (5 + \nu) A'_4 + D_0 (6 + \nu) A'_4 \} r^5
\]  

(1.1.25)

\[
M_t = -\left(\frac{A' + A'_0}{r}\right) D_0 - \{(2A'_0 + 2B')D_0 + (A' + A'_0)D_1 \}
\]

\[
- \{(2A'_0 + 2B')D_1 + (A' + A'_0)D_2 \\
+ D_0 (3A'_0 + 3C' - A' \nu - A'_0 \nu) \} r
\]

\[
- \{(2A'_0 + 2B')D_2 + (A' + A'_0)D_3 \\
+ D_1 (3A'_0 + 3C' - A' \nu - A'_0 \nu) + D_0 (4A'_0 - A'_0 \nu - B' \nu) \} r^2
\]
\[-(2A' + 2B')D_3 + D_2 (3A'_2 + 3C' - A' \nu - A'_0 \nu)\]
\[+ D_1 (4A'_3 - A'_1 \nu - B' \nu) + D_0 (5A'_4 - A'_2 \nu - C' \nu) r^3\]
\[-(2A' + 2B')D_4 + D_3 (3A'_2 + 3C' - A' \nu - A'_0 \nu)\]
\[+ D_2 (4A'_3 - A'_1 \nu - B' \nu) + D_1 (5A'_4 - A'_2 \nu - C' \nu)\]
\[+ D_0 (6A'_5 - A'_3 \nu) r^4\]
\[-D_3 (4A'_3 - A'_1 \nu - B' \nu) + D_2 (5A'_4 - A'_2 \nu - C' \nu)\]
\[+ D_1 (6A'_5 - A'_3 \nu) + D_0 (7A'_6 - A'_4 \nu) r^5\]

(1.1.26)

**NUMERICAL RESULTS**

We consider two different circular plates of radius 10 and 15 units (\(a = 10\) and \(a = 15\)). We also assume,

\[N = \frac{D_0}{10}, \quad D_1 = \frac{D_0}{20}, \quad D_2 = \frac{D_0}{30}, \quad D_3 = \frac{D_0}{40}, \quad D_4 = \frac{D_0}{50}, \quad q_0 = a_0 D_0,\]
\[q_1 = \frac{q_0}{20}, \quad q_2 = \frac{q_0}{30}, \quad q_3 = \frac{q_0}{40}, \quad q_4 = \frac{q_0}{50}\]

With these assumptions the variations of initial deflection (\(\bar{w}_0 = \frac{w_0}{a_0 10^5}\)), total deflection (\(\bar{w} = \frac{w}{a_0 10^5}\)) and bending moments (\(\bar{M}_r = \frac{M_r}{a_0 10^3}, \quad \bar{M}_t = \frac{M_t}{a_0 10^3}\)) with the radial distance \(r\) are calculated and shown graphically for both cubic and biquadratic variation of flexural rigidity and load intensity.
GRAPHS FOR CUBIC VARIATION

Figure 1.1.1: Comparative study of the initial deflection($\bar{w}_0$) and total deflection($\bar{w}$) for cubic variation with radius $a = 10$.

Figure 1.1.2: Comparative study of the initial deflection($\bar{w}_0$) and total deflection($\bar{w}$) for cubic variation with radius $a = 15$. 
Figure 1.1.3: Bending moment $M_r$ of the circular plate for cubic variation with radius $a = 10$.

Figure 1.1.4: Bending moment $M_r$ of the circular plate for cubic variation with radius $a = 15$. 

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GRAPHS FOR CUBIC VARIATION

Figure 1.1.5: Bending moment $M_r$ of the circular plate for cubic variation with radius $a = 10$.

Figure 1.1.6: Bending moment $M_r$ of the circular plate for cubic variation with radius $a = 15$. 
Figure 1.1.7: Comparative study of the initial deflection ($w_0$) and total deflection ($w$) for biquadratic variation with radius $a = 10$.

Figure 1.1.8: Comparative study of the initial deflection ($w_0$) and total deflection ($w$) for biquadratic variation with radius $a = 15$. 
GRAPHS FOR BIQUADRATIC VARIATION

Figure 1.1.9: Bending moment $\bar{M}_r$ of the circular plate for biquadratic variation with radius $a = 10$.

