CHAPTER 5

BUCKLING PROBLEMS ON

SHELLS
SECTION 5.1

BUCKLING OF \((2n+1)\)-LAYERS PLYWOOD SHELL UNDER TWO WAY COMPRESSIONS*

INTRODUCTION

Woods are anisotropic in nature. Woods display much more rigidity in the direction of the grain than across. So along the cross grain direction the rigidity of the wood is very less. Therefore to satisfy the need of such wooden material which has approximately same rigidity in both directions, the concept of plywood is introduced.

Shells are used for roof structures and large columnless areas and for storage tanks. A large number of air craft hangers, factory and car sheds, covered markets, planetarium and rail-road terminals etc. have been created with shell constructions. A detailed study of the shell of arbitrary shape is necessary for consideration of variety, economy and architectural showmanship in building construction.

Cheng and Ho\[^{10}\] (1963) investigated some problems in stability of heterogeneous aeolotropic cylindrical shells under combined loading. Tasi, Feldman and Stang\[^{52}\] (1965) discussed the buckling problem of filament- wound cylinder under axial compression. De\[^{15}\] (1983) analysed the

buckling problem of anisotropic cylindrical shell. Huang and Lu\textsuperscript{[26]}(2002) investigated buckling problem of laminated circular cylindrical shell using two surface theory. Chao-Sheng, Jing and Quan\textsuperscript{[8]}(2003) obtained the solution of non-linear bending and buckling of circular plates.

Anisotropic plywood shell of \((2n+1)\)-layers is under consideration of this work. The object of this section is to obtain all the stress resultants and to formulate the differential equations of buckling of \((2n+1)\)-layers plywood shell. The solution of differential equations of buckling problem for anisotropic plywood shell in case of two way compressions is obtained here. The buckling condition for a plywood shell in this case is obtained. The buckling diagrams for five layers plywood shell are shown graphically for different values of parameters.

**THEORY**

Here we consider a circular cylindrical shell with co-ordinates \(x, \varphi\) and \(z\). \(x\) is the distance of any point from a datum plane, \(\varphi\) is the angular distance of the point from a datum generator and \(z\) is the distance from the middle surface. \(u, v\) and \(w\) are the components of velocity in \(x, \varphi\) and \(z\) directions respectively.

Symmetric \((2n+1)\)-layers plywood shells are considered here. Wood displays much more rigidity in the direction of the grain than across, so Hooke’s law is not symmetric with respect to \(x\) and \(\varphi\) here. For the inner layer it takes the form (vide, Flugge\textsuperscript{[18]}(1973)),

\[\]
Here $\sigma_x, \sigma_y, \tau_{xy}$ are stresses and $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are strains and the four moduli $E_1, E_2, E_3, G$ are all independent of each other. If the outer symmetric layers are made of the same kind of wood and this we shall assume, then their elastic laws are the same except that the moduli $E_1$ and $E_2$ change places.

To explain the elastic behavior of plywood shell, we consider the figure of the 7 layers plywood material. In the figure 5.1.1, it has been assumed that the grain of the inner layer is running in the $x$ direction. $E_1$ is the common modulus of elasticity of wood i.e., the one for stresses in the direction of the grain, while $E_2$ is the much smaller cross grain modulus. When in a shell the grain runs circumferentially in the middle layer and lengthwise in the adjacent two layers to the middle layer then for two layers adjacent to the middle layer we must identify $E_2$ with common modulus and $E_1$ with the cross grain modulus. Similarly for the next two adjacent layers the common modulus of elasticity will be $E_1$ and $E_2$ will be the cross grain modulus and so on.
Figure 5.1.1: An element of a 7-layers plywood shell.
The strains in this case are given by (vide, Flugge\textsuperscript{[18]}(1973)),

\[
\begin{align*}
\epsilon_x &= \frac{u'}{a} - z \frac{w''}{a} \\
\epsilon_\varphi &= \frac{v'}{a} - z \frac{w''}{a} + \frac{w}{a + z} \\
\gamma_{x\varphi} &= \frac{u'}{a + z} + \frac{a + z}{a^2} \cdot v' \cdot \frac{w''}{a} \left( \frac{z}{a} + \frac{z}{a + z} \right)
\end{align*}
\] (5.1.2)

where \( (') = \frac{\alpha\frac{\partial}{\partial x}}{a} \) and \( (\varphi') = \frac{\alpha\frac{\partial}{\partial \varphi}}{a} \) and \( a \) is the radius of the shell.

