CHAPTER 3

PROBLEMS ON THERMAL STRESSES
SECTION 3.1

THERMAL STRESS IN A LONG IN-HOMOGENEOUS CIRCULAR CYLINDRICAL SHELL WITH EXPONENTIALLY VARYING MATERIAL PROPERTIES*

INTRODUCTION

Thermal stresses in cylindrical shells subject to internal heat generation are discussed by different authors. Mollah\[36\](1989) obtained the thermal stresses of an in-homogeneous aelotropc cylinder subject to \(\gamma\)-ray heating. Tarn\[50\](2001) obtained exact solution of an anisotropic cylinder subject to thermal and mechanical loads whose conductivity co-efficients vary as the \(n^{th}\) power of the radial distance. Sutradhar, Paulino and Gray\[49\](2002) discussed transient heat conduction in non-homogenous materials with exponentially varying thermal conductivity and specific heat. Shao\[45\](2005) analysed mechanical and thermal stresses of a functionally graded circular hollow cylinder with finite length. Ghosh and Kanoria\[19\](2008) determined thermo-elastic stresses in a spherical body considering power law dependence of material property. Kaczynski and Monastyryskyy\[29\](2009) discussed the thermo elastic problem of uniform heat flow disturbed by a circular rigid lamellate inclusion. Cao, Qin and Zhao\[6\](2012) investigated the transient heat conduction in functionally graded materials with variable thermal conductivity and specific heat.

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The object of this section is to obtain theoretically the thermal stresses of a long inhomogeneous circular cylindrical shell whose modulus of elasticity, co-efficient of thermal expansion and thermal conductivity all vary exponentially as the radial distance. The rate of heat generation is also considered to vary exponentially. First the conductivity equation for this shell is obtained and solved to find the temperature \( T \). Then the equation of equilibrium is obtained. The exact solution of this equilibrium equation is determined. The hoop stresses for this shell are calculated numerically and shown graphically.

**FORMULATION AND SOLUTION OF THE PROBLEM**

A non-homogeneous circular cylindrical shell with inner radius \( a \) and outer radius \( b \) is considered. We used the cylindrical co-ordinates and take the \( z \) axis coinciding with the axis of the cylindrical shell. The conductivity equation for a circular cylindrical shell is given by (vide, Nowinski\(^{40}\)(1959))

\[
K \left( \frac{d^2T}{dr^2} + \frac{1}{r} \frac{dT}{dr} \right) + \frac{dK}{dr} \frac{dT}{dr} = H
\]

\[(3.1.1)\]

where, \( H \) denotes the rate of heat generation and \( K \) denotes the thermal conductivity of the material and \( T \) is the temperature. The rate of heat generation and thermal conductivity both varies exponentially as,

\[
H = H_0 e^{m(1-r/b)}
\]

\[(3.1.2)\]

\[
K = K_0 e^{n(1-r/b)}
\]

\[(3.1.3)\]

where, \( K_0, H_0 \) are values of \( K \) and \( H \) at the outer radius \( r = b \), and \( m, n \) are real numbers.
Considering \( R = r/b \), and using equations (3.1.2), (3.1.3) in the equation (3.1.1) we obtain the simplified conductivity equation as,

\[
R \frac{d^2 T}{dR^2} + (1 - nR) \frac{dT}{dR} = k_1 \text{Re}^{-\mu R}
\]  
(3.1.4)

where, \( k_1 = \frac{b^2 H_0}{K_0} e^\mu \), \( \mu = m - n \).

To find the complementary function of the equation (3.1.4), we consider,

\[
R \frac{d^2 T}{dR^2} + (1 - nR) \frac{dT}{dR} = 0
\]

Taking \( \frac{dT}{dR} = \psi \), the above equation transforms in to,

\[
R \frac{d\psi}{dR} + (1 - nR)\psi = 0
\]  
(3.1.5)

Solving the above equation we obtain,

\[
\psi = \frac{A_1}{R} e^{-nR}
\]  
(3.1.6)

where, \( A_1 \) is an arbitrary constant.