Figure 1.1.10: Bending moment $\bar{M}_r$ of the circular plate for biquadratic variation with radius $a = 15$. 
GRAPHS FOR BIQUADRATIC VARIATION

Figure 1.1.11: Bending moment $\bar{M}_t$ of the circular plate for biquadratic variation with radius $a = 10$.

Figure 1.1.12: Bending moment $\bar{M}_t$ of the circular plate for biquadratic variation with radius $a = 15$. 
DISCUSSION AND CONCLUSION

The differential equation of equilibrium for axi-symmetric bending of an initially curved circular plate with cubically and biquadratically varying flexural rigidity and load intensity is obtained. Exact solution of the equation of equilibrium is computed using Frobenius method of series solution. Hence additional deflection ($w_i$), total deflection ($w$), radial bending moment ($M_r$) and tangential bending moment ($M_t$) are computed for a thin circular plate with cubic as well as biquadratic variation of flexural rigidity and load intensity. Circular plates of any radius can be considered. Considering some fixed values of parameters variation of initial deflection ($w_0$), total deflection ($w$), radial bending moment ($M_r$) and tangential bending moment ($M_t$) with radial distance are calculated numerically and shown graphically. For both types of variation cubic and biquadratic, graphs are drawn considering two different values of radii, $a = 10$ and $a = 15$.

First we will discuss about the figures of cubic variation. In case of cubic variation the nature of the graphs (deflections, radial bending moment and tangential bending moment) remain the same as the radius of the circular plate increases or decreases but the dimension of the graphs changes with radius. It is observed from figure 1.1.1 and figure 1.1.2 that the total deflection is higher than the initial deflection and both have similar types of variations with radial distance. Also both the deflections are maximum at the centre of the plate and zero at the outer edge. From figure 1.1.3 and 1.1.4 we see that the radial bending moments are zero at the centre as well as at the outer surface of the plate. As the radial distance increases the radial bending
moment also gradually increases and attains its maximum value than rapidly decreases towards the minimum negative value. After attaining the minimum value it again increases and become zero at the outer surface. In this case we observe that tangential bending moment is minimum at the centre of the plate and maximum at the outer edge of the plate. Figure 1.1.5 and figure 1.1.6 show that tangential bending moment \((M_t)\) gradually increases as radial distance increases. Near the edge of the plate, \(M_t\) slightly decreases and than rapidly increases as radial distance increases. In this case for all the graphs, it is found that, plate with higher radius (figure 1.1.2) has higher deflection than the plate with lower radius (figure 1.1.1). This trend remains same for radial bending moment and tangential bending moment. Plate with radius 15 has higher radial bending moment and tangential bending moment than the plate with radius 10.

In case of biquadratic variation it is observed that as the radius of the circular plate changes the nature of the curve changes absolutely. From figure 1.1.7 and figure 1.1.8 we find that, in this case also the total deflection is higher than the initial deflection and both have similar variation with radial distance. Also both the deflections are maximum at the centre of the plate and zero at the outer edge. For biquadratic variation, circular plate with higher radius (figure 1.1.8) has higher deflection whereas the circular plate with lower radius (figure 1.1.7) has much lower deflection. In this case also, radial bending moment with lower radius (figure 1.1.9) is zero at the centre of the plate and as the radial distance increases the radial bending moment also gradually increases and attains its maximum value, than rapidly decreases towards zero at the outer edge. Radial bending moment with higher radius
(Figure 1.1.10) is zero at the centre of the plate and as the radial distance increases the bending moment gradually increases attains its maximum value at the midway between the centre and the outer edge of the plate and after that it gradually decreases rapidly to a minimum negative value and after attaining minimum value it gradually increases to zero deflection at the outer edge. In case of tangential bending moment we find that, for \( a = 10 \) (figure 1.1.11) tangential bending moment increases with increasing radial distance whereas for the plate with higher radius (for \( a = 15 \)) (figure 1.1.12), the nature of the curve is almost different. In this case, as the radial distance increases tangential bending moment gradually increases, reaches maximum value and than gradually decreases reaches a minimum, after that it sharply increases towards a maximum value at the outer edge.
SECTION 1.2