The stress resultants for any anisotropic shell are defined by (vide, Flugge\textsuperscript{[18]}(1973)),

\[
\begin{align*}
N_x &= \int_{-\alpha/2}^{\alpha/2} \sigma_x (1 + \frac{z}{a}) dz, \\
N_\varphi &= \int_{-\alpha/2}^{\alpha/2} \sigma_\varphi dz, \\
N_{x\varphi} &= \int_{-\alpha/2}^{\alpha/2} \tau_{x\varphi} (1 + \frac{z}{a}) dz, \\
N_{\varphi x} &= \int_{-\alpha/2}^{\alpha/2} \tau_{\varphi x} dz, \\
M_x &= -\int_{-\alpha/2}^{\alpha/2} \sigma_x (1 + \frac{z}{a}) zdz, \\
M_\varphi &= -\int_{-\alpha/2}^{\alpha/2} \sigma_\varphi zdz, \\
M_{x\varphi} &= -\int_{-\alpha/2}^{\alpha/2} \tau_{x\varphi} (1 + \frac{z}{a}) zdz, \\
M_{\varphi x} &= -\int_{-\alpha/2}^{\alpha/2} \tau_{\varphi x} zdz
\end{align*}
\] (5.1.3)
But when we introduce elastic laws, we must use it in the form given by the equations (5.1.1) for middle layer and exchange $E_i$ and $E_2$ while integrating over the outer middle layer and again the same for the next outer layers and so on. This leads to the definition of following rigidities.

(i) **Extensional rigidities:**

when $n$ is even

$$D_x = E_1(t_1 + 2t_3 + 2t_5 + ... + 2t_{n+1}) + 2E_2(t_2 + t_4 + t_6 + ... + t_n)$$

$$D_y = E_2(t_1 + 2t_3 + 2t_5 + ... + 2t_{n+1}) + 2E_1(t_2 + t_4 + t_6 + ... + t_n)$$  \hspace{1em} (5.1.4a)

$$D_z = E_s t$$

and when $n$ is odd

$$D_x = E_1(t_1 + 2t_3 + 2t_5 + ... + 2t_n) + 2E_2(t_2 + t_4 + t_6 + ... + t_{n+1})$$

$$D_y = E_2(t_1 + 2t_3 + 2t_5 + ... + 2t_n) + 2E_1(t_2 + t_4 + t_6 + ... + t_{n+1})$$  \hspace{1em} (5.1.4b)

$$D_z = E_s t$$

(ii) **Shear rigidity:**  

$$D_{xy} = G t$$  \hspace{1em} (5.1.4c)

(iii) **Bending rigidities:**

when $n$ is even

$$K_x = \frac{1}{12} \left[ E_1 \left\{ t_1^3 - (2t_2 + t_1)^3 + (2t_3 + t_2 + t_1)^3 - ... - t_3^3 \right\} 
+ E_2 \left\{ -t_1^3 + (2t_2 + t_1)^3 - (2t_3 + t_2 + t_1)^3 + ... + (t - 2t_{n+1})^3 \right\} \right]$$

$$K_y = \frac{1}{12} \left[ E_2 \left\{ t_1^3 - (2t_2 + t_1)^3 + (2t_3 + t_2 + t_1)^3 - ... - t_3^3 \right\} 
+ E_1 \left\{ -t_1^3 + (2t_2 + t_1)^3 - (2t_3 + t_2 + t_1)^3 + ... + (t - 2t_{n+1})^3 \right\} \right]$$

$$K_z = \frac{1}{12} E_v t^3$$  \hspace{1em} (5.1.4d)
and when $n$ is odd

$$
K_x = \frac{1}{12} \left[ E_1 \{t_1^3 - (2t_2 + t_1)^3 + (2t_3 + t_2 + t_1)^3 - \ldots + (t - 2t_n + t_1)^3 \} + E_2 \{-t_1^3 + (2t_2 + t_1)^3 - (2t_3 + t_2 + t_1)^3 - \ldots - (t - 2t_n + t_1)^3 + t_1^3 \} \right]
$$

$$
K_y = \frac{1}{12} \left[ E_2 \{t_1^3 - (2t_2 + t_1)^3 + (2t_3 + t_2 + t_1)^3 - \ldots + (t - 2t_n + t_1)^3 \} + E_1 \{-t_1^3 + (2t_2 + t_1)^3 - (2t_3 + t_2 + t_1)^3 - \ldots - (t - 2t_n + t_1)^3 + t_1^3 \} \right]
$$

$$
K_v = \frac{1}{12} E_v t^3
$$

(5.1.4e)

(iv) Twisting rigidity:

$$
K_{s\phi} = \frac{1}{12} G t^3
$$

(5.1.4f)

where,

$t = t_1 + 2t_2 + 2t_3 + \ldots + 2t_{n-1}$, is the total thickness of the plate and $t_1$ is the thickness of the middle most layer, $t_2$ is the thickness of two adjacent layers of the middle most layer, $t_3$ is the thickness of the next two layers and so on.

