CHAPTER-2
FRAMES AND MATRIX REPRESENTATION OF OPERATORS

2.1 INTRODUCTION

In this chapter we obtained analogous results of Duffin and Schaeffer [36] by using a matrix representation of operators on a Hilbert space with Bessel sequences, frames and Riesz bases.

From practical experience it has been noticed that the concept of an orthonormal basis is not always useful. Sometimes it is more useful for a decomposing set to have other special properties rather than guaranteeing unique coefficients. It is impossible to have good time frequency localization for Gabor orthonormal bases or a wavelet orthonormal bases with a mother wavelet which has exponentially decay and is infinitely often differentiable with bounded derivatives [13]. Also, suitable orthonormal bases are often difficult to construct in a numerical efficient way. This led to the concept of frames, which was introduced by Duffin and Schaeffer [36] in the context of non-harmonic Fourier series. A sequence \( \varphi = (\varphi_k; k \in K) \) is called a frame for the Hilbert space H, if constants A,B > 0 exist, such that

\[
A \| f \|_H^2 \leq \sum_k \left| \left< f , \varphi_k \right> \right|^2 \leq B \| f \|_H^2 \quad \forall \ f \in H. \tag{2.1.1}
\]

The two constants A and B are called lower and upper frame bounds. The frame is tight if A=B and is said to be exact if it ceases to be a frame by removing any of its elements.

Frames have many of the properties of bases, but lack very important one uniqueness. Frames need not be linearly independent. This turns out to be useful in image and signal processing applications, since the redundancy or transmission errors. As with the Riesz bases, perturbing a frame by a small amount also yield a frame. Chui and Shi’s over sampling theorems provide methods to generate wavelet frames. The theory of frames are discussed in variety of sources, including [47,30,29,5,36,77].

Models in Physics[18] and other applications areas, such as sound vibration analysis [18], are mostly continuous models. Many problems there can be formulated as operator theory problems one way to discretize the operators to work numerically
is to find (possible infinite) matrices describing these operators using orthonormal bases. In this chapter an attempt has been made to describe an operator as a matrix using frames and obtained analogous results of Duffin and Schaeffer [36].

The chapter is organized as follows: Section 2.1, incorporates the introductory exposition of the topic. In section 2.2, we give the basic definitions and properties of Bessel sequences and frames. Some results also have been collected which are used in proving the main results. In section 2.3, we prove our main results.

2.2 DEFINITIONS AND NOTATIONS

In the sequel we will denote infinite dimensional Hilbert spaces by H and their inner product with < ..., > which is linear in first coordinate. Let B \((H_1, H_2)\) denote the set of all linear and bounded operators from H_1 to H_2. With the operator norm, \(\|A\|_\text{op} = \sup_{\|x\|_{H_1} \leq 1} \{\|A(x)\|_{H_2}\}\), this set is a Banach space. An example for a Hilbert space is the sequence space \(\ell^2\) consisting of all square summable sequences in \(C\) with the inner product \(\langle c, d \rangle = \sum_k c_k d_k\).

If only the right hand inequality in (2.1) is satisfied for all \(f \in H\), than a sequence \(\{\varphi_k, k \in K\}\) is called a Bessel sequence with Bessel bound B. One of the properties that are important is the following:

\[\varphi_k\] is a Bessel sequence with bound B if and only if, for every finite sequence of scalars \((c_k)\):

\[
\left\| \sum_k c_k \varphi_k \right\|_H^2 \leq B \sum_k \|c_k\|_2^2
\]

(c.f.[5, p.155]). As remarked by Chui and Shi in [16], it is straightforward consequence of this statement that \((\varphi_k)\) is a Bessel sequence with bound if and only if (2.2.1) is satisfied for every sequence \((c_k \in \ell^2)\).

Two sequences \((a_k)\) and \((b_k)\) in \(H\) are called bi-orthogonal if \(\langle a_k, a_\ell \rangle = \delta_{k, \ell}\)

where \(\delta_{k, \ell}\) is the Kronecker delta.

For a Bessel sequence, \(\psi = (\varphi_k)\), let \(C_\varphi : H \to \ell^2 (K)\) be the analysis operator
C_φ(f) = \left( \left\langle f, \varphi_k \right\rangle \right)_κ . Let D_φ : ℓ^2(K) \to H be the synthesis operator D_φ((c_κ)) = \sum_κ c_κ \varphi_κ . Let S_φ : H \to H be the (associated) frame operator S_φ(f) = \sum_κ \left\langle f, \varphi_κ \right\rangle \varphi_κ .

