Chapter 3

Linear and Non-Linear Heat Transfer in Biological Tissues

3.1 Study of Linear Bio-Heat Equation

In the study of non-invasive thermal diagnosis, accurate correlations between the thermal image of skin surface and interior human physiology are often desired, which require general solutions of the bio-heat equations. Thus, the appropriate solutions of the heat equation play a great role in biomedical engineering. In this section, an eigenvalue approach for the Pennes’ bio-heat equation [70], with convective boundary condition on the top of the surface was used for the solution. The effect of the convective heat transfer coefficient, the temperature of the surrounding air, the metabolic rate, the blood perfusion rate in the tissue and the sensitivity of the thermo-graphy were investigated. Pennes’ bio-heat transfer equation which describes exchange magnitude of heat transfer between tissue and blood is widely used to solve temperature distribution in a biological tissue [70]. The objective of the present study is to investigate the solution of Pennes’ bio-heat equation which can be applied to several bio-heat transfer problems often encountered due to variations of ambient temperature or other factors.

Mathematical Formulation

If $T(x, t)$ is the temperature at time $t$ and position $x$, then the 1-D heat equation in biological tissues is given by Pennes’ [70]

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \rho_b m_b c_b (T - T_a) + Q$$  (3.1.1)

where the parameters $\rho, c, k$ and $Q$ are respectively the density, specific heat, thermal conductivity and rate of metabolic heat generation in tissues, $m_b, c_b$ and $T_a$ are the blood mass flow rate, specific heat of the blood and arterial blood temperature respectively. The outer surface of the skin is exposed to the environment space and there is continuous change of heat flux between the two media, therefore the boundary
condition at the outer surface is governed by Newton’s cooling law

\[-k \frac{\partial T}{\partial x} = h_1 (T - T_A) \text{ at } x = l \tag{3.1.2}\]

where \(l\) is the thickness of skin and subcutaneous tissues, \(h_1\) is the heat transfer coefficient and \(T_A\) is the atmospheric temperature. Assuming that there is no flux at the interior core of the body therefore, the other boundary condition is given as

\[T'(0, t) = 0, \text{ at } x = 0, t \geq 0 \tag{3.1.3}\]

In the present study, it is assumed that the latent heat of evaporation at the skin surface is almost negligible. Also, body core temperature is maintained at a uniform temperature 37\(^\circ\)C. The initial condition associated to the above problem is given by

\[T(x, 0) = z(x) \tag{3.1.4}\]

**Solution of the Model**

Rearranging equation (3.1.1) and by substituting

\[a = \frac{k}{\rho c}, \quad b = \frac{M}{\rho c}, \quad d = \frac{Q - MT_A}{\rho c}, \quad M = \rho_b m_b c_b\]

the equation (3.1.1) reduces to

\[\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + bT + d \tag{3.1.5}\]

Now define \(T(x, t) = f(x, t) - \frac{d}{b}\), equation (3.1.5) reduces to

\[\frac{\partial f}{\partial t} = a \frac{\partial^2 f}{\partial x^2} + bf \tag{3.1.6}\]

along with the boundary conditions

\[
\begin{align*}
 f'(0, t) &= 0 \text{ at } x = 0, t \geq 0 \\
 f'(l, t) + hf(l, t) &= A
\end{align*}
\]

\[f'(0, t) = 0 \text{ at } x = 0, t \geq 0 \tag{3.1.7}\]

where, \(A = h \left( T_A + \frac{d}{b} \right)\) and the corresponding initial condition as

\[f(x, 0) = z(x) + \frac{d}{b} \tag{3.1.8}\]

In order to deal with the non homogeneous conditions, make use of the following transformation \(f(x, t) = F(x, t) + \phi(x)\) as the sum of transient and steady state case
of the problem. Thus, we have

\[ a\phi''(x) + b\phi(x) = 0 \]  \hspace{1cm} (3.1.9)

subject to the conditions

\[
\begin{align*}
\phi'(0) &= 0, \\
\phi(l) + h\phi(l) &= A
\end{align*}
\]  \hspace{1cm} (3.1.10)