Substituting the values of $\sigma_x, \sigma_y, \tau_{s\phi}$ from the equations (5.1.1) in the equations (5.1.3) and using the equations (5.1.2), after simplifications we get the elastic laws for the $(2n + 1)$-layers plywood shell in the following form,

$$
N_{\phi} = \frac{D_{\phi}}{a} (v^\prime + w) + \frac{D_x}{a} u^\prime + \frac{K_{\phi}}{a^3} (w + w^{\prime\prime}),
$$

$$
N_x = \frac{D_x}{a} u^\prime + \frac{D_y}{a} (v^\prime + w) + \frac{K_x}{a^3} u^\prime\prime,
$$

$$
N_{s\phi} = \frac{D_{s\phi}}{a} (u^\prime + v^\prime) + \frac{K_{s\phi}}{a^3} (v^\prime - w^{\prime\prime}),
$$

$$
N_{s\phi} = \frac{D_{s\phi}}{a} (u^\prime + v^\prime) + \frac{K_{s\phi}}{a^3} (u^\prime + w^{\prime\prime}),
$$
Thus we got all the stress resultants and the rigidities for \((2n+1)\)-layers plywood shell. These are the most general form of rigidities for arbitrary odd number of layers. Also these formulae contain the formulae for isotropic shell as a special case. We only need to replace in (5.1.1) the moduli \(E_1\) and \(E_2\) by \(\frac{E}{(1-\nu^2)}\) and \(E_v\) by \(\frac{E}{(1-\nu^2)}\) respectively and \(G\) by \(\frac{E}{2(1+\nu)}\), and \(n = 0, \ t = t_1\) and make the necessary changes in the definition of the rigidities.

**THE BASIC EQUATIONS**

We consider a \((2n+1)\)-layers plywood shell shaped as a circular cylindrical shell of length \(l\) (Figure 5.1.2) and subjected simultaneously to three simple loads:

1. A uniform normal pressure on its \(P_r = p\) wall,
2. An axial compression applied at its edge, the force per unit circumference being \(P\),
3. A shear load applied at the edges so as to produce a torque in the cylinder, the shearing force is \(T\).
Figure 5.1.2: A cylindrical shell subjected simultaneously to three simple loads
The equations of equilibrium of buckling of circular cylindrical shell (vide, Flugge[18](1973)) are given by,

\[
aN_x' + aN_{\phi x} - pa(u'' - w') - Pu'' - 2Tu'' = 0,
\]

\[
aN_x' + aN_{\phi x} - M_{\phi} - M_{\phi x} - Pa(v'' + w') - Pv'' - 2T(v'' + w') = 0,
\]

\[
M_{\phi} + M_{\phi x} + M_{\phi x} + M_{x} + aN_{\phi} + Pa(u' - v' + w')
+ Pw'' - 2T(v' - w') = 0
\]

(5.1.6)

Substituting the values of stress resultants from the equation (5.1.5) in the equations (5.1.6), the differential equations for the buckling problem of a (2n+1)-layers circular cylindrical plywood shell under three different loads appear in the following form after proper simplification:

\[
u'' + A_4u'' + A_2v'' + A_3w'' + k_1\{A_4(u'' + w'') - w''\} - q_1(u'' - w') - q_2u - 2q_3u = 0
\]

\[
A_3u'' + v'' + A_8v'' + w'' + k_1[3A_4v'' - A_8w']
- A_8[q_1(v'' + w') + q_2v'' + 2q_3(v'' + w')] = 0
\]

(5.1.7)

\[
A_{10}u' + v' + w + k_1[A_7u''' - A_8u'' - A_8v''' + A_8w'''] + 2A_4w'''
+ A_{12}(w'''' + 2w''' + w)
+ A_9[q_1(u' - v' + w'') + q_2w'' - 2q_3(v' + w')] = 0
\]
where,

\[ A_1 = \frac{D_{x\phi}}{D_x}, \quad A_2 = \frac{D_v + D_{x\phi}}{D_x}, \quad A_3 = \frac{D_v}{D_x}, \quad A_4 = \frac{K_{x\phi}}{K_x}, \]

\[ A_5 = \frac{D_v + D_{x\phi}}{D_\phi}, \quad A_6 = \frac{D_{x\phi}}{D_\phi}, \quad A_7 = \frac{D_x K_{x\phi}}{D_\phi K_x}, \quad A_8 = \frac{D_x (K_v + 3K_{x\phi})}{D_\phi K_v}, \]

\[ A_9 = \frac{D_x}{D_\phi}, \quad A_{10} = \frac{D_\phi}{D_x}, \quad A_{11} = \frac{D_x (K_v + 2K_{x\phi})}{D_\phi K_x}, \quad A_{12} = \frac{D_x K_\phi}{D_\phi K_x}, \]

\[ k_1 = \frac{K_x}{a^2 D_x}, \quad q_1 = \frac{pa}{D_x}, \quad q_2 = \frac{P}{D_x}, \quad q_3 = \frac{T}{D_x} \]

(5.1.8)

Equations (5.1.7) describe the buckling of a circular cylindrical shell under the most general homogeneous stress action in the anisotropic case. It is easy to observe that the parameters \( k_1, q_1, q_2 \) and \( q_3 \) are small quantities. For \( k_1 \) it is obvious, since we are interested in thin shells where \( t \ll a \). The three load parameters \( q_1, q_2 \) and \( q_3 \) are approximately the elastic strains, in the limiting case, caused by the corresponding basic loads. Since all our theory is based on the assumption that such strains are small as compared with unity, we shall neglect the squares and higher order terms of \( q_1, q_2 \) and \( q_3 \) whenever possible.