We shall use the notation S_φ, φ = D_φ 0 C_φ . Cand D are adjoint to each other D = C^* with \| D \|_{op} = \| C \|_{op} ≤ \sqrt{B} . The series \sum_κ c_κ \cdot \varphi_κ converges unconditionally for all c_κ ∈ ℓ^2 .

For a frame \psi = (\varphi_κ) with bounds A, B, C is bounded, injective operator with closed range and S = C^*C = DD^* is a positive invertible operator satisfying \Al_H ≤ S ≤ BI_H and B^{-1}I_H ≤ S^{-1} ≤ A^{-1}I_H. The Sequence \psi = (\tilde{φ}_κ) = (S^{-1}) \varphi_κ is a frame with frame bounds B^{-1}, A^{-1} > 0, the so called canonical dual frame. Every \( f \in H \) has the expansion and \( f = \sum_{κ \in K} \left\langle f, \varphi_κ \right\rangle \varphi_κ \) and \( f = \sum_{κ \in K} \left\langle f, \varphi_κ \right\rangle \tilde{φ}_κ \) where both sums converge unconditionally in H.

A complete sequence (\varphi_κ) in H is called a Riesz basis if there exist constants A, B > 0 such that the inequalities

\[ A \left\| c \right\|_2^2 ≤ \left\| \sum_{κ \in K} c_κ \varphi_κ \right\|_H^2 ≤ B \left\| c \right\|_2^2 \]

holds for all finite sequences c_κ .

A number of important properties of frames are given here(see[2]).

**Theorem 2.2.1.** Let \( \varphi_κ \) be a frame for H with frame bounds A and B

(a) For each sequence \( c_κ ∈ ℓ^2 \) such that \( f = \sum_κ c_κ \varphi_κ \) converges in H and \( \left\| f \right\|_2^2 ≤ B \left\| c \right\|_2^2 \).

(b) Let \( \varphi_κ \) be a frame and let v be any vector, then there exists a moment sequence \( y_κ \) such that
\[ v = \sum_{\kappa} \varphi_\kappa \cdot y_\kappa \]

and

\[ A \| v \|^2 \leq \sum_{\kappa} |y_\kappa|^2 \leq B \| v \|^2. \]

A basis of \( \varphi_\kappa \) is called unconditional if there exist a \( C > 1 \) such that for any two finite sequences of scalars \( (a_\kappa, 1 \leq \kappa \leq n) \) and \( (b_\kappa, 1 \leq \kappa \leq n) \) if \( |a_\kappa| < |b_\kappa|, 1 \leq \kappa \leq n \), then

\[ \left\| \sum_{\kappa=1}^{n} a_\kappa \varphi_\kappa \right\|_{\mathcal{H}} \leq C \left\| \sum_{\kappa=1}^{n} b_\kappa \varphi_\kappa \right\|. \]

**Theorem 2.2.** A sequence \( \varphi_\kappa \) in a Hilbert space \( \mathcal{H} \) is an exact frame for \( \mathcal{H} \) if and only if it is bounded unconditional basis for \( \mathcal{H} \).

Let \( \psi = (\varphi_\kappa) \) and \( \phi = (\phi_\kappa) \) be two sequences in \( \mathcal{H} \). The Gram matrix \( G_{\varphi, \phi} \) for these sequences is given by \( (G_{\varphi, \phi})_{j, m} = \left\langle \phi_m, \varphi_j \right\rangle \), \( j, m \in \mathbb{K} \). We denote \( G_{\varphi, \phi} \) by \( G_{\varphi} \). The operator induced by the Gram matrix for \( c \in \ell^2 \) is defined as \( (G_{\varphi, \phi} c)_j = \sum_{\kappa} c_\kappa \left\langle \phi_\kappa, \varphi_j \right\rangle \). Clearly for two Bessel sequences it is well defined as linear bounded operator, because

\[ (G_{\varphi, \phi} c)_j = \sum_{\kappa} c_\kappa \left\langle \phi_\kappa, \varphi_j \right\rangle = \left\langle \sum_{\kappa} c_\kappa \phi_\kappa, \varphi_j \right\rangle = ((C_{\phi} 0 D_{\phi}) c)_j \]

and therefore \( \| G_{\varphi, \phi} \|_{op} \leq \| C_{\phi} \|_{op} \| D_{\phi} \|_{op} \leq B \). A frame is a Riesz sequence if and only if the Gram matrix defines a bounded and invertible operator on \( \ell^2 \).