Substituting \( \frac{dT}{dR} = \psi \) in the equation (3.1.6) and solving, we obtain the complementary function of the equation (3.1.4) as,

\[
C.F. = A_0 + A_1 T_1(R)
\]  
(3.1.7)

where, \( T_1(R) \) is Exponential integral function. For real \( R \) it has a series expansion of the form,

\[
T_1(R) = \int \frac{e^{nR}}{R} dR = \log(nR) + \frac{n}{2} R + \frac{n^2}{8} R^2 + \frac{n^3}{36} R^3 + \frac{n^4}{192} R^4 + \ldots
\]

and \( A_0 \) is an arbitrary constant.

The particular integral of the equation (3.1.4) is obtained as,
where,
\[ A_2 = \frac{1}{4} k_i e_i, \quad A_3 = \frac{1}{9} k_i \left( \frac{3n}{2} - m \right), \quad A_4 = \frac{1}{96} k_i (6n^2 - 7mn + 3m^2) \]

Hence the temperature of the cylindrical shell is found to be,
\[ T = A_0 + A_1 T_1 (R) + A_2 R^2 + A_3 R^3 + A_4 R^4 \]  
(3.1.8)

Using boundary conditions (vide, Mollah\[^{[36]}\](1989)),
\[ T = T_0 \text{ on } r = a \text{ which implies, } T = T_0 \text{ on } R = \frac{a}{b} \text{ and} \]
\[ \frac{dT}{dr} = 0 \text{ on } r = b \text{ which implies, } \frac{dT}{dR} = 0 \text{ on } R = 1 \]

arbitrary constants \(A_0\) and \(A_1\) are determined.

The stress-strain relations for transversely isotopic materials in the presence of temperature are given by (vide, Sharma\[^{[46]}\](1957)).

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{zz} \\
\sigma_{rz}
\end{bmatrix} =
\begin{bmatrix}
E_{11} & E_{12} & E_{13} & 0 \\
E_{12} & E_{11} & E_{13} & 0 \\
E_{13} & E_{13} & E_{33} & 0 \\
0 & 0 & 0 & E_{44}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta\theta} \\
\varepsilon_{zz} \\
\varepsilon_{rz}
\end{bmatrix}
\]

(3.1.9)

where, \(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}\) are stress components, \(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}\) are strain components, \(E_{ij}\)'s (\(i=1,2,3,4\) and \(j=1,2,3,4\)) are modulus of elasticity and \(\alpha_1, \alpha_2\) are co-efficients of thermal expansion. Considering the axi-symmetric character of the problem, the strain components are given by,

\[
\varepsilon_{rr} = \frac{\partial u}{\partial r},
\]

\[
\varepsilon_{\theta\theta} = \frac{u}{r},
\]
\[ \varepsilon_{zz} = \frac{\partial w}{\partial z}, \]
\[ \varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}. \]

Assuming \( u \) to be dependent on \( r \) alone and \( w = 0 \), the above components reduce to:
\[ \varepsilon_{rr} = \frac{du}{dr}, \]
\[ \varepsilon_{\theta\theta} = \frac{u}{r}, \]

(3.1.10)
\[ \varepsilon_{zz} = 0, \]
\[ \varepsilon_{rz} = 0. \]

The stress equations of equilibrium are given by (vide, Timoshenko and Goodier\textsuperscript{[5]}(1955))
\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \]
\[ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial r} + \frac{\sigma_{rz}}{r} = 0 \]

(3.1.11)

From equations (3.1.9) and (3.1.10) it is observed that, the second equation of (3.1.11) automatically holds.