AXI-SYMMETRIC BENDING OF A NON-UNIFORMLY STRESSED ANNULAR PLATE WITH EXPONENTIALLY VARYING THICKNESS *

INTRODUCTION

Bending problem of axially symmetric circular plates with linearly varying thickness had been solved by Conway\textsuperscript{[13]}(1958). Basuli\textsuperscript{[3]}(1961) had solved the bending problem of uniformly compressed circular plate of variable thickness. Bose\textsuperscript{[5]}(1965) discussed the torsion of an aeolotropic cylinder having a spheroidal inclusion on its axis. Bending problem of non-homogeneous cylindrical orthotropic circular clamped plate under uniformly distributed transverse load was discussed by Qin\textsuperscript{[42]}(1994). Material properties of non-homogeneous cylindrically orthotropic circular plate had been discussed by Qin, Yan, Hu and Huang\textsuperscript{[43]}(1997). They further calculated the values of material properties which are the functions of radius. Gregory, Gu and Wan\textsuperscript{[20]}(2002) had discussed the bending problem of a cantilever strip plate of exponentially varying thickness.

The object of this section is to obtain theoretically the stresses of a composite annular plate due to axi-symmetric bending, considering

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exponentially varying thickness. The differential equation of bending of the annular plate with exponentially varying thickness is obtained and solved using the Frobenius method of series solution. Hence the stress function, deflection and both the bending moments are found. Variation of deflection and bending moments with radial distance are calculated numerically and shown graphically for different parameters.

**FORMULATION AND SOLUTION OF THE PROBLEM**

The differential equation for bending of annular plate with variable thickness is given by (vide, Timoshenko and Woinowsky-Krieger\(^{[54]}\) (1983)),

\[
M_r + \frac{dM_r}{dr} = M_t + Ph \varphi = -Q_r
\]

(1.2.1)

where, \(P\) is the uniform pressure and \(\varphi\) is the stress function.

The bending moments are given by,

\[
M_r = D \left[ \frac{d\varphi}{dr} + \varphi \frac{\varphi}{r} \right]
\]

(1.2.2)

and

\[
M_t = D \left[ \varphi \frac{\varphi}{r} + \nu \frac{d\varphi}{dr} \right]
\]

(1.2.3)

and the shear load \(Q_r\) is given by,

\[
Q_r = \frac{1}{r} \int_0^r rqdr
\]

(1.2.4)

For this problem, we consider that, the thickness of the plate varies exponentially as,

\[
h = h_0 e^{\left(\frac{n}{3}\right)r}
\]

(1.2.5)

where, \(h_0\) is a real constant and \(n\) is any real number such that, \(-1 < n < 1\).
Thus the flexural rigidity of the plate will be,

\[ D = D_0 e^{\alpha r} \]  \hspace{1cm} (1.2.6)

where, \( D_0 \) is a real constant.

Also the load intensity \((q)\) of the plate is considered as,

\[ q = q_0 e^{\alpha r} \]  \hspace{1cm} (1.2.7)

\( q_0 \) and \( m \) are real constants and \( m \) is non-zero.

Using equations (1.2.2), (1.2.3), (1.2.4), (1.2.6) and (1.2.7) in the equation (1.2.1) we obtain the equation of equilibrium as,

\[
r^2 \frac{d^2 \phi}{dr^2} + r(nr + 1) \frac{d\phi}{dr} - \left(1 - n \nu r + k_0 e^{\alpha r} r^2 \right) \phi = k_1 \left(e^{\alpha r} r^2 - \frac{r}{m} e^{\alpha r} + \frac{r}{m} e^{-\alpha r} \right)
\]  \hspace{1cm} (1.2.8)

where, \( k_0 = \frac{P}{D_0}, \quad k_1 = -\frac{q_0}{D_0m}, \quad \lambda_0 = \frac{2n}{3}, \quad l = m - n; \)

This is a second order differential equation with exponential co-efficient. To find the complementary function of the equation (1.2.8) we consider,

\[
r^2 \frac{d^2 \phi}{dr^2} + r(nr + 1) \frac{d\phi}{dr} - \left(1 - n \nu r + k_0 e^{\alpha r} r^2 \right) \phi = 0 \]  \hspace{1cm} (1.2.9)

The solution of the equation (1.2.9) is obtained using Frobenius method of series solution. The trial solution for this equation is considered as,

\[ \varphi = \sum_{i=0}^{\infty} (a_i r^{i+c}) \]

Substituting \( \varphi \) in the equation (1.2.9) and equating to zero we obtain,

\[
\sum_{i=0}^{\infty} (i + c + 1)(i + c - 1)r^{(i+c)} a_i + n \sum_{i=0}^{\infty} (i + c + \nu)r^{(i+c+1)} a_i
\]

\[
- \sum_{i=0}^{\infty} r^{(i+c)} a_i e^{-\alpha r} = 0
\]  \hspace{1cm} (1.2.10)
This is an identity. Equating the smallest power of \( r \) to zero, the indicial equation is obtained as,

\[(c + 1)(c - 1) = 0\]  \hspace{1cm} (1.2.11)

which gives, \( c = -1, \ 1 \).