**SOLUTION FOR SHELL UNDER TWO WAY COMPRESSIONS (WITHOUT SHEAR LOAD)**

We consider that the shell is under two way compressions and there is no shear load \((T = 0, \text{ hence } q_3 = 0)\). Therefore from equations (5.1.7) the
differential equations for the buckling of a \((2n+1)\)-layers plywood shell for
two way compressions are obtained as,

\[
\begin{align*}
    u'' + A_1u' + A_2v' + A_3w' + k_1 \{ A_4(u'' + w''') - w'''' \} \\
    - q_1(u'' - w') - q_2u = 0
\end{align*}
\]

\[
\begin{align*}
    A_3u'' + v'' + A_6v'' + w' + k_1[3A_5v'' - A_8w'''] \\
    - A_9[q_1(v'' + w') + q_2v'''] = 0 \tag{5.1.9}
\end{align*}
\]

\[
\begin{align*}
    A_{10}u' + v' + w + k_1[A_7u''' - A_8u'' - A_9v''' + A_9w'''] + 2A_9w'''
    + A_{12}[w'''' + 2w''' + w] \\
    + A_9[q_1(u' - v' + w'') + q_2w'''] = 0
\end{align*}
\]

The equations (5.1.9) admit a solution of the form (vide, Flugge\(^{18}\)(1973)),

\[
\begin{align*}
    u &= ACos(m\phi)Cos(\frac{\lambda x}{a}) \\
    v &= BSin(m\phi)Sin(\frac{\lambda x}{a}) \\
    w &= CCos(m\phi)Sin(\frac{\lambda x}{a}) \tag{5.1.10}
\end{align*}
\]

where,

\[
\lambda = \frac{s\pi a}{l} \tag{5.1.11}
\]

\(l\) is the length of the shell and \(s\) is an integer.

The solution (5.1.10) describes a buckling mode with \(s\) half waves
along the length of the cylinder and \(2m\) half waves around its circumference.

Although this is far from being the most general solution, it is the one which
fulfills reasonable boundary conditions.
It is evident that the solution (5.1.10) satisfies the boundary conditions \( v = w = 0 \) at \( x = 0 \) and \( x = l \) also \( N_x = M_x = 0 \) at \( x = 0 \) and \( x = l \).

This shows that the solution (5.1.10) represents the buckling of a shell whose edges are supported in tangential and radial directions, but are neither restricted in the axial direction nor clamped.

Substituting the solution (5.1.10) into the differential equations (5.1.9), the trigonometric functions drop out entirely and we are left with the following equations:

\[
\begin{align*}
A[\lambda^2 + (A_i + k_1 A_4) m^2 - q_i m^2 - q_z \lambda^2] + B[-A_5 \lambda m] + C[-A_5 - k_1 (\lambda^3 - A_5 \lambda m^2) - q_i \lambda] &= 0 \\
A[-A_5 \lambda m] + B[m^2 + (A_5 + 3k_1 A_4) \lambda^2 - q_1 A_9 m^2 - q_2 A_9 \lambda^2] + C[m + k_1 A_9 \lambda^2 m - q_i A_9 \lambda] &= 0 \\
A[-A_9 \lambda - k_1 (A_9 \lambda^2 - A_5 \lambda m^2) - q_1 A_9 \lambda] + B[m + k_1 A_9 \lambda^2 m - q_i A_9 \lambda] + C[1 + k_1 \{A_9 \lambda^4 + 2A_1 \lambda^2 m^2 + A_1 (m^2 - 1)^2\} - A_9 (q_1 m^2 + q_2 \lambda^2)] &= 0
\end{align*}
\]

(5.1.12)

The equations (5.1.12) are three linear equations with buckling amplitudes \( A, B, C \) as unknowns and with the brackets as coefficients. Since the equations are homogeneous, they admit, in general, only the solution \( A = B = C = 0 \), which shows that the shell is not in neutral equilibrium. The non-vanishing solution \( A, B, C \) is possible if and only if the determinant of the nine coefficients of the equations (5.1.12) is equal to zero. Thus the vanishing of this determinant is the buckling condition of the shell. Whenever the buckling condition is fulfilled, any two of the three equations (5.1.12) determine the
ratios $\frac{A}{C}$ and $\frac{B}{C}$ and thus the buckling mode according to the equation (5.1.10). As in all cases of neutral equilibrium, the magnitude of the possible deformation remains arbitrary.