For this non-homogeneous circular cylindrical shell modulus of elasticity, coefficient of thermal expansion are considered to vary as follows,
\[ E_y = E_y' e^{\lambda_2 (1 - \gamma)}, \]
\[ \alpha_i = \alpha_i' e^{\lambda_2 (1 - \gamma)} \]

(3.1.12)

where, \( E_y', \alpha_i' \) are constants and \( \lambda_1, \lambda_2 \) are real numbers.
For exponential variation of modulus of elasticity, co-efficient of thermal expansion and considering $R = r/b$, after proper simplification the stresses are obtained as,

$$
\sigma_{rr} = \left\{ \frac{E_{11}}{b} \frac{du}{dR} + \frac{E_{12}}{b} \frac{u}{R} \right\} e^{\lambda_2 (1-R)}
$$

$$
\sigma_{\theta \theta} = \left\{ \frac{E'_{12}}{b} \frac{du}{dR} + \frac{E'_{11}}{b} \frac{u}{R} \right\} e^{\lambda_2 (1-R)}
$$

$$
\sigma_{zz} = \left\{ \frac{E'_{13}}{b} \frac{du}{dR} + \frac{E'_{12}}{b} \frac{u}{R} \right\} e^{\lambda_2 (1-R)}
$$

(3.1.13)

where, $b_1 = (E'_{11} + E'_{12}) \lambda_1 + E'_{13} \lambda_2$ and $b_2 = 2E'_{13} \lambda_1 + E'_{33} \lambda_2$

Substituting equation (3.1.13) in the first equation of (3.1.11), the equation of equilibrium of the non-homogeneous circular cylindrical shell is obtained as,

$$
R^2 \frac{d^2 u}{dR^2} + (R - \lambda_1 R^2) \frac{du}{dR} - (1 + \lambda_3 R)u = \gamma_1 e^{\lambda_2 (1-R)} R^2 [\gamma_2 A_0
$$

$$
+ \gamma_2 A_1 T_1(R) + \gamma_2 R^2 (A_2 + A_3 R + A_4 R^2) - \frac{A_1}{R} e^{\alpha R} - 2A_2 R - 3A_3 R^2
$$

$$
- 4A_4 R^3]
$$

(3.1.14)

where, $\gamma_1 = \frac{b_1 b}{c_{11}}$, $\gamma_2 = \lambda_1 + \lambda_2$, $\lambda_3 = \frac{E'_{12}}{E'_{11}} \lambda_1$.

To find the complementary function of the equation (3.1.14) we consider,

$$
R^2 \frac{d^2 u}{dR^2} + (R - \lambda_1 R^2) \frac{du}{dR} - (1 + \lambda_3 R)u = 0
$$

(3.1.15)

Changing the dependent variable $u$ by $u = Ry$, equation (3.1.15) is transformed to the following equation,

$$
R \frac{d^2 y}{dR^2} + (3 - \lambda_1 R) \frac{dy}{dR} - (\lambda_1 + \lambda_3) y = 0
$$

(3.1.16)

This is a generalised hypergeometric differential equation of the form,
\[ R \frac{d^2 y}{dz^2} + (a_i + b_i R) \frac{dy}{dR} + a_0 y = 0 \]

where,
\[ a_i = 3, \]
\[ b_i = -\lambda_i, \]
\[ a_0 = -(\lambda_i + \lambda_j). \]

Hence the solution of the equation (3.1.16) is found to be,
\[ y = C_1 U_1(R) + C_2 U_2(R) \quad (3.1.17) \]
where, \( C_1 \) and \( C_2 \) are arbitrary constants. \( U_1(R) \) is confluent hypergeometric function \( U(\mu_0,3,\lambda_i R) \) and \( U_2(R) \) is the generalised Laguerre polynomial \( L(-\mu_0,3,\lambda_i R) \) and \( \mu_0 = 1 + \frac{\lambda_3}{\lambda_1} \).

Hence the complementary function of the equation (3.1.14) is given by,
\[ u = C_1 R U_1(R) + C_2 R U_2(R) \]