For \( c = -1 \), \( a_2 \) becomes indeterminate. Hence the series corresponding to \( c = -1 \), will contain two arbitrary constants. Also the series corresponding to \( c = 1 \) is merely a numerical multiple of one of the series contained in the first solution. Therefore the series obtained for \( c = 1 \) is rejected. Comparing coefficients of different powers of \( r \) from both sides in (1.2.9) \( a_j^s \) are obtained.

Hence the complementary function is given by,

\[ C.F. = \frac{A}{r} f(r) + Br g(r) \]  \hspace{1cm} (1.2.12)

where, \( A \) and \( B \) are arbitrary constants and

\[ f(r) = (1 + b_1 r + b_2 r^3 + b_3 r^4 + b_4 r^5 + ............) \]
\[ g(r) = (1 + d_1 r + d_2 r^2 + d_3 r^3 + d_4 r^4 + ............) \]

\[ b_1 = n(\nu - 1), \quad b_2 = \frac{k_0}{3} (10 + n - n\nu), \]

\[ b_3 = \frac{k_0}{48} \{ -3\lambda_0^2 + 2\lambda_0 n(2\nu - 5) + 2n^2 (\nu^2 + \nu - 2) \}, \]

.............

\[ d_1 = -\frac{1}{3} a_z n(1 + \nu), \quad d_2 = \frac{1}{24} \{ -3k_0 + n^2 (\nu^2 + 3\nu + 2) \}, \]

\[ d_3 = \frac{1}{360} \{ -n^3 (6 + 11\nu + 6\nu^2 + \nu^3) + k_0 \{ 24\lambda_0 + n(17 + 11\nu) \}, \]

.............
The particular integral of the equation (1.2.8) is found to be,

\[ P.I. = pr^2 + p_2r^3 + p_3r^4 + p_4r^5 \]  

(1.2.13)

where,

\[ p_1 = \frac{k_1}{3} \left( l - \frac{(l+n)m}{m} \right), \quad p_2 = \frac{1}{8} \left\{ k_1 \left( l - \frac{(l^2-n^2)}{2m} \right) - (2+\nu)p_3 \right\} \]

\[ p_3 = \frac{1}{15} \left\{ k_1 \left( \frac{l^2}{2} - \frac{(l^2+n^2)}{6m} \right) - n(3+\nu)p_2 - k_0p_1 \right\} \]

\[ p_4 = \frac{1}{24} \left\{ k_1 \left( \frac{l^3}{6} - \frac{(l^4-n^4)}{24m} \right) - n(4+\nu)p_3 - k_0p_2 - k_0^2p_1 \right\} \]

Hence the complete solution is given by,

\[ \varphi = \frac{A}{r} f(r) + Brg(r) + pr^2 + p_2r^3 + p_3r^4 + p_4r^5 \]  

(1.2.14)

Now stress function and deflection has the following relation,

\[ \varphi = -\frac{dw}{dr} \]

Thus integrating (1.2.14), deflection of the annular plate is obtained as

\[ w = AF(r) + BG(r) - \frac{p_1}{3}r^3 - \frac{p_2}{4}r^4 - \frac{p_3}{5}r^5 - \frac{p_4}{6}r^6 - K \]  

(1.2.15)

where, \( F(r) = -\log r - r(b_1 + \frac{b_2}{3}r^2 + \frac{b_3}{4}r^3 + \frac{b_4}{5}r^4 + ...) \)

\[ G(r) = \frac{r^2}{2} - r^3 \left( \frac{d_1}{3} + \frac{d_2}{4}r + \frac{d_3}{5}r^2 + \frac{d_4}{6}r^3 + ... \right) \]

and \( K \) is arbitrary constant.