The buckling condition contains four unknowns, the dimensionless loads $q_1$ and $q_2$ and the modal parameters $m$ and $\lambda$. Also we know that $m$ must be an integer (0, 1, 2, 3, 4,...) and $\lambda$ must be an integer multiple of $\pi n/l (s = 1, 2, 3, 4...).$ Thus we can write the buckling condition separately for every pair $m$, $\lambda$ fulfilling these requirements, and consider it as a relation between $q_1$ and $q_2$, which describes those conditions of the two loads for which the shell is in neutral equilibrium.

The coefficients of the equations (5.1.12) are linear functions of $k_1$, $q_1$ and $q_2$. The expanded determinant is, therefore, a polynomial of the third degree in these parameters. Since they are very small quantities it is sufficient to keep only the linear terms and to write the buckling condition in the following form:

$$C_1 + C_2 k_1 = C_3 q_1 + C_4 q_2$$

(5.1.13)

The equation (5.1.13) describes a straight line in the $q_1 - q_2$ plane and the limit of the stable domain is a polygon consisting of the sections of straight lines for various pairs of $m$, $\lambda$.

The coefficients $C_1, C_2, C_3$ and $C_4$ of the equation (5.1.13) can be found by expanding the determinant and putting it equal to zero. Thus we have,
\[ C_1 = A_6(1 - A_3 A_{10}) \lambda^4, \]

\[ C_2 = [A_6 \lambda^4 + 2 A_{11} \lambda^2 m^2 + A_{12} m^4][A_6 \lambda^4 + 2 A_{13} \lambda^2 m^2 + A_{14} m^4] - A_6 (A_3 A_6 + A_{10}) \lambda^6 - 2 \lambda^4 m^2 [A_6 + A_{10} - A_5 - A_6 (A_2 A_6 + A_4 A_6)] - \lambda^2 m^2 [2 A_4 A_8 + 4 A_{12} A_{13} + A_4 (A_5 + A_6 - A_{10})] - 2 A_4 A_{12} m^6 + [3 A_4 A_7 + A_4 A_6 + 2 A_{12} A_{13}] \lambda^2 m^2 + A_4 A_{12} m^4, \]

\[ C_3 = m^2 [A_6 \{A_1 m^4 + A_6 \lambda^4 + (1 + A_1 A_6 - A_3^2) \lambda^2 m^2\}] + \lambda^2 m^2 [2 (A_4 + A_{10}) + A_{10} (2 A_5 - A_{10}) + A_6 - A_9] - A_4 A_6 m^4, \]

\[ C_4 = \lambda^2 [A_6 \{A_2 \lambda^4 + A_4 m^4 + 2 A_{13} \lambda^2 m^2\} + A_4 m^2] \]

and \[ A_{13} = 1 + A_1 A_6 - A_2 A_3 \]

(5.1.14)

From the equations (5.1.13), (5.1.14) and (5.1.15) the stability curve may easily be constructed when \( l \) and \( k_1 \) are given.

**NUMERICAL RESULTS**

Using the formulae (5.1.13) and (5.1.14) the stability curve in case of two ways compression may easily is drawn here for fixed values of parameters. We consider the shell to be made of the same kind of material as that of Gaboon (Okoume), so that,

\[ E_1 = 1.28 \times 10^6 \text{ psi}, \quad E_2 = 0.11 \times 10^6 \text{ psi}, \]

\[ E_v = 0.014 \times 10^6 \text{ psi}, \quad G = 0.085 \times \text{psi}. \]

vide, Timoshenko and Woinowsky-Krieger\(^{[54]}\)(1983) and \( k_1 = 10^{-6} \).

The stable and unstable region for 5-layers circular cylindrical plywood shell are drawn and shown in figure 5.1.3 and figure 5.1.4 for different values of \( \lambda \).
Figure 5.1.3: Buckling diagram for a 5-layers cylindrical shell for $\lambda=20$. 
Figure 5.1.4: Buckling diagram of a 5-layers cylindrical shell for $\lambda = 30$. 
DISCUSSION AND CONCLUSIONS

Extensional rigidities, shear rigidity, bending rigidities and twisting rigidity for plywood shell consisting of any odd number of layers are obtained here. The differential equations of equilibrium for a \((2n+1)\)-layers \((n\) is a whole number) circular cylindrical plywood shell under three simultaneous forces, namely uniform normal pressure, axial compression and shear load are obtained here. Hence the differential equations of equilibrium for a circular cylindrical shell under two way compressions are solved and the buckling condition of the shell is determined. Taking some fixed values of parameters, and using the buckling condition buckling diagrams for 5-layers plywood shell are drawn and shown here.