For orthonormal sequence it is well known, that operators can be uniquely described by matrix representation. The same can be constructed with frames and their duals. In view of the definition of the operator defined by a (possibly infinite) matrix : \( (M c)_j = \sum_{\kappa} M_{j, \kappa} c_\kappa \). We will discuss the general case of Bessel sequences and use the notation \( \| . \|_{\mathcal{H}_1 \to \mathcal{H}_2} \) for the operator norm in \( \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \).
Definition 2.2.1. (See[62]). A necessary and sufficient condition for a matrix \( M = (M_{i,k}) \) is said to be regular if and only if

1. \( \sup_i \sum_k |M_{i,k}| < \infty \)
2. \( \lim_{i \to \infty} M_{i,k} = 0 \)
3. \( \lim_{i \to \infty} \sum M_{i,k} = 1 \)

Definition 2.2.2. A M-frame for an infinite non-negative regular matrix \( M = (M_{i,k}) \) is an infinite sequence \( (M c)_j = \sum_k M_{i,k} c_k \) such that for every \( f \in H \).

\[
A \|f\|_H^2 \leq \sum_j \left| \langle f, (M c)_j \rangle \right|^2 \leq B \|f\|_H^2
\]

(2.2.2)

The numbers \( \beta_j = \langle f, (M c)_j \rangle \) are called M-moment sequences of \( f \in H \) relative to the frame.

Definition 2.2.3. For an operator \( O : H_1 \to H_2 \) and a matrix \( M \) defined above, we call \( M^{\phi, \psi} (O) \) the matrix induced by the operator \( O \) with respect to the Bessel sequences \( \psi = (\psi_k) \) and \( \Phi = (\phi_k) \) and \( O^{\phi, \psi} (M) \) the operator induced by the matrix \( M \) with respect to the Bessel sequences \( \psi \) and \( \Phi \).

2.3 MAIN RESULTS

Let \( \psi = (\psi_k) \) and \( \Phi = (\phi_k) \) be frames in \( H_1 \) and \( H_2 \) respectively. Then we define

1. \( (M^{\phi, \psi} (O) (c))_j = (G_{\psi, \phi} c)_j \)
2. \( O^{\phi, \psi} (M (c) (f)) = S_{\phi, \psi} (f) \)

\[
S_{\phi, \psi} (f) = \sum_k \left( \sum_j M_{k,j} \langle f, \phi_j \rangle \right) \tilde{\phi}_k \text{ for } f \in H_1.
\]

Since \( M \) is an infinite matrix and we have \( O^{\phi, \psi} (M) = D_{\phi} o M o C_{\psi} \) it gives

\[
\| O^{\phi, \psi} (M) \|_{H_1 \to H_2} \leq \| D_{\phi} \|_{l^2 \to H_2} \cdot \| M \|_{l^2 \to l^2} \cdot \| C_{\psi} \|_{H_1 \to l^2}
\]
\[ \leq \sqrt{BB'} \| M \| _{l^2}^2 \rightarrow l^2 \]  

Here \(S_{\varphi, \tilde{\varphi}}\) is a bounded operator from \(H_1 \to H_2\). The operator is self adjoint, and therefore

\[ \langle S_{\varphi, \tilde{\varphi}} f, f \rangle \leq \sqrt{BB'} \| M \| _{l^2}^2 \| f \| _{H_1}^2. \]

**Theorem 2.3.1.** If \((G_{\varphi, \varphi} c)_j\) is M-frame and for any \(f \in H_1\). There exists a moment sequence \((\beta_j)\) such that

\[ \beta_j = \langle f, (G_{\varphi, \varphi} c)_j \rangle \]

\[ f = \sum_j \beta_j (G_{\varphi, \varphi} c)_j \]

and

\[ C_2^{-1} \| f \| _{H_1}^2 \leq \sum_j |\beta_j|^2 \leq C_1^{-1} \| f \| _{H_1}^2 \]

**Proof.** For any \(f \in H_1\). Let us define a linear transform \(S_{\varphi, \tilde{\varphi}}\) by the relation

\[ S_{\varphi, \tilde{\varphi}} f = \sum_i \langle f, (G_{\varphi, \varphi} c)_i \rangle O^{(\varphi, \tilde{\varphi})} (M (c))_i \]  

(2.3.1)

Then transformation is self adjoint, and if we use (2.2.2) we get

\[ \langle S_{\varphi, \tilde{\varphi}} f, f \rangle = \sum_i \langle f, (G_{\varphi, \varphi} c)_i \rangle \langle O^{(\varphi, \tilde{\varphi})} (M (c))_i, f \rangle \]

or

\[ C_1 \| f \| _{H_1}^2 \leq (S_{\varphi, \tilde{\varphi}} f, f) \leq C_2 \| f \| _{H_1}^2 \]