Using boundary conditions (vide, Conway\textsuperscript{[13]}(1958))

\[ \varphi = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = b \quad \text{and} \]

\[ w = 0 \quad \text{at} \quad r = b \]

arbitrary constants are determined.
Substituting the value of $D$ from (1.2.6) and the value of $\phi$ from (1.2.14) in (1.2.2) and (1.2.3), bending moments are obtained as,

$$M_r = D_0 e^{\omega r} \int \frac{A}{r^2} ((\nu - 1) + \nu rb_1 + (\nu + 2) b_2 r^3 + (\nu + 3) b_3 r^4 + (\nu + 4) b_4 r^5 + \ldots)$$

$$+ B((\nu + 1) + (2 + \nu) d_1 r + (\nu + 3) d_2 r^2 + (\nu + 4) d_3 r^3$$

$$+ (\nu + 5) d_4 r^4 + \ldots)$$

$$+ (2 + \nu) M_2 r + (3 + \nu) M_3 r^2 + (4 + \nu) M_4 r^3 + (5 + \nu) M_5 r^4]$$

(1.2.16)

$$M_t = D_0 e^{\omega r} \int \frac{A}{r^2} ((1 - \nu) + rb_1 + (2\nu + 1)b_2 r^3 + (3\nu + 1) b_3 r^4 + (4\nu + 1) b_4 r^5 + \ldots)$$

$$+ B((\nu + 1) + (2\nu + 1) d_1 r + (3\nu + 1) d_2 r^2 + (4\nu + 1) d_3 r^3$$

$$+ (\nu + 5) d_4 r^4 + \ldots)$$

$$+ (2\nu + 1) M_2 r + (3\nu + 1) M_3 r^2 + (4\nu + 1) M_4 r^3 + (5\nu + 1) M_5 r^4]$$

(1.2.17)

**NUMERICAL RESULTS**

Considering $\nu = 0.3$, $k_0 = \frac{1}{3}$, outer radius $b = 1$ and inner radius $a = 0.2$, deflection $(\bar{w} = w \frac{D_0}{q_0})$ and bending moments $(\bar{M}_r = \frac{M_r}{q_0}, \bar{M}_t = \frac{M_t}{q_0})$ of the annular plate are calculated numerically and shown graphically for different values of $m$ and $n.$
VARIATION OF DEFECTION

Figure 1.2.1: Variation of $\bar{w}$ with radial distance($r$) and $m$, for $n = 0.1$.

Figure 1.2.2: Variation of $\bar{w}$ with radial distance($r$) and $n$, for $m = 0.1$. 
VARIATION OF RADIAL BENDING MOMENTS

Figure 1.2.3: Variation of $\overline{M}_r$ with radial distance ($r$) and $m$, for $n = 0.1$.

Figure 1.2.4: Variation of $\overline{M}_r$ with radial distance ($r$) and $n$, for $m = 0.1$. 
VARIATION OF TANGENTIAL BENDING MOMENTS

Figure 1.2.5: Variation of $M_t$ with radial distance ($r$) and $m$, for $n = 0.1$

Figure 1.2.6: Variation of $M_t$ with radial distance ($r$) and $n$, for $m = 0.1$. 
DISCUSSION AND CONCLUSION

Bending problem of an annular plate with exponentially varying thickness is discussed here. Deflection, radial bending moment and tangential bending moment are obtained theoretically for the above said plate. For some fixed values of different parameters deflection and both the bending moments are calculated and shown graphically. Figures 1.2.1 and 1.2.2 show the variation of deflection for fixed \( n \) and \( m \) respectively. For any value of \( n \) and \( m \), (figures 1.2.1 and 1.2.2) it is observed that, magnitude of the deflection of the annular plate increases as radial distance increases. Also the deflection at any point of the plate is in negative direction. Figure 1.2.1 shows that, for fixed \( n \), as \( m \) increases magnitude of the deflection increases. For fixed \( m \), as \( n \) increases magnitude of the deflection decreases (figure 1.2.2). Figures 1.2.3(for fixed \( n \)) and 1.2.4(for fixed \( m \)) show the variation of radial bending moment. The radial bending moment initially increases slightly and then decreases gradually. Figures 1.2.5 (for fixed \( n \)) and 1.2.6 (for fixed \( m \)) show the variation of tangential bending moment. The tangential bending moment of the plate gradually increases and attains its maximum value. Then it decreases gradually and obtains the minimum value. For the area near the inner edge all the bending moments are positive but for the rest of the plate bending moments are negative.