From the Figure 5.1.3 and figure 5.1.4 we observe that, when a load is applied, the corresponding diagram point moves along some path, as shown by the bold line. As long as it does not touch any of the curves, the shell is in stable equilibrium. But as soon as one of the curves is reached, equilibrium becomes neutral. The region beyond that bold line is unstable region. Therefore the stable domain in the \(q_1 - q_2\) plane is bounded by the envelope of all the curves which is shown in figure 5.1.3 and figure 5.1.4. The stable and unstable domain for \(\lambda = 20\) and \(\lambda = 30\) are shown for 5-layers plywood shell. Using equations (5.1.13) and (5.1.14) buckling diagram of a circular cylindrical shell with any number of layers can be drawn and hence the stable region can be obtained.
SECTION 5.2

BUCKLING OF CIRCULAR CYLINDRICAL LONG PLYWOOD SHELL DUE TO SHEAR LOAD*

INTRODUCTION

The last section is the continuation of the previous section. Cheng and Kuenzi[11](1963) analysed the buckling problem of plywood cylindrical shells under external radial pressure. The effect of heterogeneity on stability of composite cylindrical shell under axial compression was discussed by Tasi[5](1966). Singer and Fersh-Scher[47](1964) investigated the buckling problem of orthotropic conical shell under external pressure. De[15](1983) solved buckling problem of 3-layers plywood shell under shear load. Li and Zhou[33](2001) used shooting method to analyse the buckling of heated orthotropic circular plate.

This section deals with the problem of buckling of a long plywood shell under shear load. The object of this section is to formulate the differential equations of buckling of \((2n + 1)\)-layers plywood shell under shear load. The solution of differential equations of the buckling problem for plywood shell with shear load is obtained here. The values of critical shear load and critical torque are obtained in the direction of the grain as well as in the cross

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grain direction. Critical shear load for different layered plywood shell are calculated numerically and shown in a diagram comparing with each other.

FORMULATION AND SOLUTION OF THE PROBLEM

The differential equations for the buckling problem of a $(2n+1)$-layers plywood shell under three different loads are given by,

\[ u'' + A_1 u'' + A_2 v'' + A_3 w' + k_1 \{ A_4 (u'' + w''') - w'''} \]

\[- q_1 (u'' - w') - q_2 u - 2q_3 u = 0 \]

\[ A_5 u'' + A_6 v'' + w' + k_1 [3A_7 v'' - A_8 w'''] \]

\[- A_9 [q_1 (v'' + w') + q_2 v'' + 2q_3 (v'' + w')] = 0 \]

\[ A_{10} u' + v' + w + k_1 [A_{11} u''' - A_9 u'''' - A_8 w'''' + A_7 w'''' + 2A_{12} w'''''] \]

\[ + A_{12} (w'''' + 2w'' + w)] \]

\[ + A_9 [q_1 (u' - v' + w'') + q_2 w'' - 2q_3 (v' + w')] = 0 \]

(5.2.1)

where,

\[ A_1 = \frac{D_x \varphi}{D_x}, \quad A_2 = \frac{D_x + D_x \varphi}{D_x}, \quad A_3 = \frac{D_x}{D_x}, \quad A_4 = \frac{K_x \varphi}{K_x}, \]

\[ A_5 = \frac{D_x + D_x \varphi}{D_x}, \quad A_6 = \frac{D_x K_x \varphi}{D_x K_x}, \quad A_7 = \frac{D_x (K_v + 3K_x \varphi)}{D_x K_v}, \quad A_8 = \frac{D_x (K_v + 3K_x \varphi)}{D_x K_v}, \]

\[ A_9 = \frac{D_x}{D_x}, \quad A_{10} = \frac{D_x}{D_x}, \quad A_{11} = \frac{D_x (K_v + 2K_x \varphi)}{D_x K_x}, \quad A_{11} = \frac{D_x (K_v + 2K_x \varphi)}{D_x K_v}, \]

\[ k_1 = \frac{K_x \varphi}{a^2 D_x}, \quad q_1 = \frac{p a}{D_x}, \quad q_2 = \frac{P}{D_x}, \quad q_3 = \frac{T}{D_x} \]

(5.2.2)
Figure 5.2.1: Circular cylindrical Shell with shear load.
We consider the shell is subjected to the shear load only. Therefore $q_3 \neq 0$ but $q_1 = q_2 = 0$. Hence the differential equations for the buckling of a $(2n + 1)$-layers plywood shell under shear load are obtained as,

\[
\begin{align*}
    u'' + A_4 u''' + A_3 v''' + A_2 w' + k_1 \left\{ A_4 \left( u''' + w''\right) - w''' \right\} - 2q_3 u &= 0 \\
    A_5 u'' + v''' + A_6 v'' + w' + k_1 \left\{ 3A_7 v'' - A_8 w'' \right\} - 2A_9 q_3 (v' + w') &= 0 \\
    A_{10} u' + v + w + k_1 \left[ A_4 u''' - A_3 u'' - A_2 v'' + A_1 w''' \\
    + 2A_7 w'''' + A_8 \left( w'''' + 2w'' + w \right) \right] - 2A_9 q_3 (v' - w') &= 0
\end{align*}
\]

(5.2.3)

The solution which is applicable in this case is (vide, Flugge\cite{Flugge18}(1973)),

\[
\begin{align*}
    u &= A \sin(\frac{\lambda x}{a} + m\varphi), \\
    v &= B \sin(\frac{\lambda x}{a} + m\varphi), \\
    w &= C \cos(\frac{\lambda x}{a} + m\varphi)
\end{align*}
\]

(5.2.4)

where $\lambda = \frac{s\pi a}{l}$,

and $l$ is the length of the shell and $s$ is an integer.