And from (3.1.6), we have

\[ \frac{\partial F}{\partial t} = a \frac{\partial^2 F}{\partial x^2} + bF \]  \hspace{1cm} (3.1.11)

subject to the condition

\[ F'(l, t) + hF(l, t) = 0 \]  \hspace{1cm} (3.1.12)

along with the initial condition

\[ F(x, 0) = g(x) \]  \hspace{1cm} (3.1.13)

where \( g(x) = z(x) + \frac{d}{b} - \phi(x) \)

Solving equation (3.1.9) along with its associated boundary conditions (3.1.10), we have

\[ \phi(x) = C_1 \cos(mx) \]  \hspace{1cm} (3.1.14)

where, \( m^2 = \frac{b}{a} \) and \( C_1 = \frac{A}{h\cos(ml) - m\sin(ml)} \)

In order to make (3.1.11) in a more generalised form, we make use of the substitution \( F(x, t) = e^{kt}u(x, t) \) and hence,

\[ \frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} \]  \hspace{1cm} (3.1.15)

and the associated conditions becomes

\[
\begin{align*}
u'(0, t) &= 0; \\
u'(l, t) + hu(l, t) &= 0; \\
and \hspace{1cm} u(x, 0) &= g(x)
\end{align*}
\]

We separate the space and time variables of \( u(x, t) \) as follows.

Let \[ u(x, t) = X(x)G(t) \]  \hspace{1cm} (3.1.16)
be the solution of differential equation (3.1.15), we obtain

\[
\frac{G'}{aG} = \frac{X''}{X} = -\lambda \quad (a \text{ separation constant})
\]

\[\Rightarrow \quad X'' + \lambda X = 0 \quad (3.1.17)\]

with boundary condition

\[
\begin{aligned}
X'(0) & = 0 \\
X'(l) + hX(l) & = 0
\end{aligned}
\]

For \( \lambda \geq 0 \) we have the trivial solution of the equation (3.1.17), and for \( \lambda > 0 \), we choose \( \lambda = \alpha^2 \). And then solving the equation (3.1.17), the standard solution is given by

\[
X(x) = A \cos \alpha x + B \sin \alpha x \quad (3.1.18)
\]

Using \( X'(0) = 0 \), we have

\[
X(x) = A \cos \alpha x \quad (3.1.19)
\]

Now using \( X'(l) + hX(l) = 0 \), we have

\[
\tan \alpha l = \frac{h}{\alpha} \quad (3.1.20)
\]

The graphs of \( y = \tan \alpha l \) and \( y = \frac{h}{\alpha} ; \alpha \neq 0 \) are plotted in the Figure-(3.1)

The above equation has infinite number of positive roots \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \ldots \), where \( \alpha_n < \alpha_{n+1} \). Now the eigenvalues of system (3.1.19) in this case are given by

\[
\lambda_n = \alpha_n^2 \quad (n = 1, 2, 3, \ldots)
\]

where, \( \tan \alpha_n l = \frac{h}{\alpha_n} \)

Clearly as \( n \to \infty, \alpha_n \to (n - 1)\frac{\pi}{l} \)

Therefore \( \lambda_n \to \left[(n - 1)\frac{\pi}{l}\right]^2 \).

The eigenfunctions corresponding to these eigenvalues are given by

\[
X_n = A \cos \alpha_n x
\]
and their normalized form is given by

\[ \phi_n(x) = \frac{X_n}{||X_n||} = \sqrt{\frac{2h}{hl + \sin^2 \alpha_n l}} \cos \alpha_n x \]  \hspace{1cm} (3.1.21)

Also from equation (3.1.16), we have

\[ \frac{G'}{G} = -a\alpha_n^2 \]  \hspace{1cm} (3.1.22)

or,

\[ G_n(t) = e^{-(a\alpha_n^2)t} \]  \hspace{1cm} (3.1.23)

The solution of the (3.1.15) is therefore given by

\[ u(x, t) = \sum_{n=1}^{\infty} A_n e^{-a\alpha_n^2 t} \cos \alpha_n x \]  \hspace{1cm} (3.1.24)

where, \[ A_n = \frac{2h}{hl + \sin^2 \alpha_n l} \int_0^l g(x) \cos \alpha_n x \, dx \] are the coefficients, to be determined from the initial condition.