The particular integral of the equation (3.1.14) is found to be,
\[ P.I. = (B_0 R + B_1 R^2 + B_2 R^3 + B_3 R^4 + B_4 R^5) \log[R] \]
\[ + (B_5 R^2 + B_6 R^3 + B_7 R^4 + B_8 R^5) \quad (3.1.18) \]

where,
\[ B_0 = A_i \frac{y_1 y_2 e^\lambda}{2}, \quad B_1 = \frac{1}{3} \{(\lambda_i + \lambda_j)B_0 + \psi_0\}, \quad B_2 = \frac{1}{8} \{(2\lambda_i + \lambda_j)B_1 - \lambda_2 \psi_0\}, \]
\[ B_3 = \frac{1}{15} \{(3\lambda_i + \lambda_j)B_2 + \lambda_i^2 \psi_0\}, \quad B_4 = \frac{1}{24} \{(4\lambda_i + \lambda_j)B_3 - \frac{\lambda_i^2}{6} \psi_0\}, \]
\[ B_5 = \{(\psi_1 - \lambda_2 \psi_2) + \lambda_i B_0 + (\lambda_i + \lambda_j)B_5 - 4B_1\} \frac{1}{7}, \]
The complete solution of the equation (3.1.14) is obtained as,

\[
\begin{align*}
B_6 &= \left( \psi_3 - \lambda_3 \psi_2 + \psi_1 \frac{\lambda_3^2}{2} - \psi_1 \frac{\lambda_2^3}{6} \right) + (2 \lambda_1 + \lambda_3) B_1 + (2 \lambda_1 + \lambda_2) B_5 - 6B_2 \frac{1}{8}, \\
B_7 &= \left( \psi_4 - \lambda_2 \psi_3 + \psi_2 \frac{\lambda_2^2}{2} - \psi_1 \frac{\lambda_2^3}{6} \right) + (3 \lambda_1 + \lambda_3) B_6 + \lambda_2 B_2 - 8B_3 \frac{1}{15}, \\
B_8 &= \frac{1}{15} \left( (3 \lambda_1 + \lambda_3) B_2 + \frac{\lambda_2^2}{2} \psi_0 \right), \\
\psi_0 &= A_1 \gamma_1 \gamma_2 e^{\kappa_1}, \\
\psi_1 &= -\{ A_0 \gamma_2 + A_1 \gamma_2 (\log n + +0.2886) - A_1 n \} \gamma_1 e^{\kappa_1}
\end{align*}
\]

Substituting the values of \(T\) and \(u\) from the equation (3.1.8) and (3.1.19) in the equation (3.1.12) the radial stress and the hoop stress are found to be,

\[
\sigma_r = \begin{pmatrix}
E'_{11} B_0 + \{ E'_{11} B_1 + (2E'_{11} + E'_{12}) B_5 \} R \\
+ \{ E'_{11} B_2 + (3E'_{11} + E'_{12}) B_6 \} R^2 \\
+ \{ E'_{11} B_3 + (AE'_{11} + E'_{12}) B_7 \} R^3 \\
+ E'_{11} B_4 + (5E'_{11} + E'_{12}) B_8 \} R^4 \\
+ C_1 U_1 (R) (E'_{11} + E'_{12}) + C_2 U_2 (R) (E'_{11} + E'_{12}) \\
- C_1 R U_3 (R) E'_{11} (\lambda_1 + \lambda_2) - C_2 \lambda_2 R E'_{11} U_4 (R) \\
- b b_1 \left( A_0 + A_1 T_1 (R) + A_2 R^2 + A_3 R^3 + A_4 R^4 + A_5 R^5 \right) e^{\kappa_2 (1-R)}
\end{pmatrix} \frac{e^{\kappa_2 (1-R)}}{b}
\]  

(3.1.20)
where, $U_3(R)$ is the confluent hypergeometric function $U(\mu_i, A_i, \lambda_i, R)$ and $U_4(R)$ is the generalized Laguerre polynomial $L(\mu_i, 3, \lambda_i, R)$ and

$$\mu_i = 2 + \frac{\lambda_i}{\lambda_i}.$$

To find the arbitrary constants we consider the following sets of boundary conditions.

$\sigma_\nu = 0$ at $r = a$ that implies $\sigma_\nu = 0$ at $R = a/b$ and

$\sigma_\nu = 0$ at $r = b$ that implies $\sigma_\nu = 0$ at $R = 1$.