Substituting the solution (5.2.4) in the equation (5.2.3) we get three homogeneous linear equations in $A, B, C$ as,
\[ A[\lambda^2 + (A_i + k_i A_j)m^2 - 2q_3 m \lambda] + B[-A_i \lambda m] + C[-A_j \lambda - k_i (\lambda^3 - A_i \lambda m^2)] = 0 \]

\[ A[A_i \lambda m] + B[m^2 + (A_i + 3k_i A_j) \lambda^2 - 2q_3 m \lambda] + C[m + k_i A_j \lambda^2 m - 2q_3 \lambda] = 0 \quad (5.2.5) \]

\[ A[A_{10} \lambda + k_i (A_q \lambda^3 - A_i \lambda m^2)] + B[m + k_i A_j \lambda^2 m - 2q_3 \lambda] + C[1 + k_i (A_q \lambda^4 + 2A_{11} \lambda^2 m^2 + A_{12} (m^2 - 1)^2) - 2q_3 m \lambda] = 0 \]

Non-vanishing solutions \( A, B, C \) are possible if and only if the determinant of the nine coefficients of the equations (5.2.5) is equal to zero. Thus the vanishing of this determinant is the buckling condition of the shell. The buckling condition contains three unknowns, the dimensionless parameter \( q_3 \) and the modal parameters \( m \) and \( \lambda \). Also we know that \( m \) must be an integer \( 0, 1, 2, 3 \ldots \), and \( \lambda \) must be an integer multiple of \( (\pi a)/l (s = 1, 2, 3, 4\ldots) \). \( q_3 \) is a small quantity. \( k_i \) is also a small quantity. So we can neglect the square and higher powers of \( q_3 \) and \( k_i \) whenever opportunity comes. Expanding the determinant of nine coefficients and equating the determinant to zero and neglecting small quantities of higher order we find the condition of neutral equilibrium as:

\[ C_1 + C_2 k_i = C_5 q_3 \quad (5.2.6) \]

where \( C_1, C_2 \) and \( C_5 \) are given by,

\[ C_1 = A_q (1 - A_3 A_{10}) \lambda^4, \]
\[
C_2 = [A_6 \lambda^4 + 2A_{11} \lambda^2 m^2 + A_{12} m^4] [A_6 \lambda^4 + 2A_{13} \lambda^2 m^2 + A_4 m^4]
\]
\[
- A_6 (A_3 A_9 + A_{10}) \lambda^6 - 2\lambda^4 m^2 [A_8 + A_{10} - A_3 (A_5 A_8 + A_6 A_7)]
\]
\[
- \lambda^4 m^2 [2A_i A_8 + 4A_{12} A_{13} + A_4 (A_5 + A_6 - A_{10})] - 2A_4 A_{12} m^6
\]
\[
+ [3A_4 A_7 + A_4 A_6 + 2A_{12} A_{13}] \lambda^2 m^2 + A_4 A_{12} m^4,
\]
\[
C_5 = 2\lambda m[(A_6 m^4 + A_6 \lambda^4 + 2A_{13} \lambda^2 m^2) - A_4 m^2 -(1 + A_4 A_{10} - 2A_4 A_{10} - A_6) \lambda^2]
\]

(5.2.7)

It is evident that neither \( \lambda \) nor \( m \) can be zero, because in both the cases \( C_5 = 0 \). It is also without interest to consider negative values of \( \lambda \) or \( m \). When both are negative nothing is changed in the equation (5.2.6) while if either \( \lambda \) or \( m \) is negative, the buckling mode (5.2.4) is altered so that the modal lines (Figure 5.2.1) become right-handed screws. One would expect that the buckling load \( T \) must be applied in the opposite sense, and this is exactly what happens.

The discussion of the buckling formula (5.2.6) is now restricted to positive values of \( \lambda \) and to integers \( m \). We may solve it for \( q_3 \), differentiating the expression with respect to \( \lambda \) and \( m \), and putting the first partial derivative equal to zero. This would yield two algebraic equations for \( \lambda \) and \( m \), and their solutions (or one of them) would lead to the smallest possible \( q_3 \). This procedure, however, is rather tiresome and may be avoided. By some trial computations we may find out that any \( m > 2 \) yields a higher buckling load than does \( m = 2 \) and that \( \lambda \) must be chosen rather small, \( \lambda << 1 \), to obtain low \( q_3 \).