Hence \(S_{\varphi, \tilde{\varphi}}^{-1}\) exist as a self adjoint transformation and

\[ C_1 \| f \| _{H_1}^2 \leq (S_{\varphi, \tilde{\varphi}}^{-1} f, f) \leq C_2 \| f \| _{H_1}^2 \]

**Theorem 2.3.2.** If \((G_{\varphi, \varphi} c)_j\) is an exact M-frame, then \((G_{\varphi, \varphi} c)_j\), where

\[ O^{(\varphi, \tilde{\varphi})} (M (c))_j = S_{\varphi, \tilde{\varphi}}^{-1} (G_{\varphi, \varphi} c)_j \]

are biorthogonal. Any sequence of numbers \((c_k) \in \ell^2\)

is a M-moment sequence of any function \(f \in H_1\) with respect to \((G_{\varphi, \varphi} c)_j\) and

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\[ C_1 \sum_{\kappa} |c_{\kappa}|^2 \leq \left\| \sum_j c_j (G_{\phi, \varphi})_j \right\|_{H_1 \rightarrow H_2}^2 \leq C_2 \sum_{\kappa} |c_{\kappa}|^2 \quad (2.3.2) \]

**Proof.** If \((G_{\phi, \varphi})_j\) is an exact M-frame then \(\left\langle (G_{\phi, \varphi})_i, (G_{\phi, \varphi})_j \right\rangle = \delta_{i,j}\) for all \(i\) and \(j\), so \((G_{\phi, \varphi})_j\) and \(O^{\phi, \varphi}(M(c))_j\) are biorthogonal. Given a sequence \((c_{\kappa} \in \ell^2)\) and for any \(f \in H_1\) such that \(f = \sum_{\kappa} c_{\kappa} (G_{\phi, \varphi})_\kappa\) has a finite norm then

\[
\sum_j c_j (G_{\phi, \varphi})_j = \sum_j c_j (M^{(\phi, \varphi)}(O)(c))_j
\]

\[
= \sum_j c_j \sum_{\kappa} \left\langle O \bar{\varphi}_\kappa, \varphi_i \right\rangle \left\langle f, \varphi_\kappa \right\rangle
\]

\[
= \sum_j c_j \left( \sum_{\kappa} \left\langle f, \bar{\varphi}_\kappa \right\rangle O \varphi_\kappa, \varphi_i \right) \right)
\]

\[
= \sum_j c_j \left( \left\langle O f, \varphi_i \right\rangle \right)
\]

\[
\sum_j c_j C_{\phi}(O f)
\]

\[
\left\| \sum_j c_j (G_{\phi, \varphi})_j \right\|_{H_1 \rightarrow H_2}^2 \leq \|O\|^2 \sum_j |c_j|^2 \quad (2.3.3)
\]

Assume \((G_{\phi, \varphi})_j\) is an exact M-frame with bounds \(C_1, C_2 > 0\), then

\[ M^{(\phi, \varphi)}(O^{p, \varphi})\) and \(O^{(\bar{\phi}, \bar{\varphi})}(M^{(\phi, \varphi)})(c)\) are biorthogonal, so

\[
C_1 \|O^{(\bar{\phi}, \bar{\varphi})}(M^{(\phi, \varphi)})(c)\|^2 \leq \sum_j |O^{(\bar{\phi}, \bar{\varphi})}(M^{(\phi, \varphi)})(c)_j (G_{\phi, \varphi})_j|^2
\]

\[
= \left\| \left( O^{(\bar{\phi}, \bar{\varphi})}(M^{(\phi, \varphi)})(c) \right)_j (G_{\phi, \varphi} C)_j \right\|^2
\]

\[
= \left\| O^{(\bar{\phi}, \bar{\varphi})}(M^{(\phi, \varphi)})(c) \right\|^2 \| (G_{\phi, \varphi} C) \|^2
\]

\[ \Rightarrow C_1 \leq \| (G_{\phi, \varphi} C) \|^2_{l^2 \rightarrow \ell^2}
\]

or

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\[ C_1 \sum_j |c_j|^2 \leq \| \sum_j c_j (G_{\phi,\phi}(c))_j \|^2 \]  

(2.3.4)

Combining (2.3.3) and (2.3.4), we get (2.3.2). Hence the proof is completed.