Using above set of boundary conditions, arbitrary constants $C_1$ and $C_2$ are obtained.
NUMERICAL RESULTS

For the following set of values numerical results have been obtained. If we consider the material to be made of magnesium, then the elastic constants are given by,

\[
E'_{11} = 0.565 \times 10^{12} \text{ dyne/cm}^2,
\]
\[
E'_{12} = 0.232 \times 10^{12} \text{ dyne/cm}^2,
\]
\[
E'_{13} = 0.181 \times 10^{12} \text{ dyne/cm}^2,
\]
\[
E'_{33} = 0.587 \times 10^{12} \text{ dyne/cm}^2,
\]
\[
E'_{44} = 0.168 \times 10^{12} \text{ dyne/cm}^2,
\]

and the coefficients of thermal expansion of the said material are,

\[
\alpha'_1 = 27.7 \times 10^{-6} \text{ cm/s},
\]
\[
\alpha'_2 = 27.7 \times 10^{-6} \text{ cm/s}.
\]

Hence the variation of radial stress(\(\sigma_{rr}\)) and hoop stress(\(\sigma_{\theta\theta}\)) with radial distance are calculated numerically and shown graphically for \(T_0 = 1, \ m = 2, \ n = 1\).
Figure 3.1.1: Variation of radial stress with radial distance for $\lambda_1 = 1$ and different $\lambda_2$. 
Figure 3.1.2: Variation of hoop stress with radial distance for $\lambda_1 = 1$ and different $\lambda_2$. 
Figure 3.1.3: Variation of radial stress with radial distance for $\lambda_2 = 1$ and different $\lambda_1$. 
Figure 3.1.4: Variation of hoop stress with radial distance for $\lambda_2 = 1$ and different $\lambda_1$. 

**HOOP STRESS**
DISCUSSION AND CONCLUSION

Conductivity equation for a circular cylindrical shell with exponentially varying heat generation and thermal conductivity is obtained here. Solving this equation the temperature is obtained. The equation of equilibrium is obtained for the above mentioned shell with exponentially varying modulus of elasticity and co-efficient of thermal expansion and the solution is also obtained. Thermal stresses of that non-homogeneous shell are computed. For some fixed values of parameters exact value of radial stress and hoop stress are calculated numerically and shown graphically. Figure 3.1.1 and figure 3.1.3 show the variation of radial stress of the circular cylindrical shell with changing radial distance. These two figures show that radial stress gradually increases as radial distance increases and attain the maximum value and then gradually decreases. Also as the parameters $\lambda_1$ and $\lambda_2$ increases the radial stress increases gradually. Figure 3.1.2 and figure 3.1.4 show the variation of hoop stress with radial distance. Here we find that hoop stress of the shell gradually decreases as radial distance increases. Also with increasing $\lambda_1$ and $\lambda_2$ the hoop stress gradually increases.
SECTION 3.2

THERMOELASTIC STRESS ANALYSIS OF A ROTATING DISC WITH VARIABLE THICKNESS AND VARIABLE DENSITY*

INTRODUCTION

Theoretical investigation of stresses of rotating annular discs has received a lot of attention in recent time. Thermal stresses in infinite elastic disc was discussed by Sharma\textsuperscript{46}(1957). Then Jahed and Shirazi\textsuperscript{27}(2001) obtained the stresses, strains and displacements for shrink fitted rotating discs at elevated temperature. Warade and Deshmukh\textsuperscript{59}(2004) investigated thermal deflection of thin clamped circular plate due to a partially distributed heat supply. Vivio and Vullo\textsuperscript{56}(2007), studied the elastic stress of a rotating conical disc under thermal load and having variable density. The same authors\textsuperscript{57}(2008) also investigated the stresses and strains in a circular rotating disc subject to thermal loads with varying thickness and varying density. Ghosh and Kanoria\textsuperscript{19}(2008) determined the thermoelastic displacement and stress of a functionally graded spherically isotropic solid. Nie and Batra\textsuperscript{38}(2010) analysed the stress of a thermoelastic functionally graded rotating disc.