For \( m = 2 \), the equation (5.2.6) yields,
\[ q_3 = \frac{A_6 (1 - A_5 A_{10}) \lambda^4 + k_1 [A_6 A_6 \lambda^8 + A_6 \lambda^6 + A_{12} \lambda^4 + A_{12} \lambda^2 + 144 A_4 A_{12}]}{4 \lambda [12 A_4 + A_{12} \lambda^2 + A_6 \lambda^4]} \]  

(5.2.8)

where

\[ A_{14} = 2(4 A_6 A_{11} + A_4 A_{10}), \]

\[ A_{15} = 16 A_4 A_5 + 17 A_6 A_{12} + 64 A_{13} A_{11} + 3 A_7 - 8(A_8 + A_4 A_{10} - A_5) \]

\[ + 8 A_3 (A_5 A_8 + A_6 A_7) + 128 A_4 A_{12} [2 A_4 A_8 + 4 A_{12} A_{13}] \]

\[ + A_4 (A_5 + A_6 - A_{10})], \]

\[ A_{16} = 128(A_4 A_{11} + A_{12} A_{13}) - 4(3 A_4 A_7 + A_4 A_6 + A_{12} A_{13}), \]

\[ A_{17} = 8 A_3 - 1 + A_{10} (2 A_3 - A_8) + A_8 \]  

(5.2.9)

Neglecting \( \lambda^2 \) as compared to unity, we have

\[ q_3 = \frac{A_6 (1 - A_5 A_{10}) \lambda^3 + 3 A_{12}}{48 A_4} k_1 \]  

(5.2.10)

Now it is easy to find from \( \frac{\partial q_3}{\partial \lambda} = \frac{A_6 (1 - A_5 A_{10})}{16 A_4} \lambda^2 - \frac{3 A_{12}}{\lambda^2} k_1 = 0 \)

that \( \lambda^4 = \frac{48 A_4 A_{12}}{A_6 (1 - A_5 A_{10})} k_1 \)  

(5.2.11)

which yields the lowest possible value of \( q_3 \),

\[ q_3 \bigg|_{\text{min}} = 2 \left[ \frac{A_6 (1 - A_5 A_{10}) A_{12}^3}{3 A_4} \right] \frac{K_{x/2}^{3/2}}{a^{3/2}} D_x^{3/2} \]  

(5.2.12)

Using the last of the equations (5.2.2) we may now return to the real shear load \( T \) and find the critical value,

\[ T_{cr} = 2 \left[ \frac{A_6 (1 - A_5 A_{10}) A_{12}^3}{3 A_4} \right] \frac{K_{x/2}^{3/2}}{a^{3/2}} D_x^{3/2} \]  

(5.2.13)
The total torque applied to the tube is given by

\[ M = 2\pi a^2 T \quad (5.2.14) \]

The critical value for this torque is,

\[ M_{cr} = 4\pi \left( \frac{A_6 (1 - A_2 A_{10}) A_{12}^3}{3A_1} \right)^{\frac{1}{2}} a^2 \quad (5.2.15) \]

All these results have been derived for an infinitely long cylinder of a (2n + 1)-layers plywood shell. Since they do not contain any wavelength, we are tempted to apply them to cylinders of finite length.

**NUMERICAL RESULTS**

We consider the shell to be made of the same kind of material as that of Gaboon (Okoume), so that,

\[ E_1 = 1.28 \times 10^6 \text{ psi}, \]

\[ E_2 = 0.11 \times 10^6 \text{ psi}, \]

\[ E_v = 0.014 \times 10^6 \text{ psi}, \]

\[ G = 0.085 \times \text{psi.} \]

vide, Timoshenko and Woinowsky-Krieger [54](1983).

The critical load in the direction of grain and cross-grain direction are calculated and shown them graphically as a comparative study.
Figure 5.2.2: The comparison of critical load in the direction of the grain and in the cross grain direction for different layers.
DISCUSSION AND CONCLUSIONS

Here we have computed the expressions for critical values of shear load and the critical values of torque along the direction of the grain of the middle layer. Replacing $E_1$ by $E_2$ and $E_2$ by $E_1$ and by following the same procedure we will get the expressions for critical shear load and critical torque in the cross grain direction. Here we have also shown the algorithm for finding the critical shear load and critical torque for any number of layers and for the layer of any width. Applying this, we can find the critical shear load and torque in both directions and can find values of required configuration of plywood, according to the need. To show its applications, we have computed critical shear loads for shells having same width but different number of layers. From the figure 5.2.2 we observe that critical shear load in both the directions increases with increasing number of layers. Also we notice that for higher number of layers the values of critical shear load in both the directions are almost equal. Another important thing we notice is that after a certain number of layers, the value for critical shear load does not increase so much. So to get harder plywood, we don’t need to increase the number so many. Just increasing it up to certain number we will get our plywood of required strength.