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Thermo-elastic problem of a rotating annular disc is considered here. The object of this section is to find theoretically the stresses of a composite annular rotating disc which is simultaneously subject to mechanical and thermal effect with exponentially varying coefficient of thermal expansion \( \alpha(r) \), thickness \( h(r) \), density \( \rho(r) \) and shear modulus \( G(r) \). The differential equation of equilibrium for rotating disc with above types of variation is determined. Exact solution of this equation is calculated. Hence deflection of the rotating circular disc is obtained. Radial stress and hoop stress of the circular disc are also obtained. Numerical values of stresses for different values of parameters are calculated for a plate with both edges free and shown them graphically.

FORMULATION AND SOLUTION OF THE PROBLEM

An annular rotating non-homogeneous disc with outer radius \( b \) and inner radius \( a \) is considered here. The compatibility equation of a rotating disc is given by (vide, You, Tang, Zhang and Zheng\[63\](2000))

\[
\frac{d}{dr}(r \varepsilon_{\theta \theta}) - \varepsilon_{rr} = 0
\]  
(3.2.1)

The elastic strains of a plate with variable coefficient of thermal expansion, thickness, density and shear modulus are given by (vide, Nie and Batra\[38\](2010)),

\[
\varepsilon_{rr} = \frac{2\sigma_{rr} - \sigma_{\theta \theta}}{6G(r)} + \alpha(r)T,
\quad
\varepsilon_{\theta \theta} = \frac{2\sigma_{\theta \theta} - \sigma_{rr}}{6G(r)} + \alpha(r)T
\]  
(3.2.2)

where, \( T \) is the temperature and the stresses are given by,

\[
\sigma_{rr} = \frac{1}{rh(r)} \varphi(r),
\quad
\sigma_{\theta \theta} = \frac{1}{h(r)} \frac{d \varphi(r)}{dr} + \rho(r) \omega^2 r^2
\]  
(3.2.3)
\( \varphi(r) = -\frac{dw}{dr} \) is the stress function and \( w \) is the deflection.

For this problem coefficients of thermal expansion, thickness, density and shear modulus are considered as follows.

\[
\begin{align*}
\alpha(r) &= \alpha_0 e^{\frac{r}{b}(\frac{r}{b}-1)}, & h(r) &= h_0 e^{-\frac{r}{b}(\frac{r}{b}-1)}, \\
\rho(r) &= \rho_0 e^{\frac{r}{b}(\frac{r}{b}-1)}, & G(r) &= G_0 e^{\frac{r}{b}(\frac{r}{b}-1)}
\end{align*}
\]  

(3.2.4)

\( r \) is the radial distance and \( b \) is the radius of the outer wall. \( G_0, h_0, \rho_0, \alpha_0 \) are real numbers. \( k, l, m \) and \( n \) are real constants.

Substituting equations (3.2.2), (3.2.3), (3.2.4) in the equation (3.2.1) and considering \( R = \frac{r}{b} \) the equation of equilibrium for a rotating annular disc after proper simplification is obtained as,

\[
R^2 \frac{d^2 w}{dR^2} + (1-k_1 R)R \frac{d^2 w}{dR^2} - (1-k_1 R) \frac{dw}{dR} = (\beta_1 R^4 + \beta_2 R^4 e^{\lambda_1 R} + \beta_3 R^2 e^{\lambda_2 R}
\]  

(3.2.5)

where,

\[
\begin{align*}
k_1 &= k - n, & \beta_1 &= -\omega^2 h_0 \rho_0 b^3 (k-m)e^{(n-m)}, & \beta_2 &= \frac{7}{2} \omega^2 h_0 \rho_0 b^3 e^{(n-m)}, \\
\beta_3 &= 3TG_0 h_0 \alpha_0 l b e^{(n-k-l)}, & \lambda_1 &= m-n & \text{and} & \lambda_2 &= k+l-n.
\end{align*}
\]

To find the complementary function of the equation (3.2.5) we consider,

\[
R^2 \frac{d^3 w}{dR^3} + (1-k_1 R)R \frac{d^2 w}{dR^2} - (1-k_1 R) \frac{dw}{dR} = 0
\]  

(3.2.6)

This is a generalised hypergeometric differential equation of the form,

\[
R^2 \frac{d^3 w}{dR^3} + (a_2 + b_2 R)R \frac{d^2 w}{dR^2} + (a_1 + b_1 R) \frac{dw}{dR} + a_0 w = 0
\]

where, \( a_0 = 0, a_1 = -1, b_1 = \frac{k_1}{2}, a_2 = 1, b_2 = -k_1, .. \)
Hence, the solution of the equation (3.2.6) is given by,

\[ C. \quad F_r = D_1 + \frac{2D_2}{12k_1^2} \left\{ 6 + e^{\frac{k_1R}{2}} (-6 + 3k_1R + 2k_1^2R^2)I_0\left(\frac{k_1}{2}R\right) \right\} \]

\[ -e^{\frac{k_1R}{2}} k_1R(5 + 2k_1R)I_1\left(\frac{k_1}{2}R\right) + 9k_1^2R^2 D_3 W(R) \]

where, \(D_1\), \(D_2\) and \(D_3\) are arbitrary constants.

\(I_0\left(\frac{k_1}{2}R\right)\) and \(I_1\left(\frac{k_1}{2}R\right)\) are the modified Bessel functions of the first kind \((I_n(z))\) and \(W(R)\) is the MeijerG function \(G(\{\{0.5\},\{-2\}\},\{\{-2,-2,0\},\{\}\},k_1R)\).

The particular integral of the equation (3.2.5) is obtained as,

\[ P.I. = \frac{R^2}{2} + D_4 \frac{R^3}{3} + D_5 \frac{R^4}{4} + D_6 \frac{R^5}{5} \]

where,

\[ D_4 = \frac{2\beta_1 + k_1^2}{6}, \quad D_5 = \frac{2\beta_2 + 2\lambda_1\beta_1 + 3k_1A_2}{16}, \]

\[ D_6 = \frac{2\beta_3 + 2\lambda_1\beta_2 + \lambda_2^2\beta_1 + 5k_1A_3}{30}. \]

Hence the deflection of the disc is found to be,

\[ w = D_1 + \frac{2D_2}{12k_1^2} \left\{ 6 + e^{\frac{k_1R}{2}} (-6 + 3k_1R + 2k_1^2R^2)I_0\left(\frac{k_1}{2}R\right) \right\} \]

\[ -e^{\frac{k_1R}{2}} k_1R(5 + 2k_1R)I_1\left(\frac{k_1}{2}R\right) \]

\[ + 9k_1^2R^2 D_3 W(R) + \frac{R^2}{2} + D_4 \frac{R^3}{3} + D_5 \frac{R^4}{4} + D_6 \frac{R^5}{5} \]

(3.2.9)

In the equation (3.2.3) using the relation \(\varphi(r) = -\frac{dw}{dr}\) and the equation (3.2.9), both the stresses of the annular plate are obtained as,
\[\begin{align*}
\sigma_{nn} &= \frac{e^{\eta(R_\text{w})}}{h_0 b^2} \frac{1}{24 k_i^2 R^2} \left\{ \begin{array}{l}
2 k_i^3 R^3 \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} + 5 k_i^2 R^2 \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} \\
D_2 e^{\frac{k_i R}{2}} \left\{ -k_i^2 R^2 (17 - 2 k_i R) \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} \\
+ 4 k_i R (8 + 5 k_i R) \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} \\
\end{array} \right\} \\
& - 24 D_3 U \left( -\frac{3}{2}, -1, k_i R \right) - 24 k_i^3 \left\{ R^2 + D_4 R^3 \\
& + D_5 R^4 + D_6 R^5 \right\}
\end{align*}\]

(3.2.10)

\[\begin{align*}
\sigma_{\phi\phi} &= \frac{e^{\eta(R_\text{w})}}{h_0 b^2} \frac{1}{96 k_i^2 R^2} \left\{ \begin{array}{l}
-(18 + 15 k_i R + 2 k_i^2 R^2) \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} \\
+ (144 + 11 k_i R - 2 k_i^2 R^2) \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} \\
D_2 e^{\frac{k_i R}{2}} \left\{ -k_i^2 R^2 \frac{1}{2} \left\{ \frac{k_i R}{2} \right\} + 52 I_2 \left( \frac{k_i R}{2} \right) + 46 k_i R I_2 \left( \frac{k_i R}{2} \right) \\
+ 4 k_i^2 R^2 I_2 \left( \frac{k_i R}{2} \right) + 5 k_i R I_3 \left( \frac{k_i R}{2} \right) \\
+ 2 k_i^2 R^2 I_3 \left( \frac{k_i R}{2} \right) + 96 U \left( -\frac{3}{2}, -1, k_i R \right) - 144 k_i R U \left( -\frac{1}{2}, 0, k_i R \right) \right\} \\
& - 24 k_i^3 \left\{ R^2 + D_4 R^3 + D_5 R^4 + D_6 R^5 \right\}
\end{array} \right\}
\end{align*}\]

(3.2.11)
where, \( U(a,b,x) \) is confluent hypergeometric function. For the convergence of the series we obtain \( k_i > 0 \). To find the arbitrary constants, we consider that, both the edges are free. Hence the boundary conditions are given by,

\[
\sigma_{rr} = 0 \quad \text{at} \quad r = a \quad \text{that implies,} \quad \sigma_{rr} = 0 \quad \text{at} \quad R = \frac{a}{b} \quad \text{and}
\]

\[
\sigma_{rr} = 0 \quad \text{at} \quad r = b \quad \text{that implies,} \quad \sigma_{rr} = 0 \quad \text{at} \quad R = 1. \tag{3.2.12}
\]

Using above set of boundary conditions arbitrary constants are determined.

**NUMERICAL RESULTS**

For an annular disc with inner radius \( a = 2 \) and outer radius \( b = 10 \),

radial stress and hoop stress \( \left( \sigma_{rr} = \frac{\sigma_{rr}}{\rho_0 \omega^2}, \sigma_{\theta \theta} = \frac{\sigma_{\theta \theta}}{\rho_0 \omega^2} \right) \) are calculated numerically for different set of values of \( k, l, m, n \). Variation of radial stress and hoop stress with radial distance and other parameters are shown graphically here.
Figure 3.2.1: Radial stress for the annular disc with $l=1$, $m=1$, $n=-1$.

Figure 3.2.2: Radial stress for the annular disc with $k=1$, $m=1$, $n=-1$. 
Figure 3.2.3: Radial stress for the annular disc with $l=1$, $k=1$, $n=-1$.

Figure 3.2.4: Radial stress for the annular disc with $l=1$, $k=1$, $m=1$. 

94
Figure 3.2.5: Hoop stress for the annular disc with $l=1$, $m=1$, $n=-1$.

Figure 3.2.6: Hoop stress for the annular disc with $k=1$, $m=1$, $n=-1$.
Figure 3.2.7: Hoop stress for the annular disc with $k=1$, $l=1$, $n=-1$.

Figure 3.2.8: Hoop stress for the annular disc with $k=1$, $l=1$, $m=1$. 
DISCUSSION AND CONCLUSION

Thermal stresses of a non-homogeneous rotating annular disc with exponentially varying thermal expansion, thickness, density and shear modulus are obtained. For a free annular disc arbitrary constants are determined. Considering some fixed values of parameters radial stress and the hoop stress of the disc are calculated numerically and shown graphically.

Figures 3.2.1, 3.2.2, 3.2.3, 3.2.4 show the variation of radial stress for different values of parameters with radial distance. Figures 3.2.5, 3.2.6, 3.2.7, 3.2.8 show the variation of hoop stress with radial distance.

From figures 3.2.1, 3.2.2, 3.2.3, 3.2.4 we observe that, the radial stress gradually increases with increasing radial distance and attains its maximum value and then gradually decreases. Figures 3.2.5, 3.2.6, 3.2.7, 3.2.8 show that, hoop stress of the annular disc gradually decreases with increasing radial distance. After attaining minimum value stress of the annular disc slightly increases.