Hence the solution of the bio-heat equation is given by

\[ T(x, t) = \sum_{n=1}^{\infty} A_n e^{-(a\alpha_n^2-b)t} \cos \alpha_n(x) + C_1 \cos mx - \frac{d}{b} \]  \hspace{1cm} (3.1.25)
Table 3.1: Numerical values of the parameters

<table>
<thead>
<tr>
<th>S.No</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( k ) (thermal conductivity of the tissue)</td>
<td>0.5</td>
<td>( W/m^0C )</td>
</tr>
<tr>
<td>2.</td>
<td>( \rho_b ) (density of the blood)</td>
<td>1.05</td>
<td>( gm/cm^3 )</td>
</tr>
<tr>
<td>3.</td>
<td>( \rho ) (density of the tissue)</td>
<td>1.060</td>
<td>( gm/cm^3 )</td>
</tr>
<tr>
<td>4.</td>
<td>( c ) (specific heat of the tissue)</td>
<td>0.830</td>
<td>( Cal/gm - ^0C )</td>
</tr>
<tr>
<td>5.</td>
<td>( c_b ) (specific heat of blood)</td>
<td>0.003</td>
<td>( W/Cal/cm^3 \ min ^0C )</td>
</tr>
<tr>
<td>6.</td>
<td>( T_a ) (atrial temperature)</td>
<td>37</td>
<td>( ^0C )</td>
</tr>
</tbody>
</table>

Table 3.2: Numerical values of the parameters

<table>
<thead>
<tr>
<th>S.No</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( M )</td>
<td>1.26, 1.38, 1.42</td>
<td>( cal/cm^3 - min^0C )</td>
</tr>
<tr>
<td>2.</td>
<td>( Q )</td>
<td>0.082, 0.067, 0.042</td>
<td>( cal/cm^3 - min )</td>
</tr>
<tr>
<td>3.</td>
<td>( h_1 )</td>
<td>0.009, 0.006, 0.004</td>
<td>( cal/cm^2 - min^0C )</td>
</tr>
<tr>
<td>4.</td>
<td>( x )</td>
<td>0.25, 0.45, 0.60</td>
<td>( cm )</td>
</tr>
</tbody>
</table>

Discussion and Conclusion

A mathematical model based on bio-heat equation has been established by means of eigenvalue expansion method. The disturbance on the human thermoregulatory system due to various adverse environmental conditions has been computed analytically. The domain of the study consists of skin and the subcutaneous tissue, which has been discretized on the basis of the physiological parameters for the estimation of heat regulation. Moreover, in the present study we have taken a suitable initial condition as a linear function of the depth of underlying tissues. The estimation of the temperature profiles may be helpful to prevent various thermal stress problems especially due to hypothermia, hyperthermia and other peripheral injuries. The work presented in this section may be applicable to various problems of heat diffusion in biomedical engineering and other medical sciences. The model is also helpful to understand the thermal behaviour of skin and subcutaneous tissues when exposed to severe environmental temperatures.

Case Studies: We have discussed two cases to identify the applications of the model. In the first case, we have taken the values of the parameters given in Tables-(3.1), (3.2) at atmospheric temperature \( T_A = 20^0C \). Also in second case, the physiological parameters were taken from the Tables-(3.1), (3.2) at \( T_A = 10^0C \). Moreover, the temperature profiles were computed by using MATLAB software at different depths of SST region and are interpreted in the Figures (3.2) and (3.3).
Figure 3.2: Temperature distribution in skin w.r.t Tables-(3.1)-(3.2) at $T_a$(atmospheric temperature) = $20^\circ C$

Figure 3.3: Temperature distribution in skin w.r.t Tables-(3.1)-(3.2) at $T_a$(atmospheric temperature) = $10^\circ C$

3.2 Study of Non-linear Bio-Heat Equation

It has been earlier revealed that the body surface temperature is controlled by factors such as blood circulation underneath the skin, heat exchange between the skin and its environment, and so on. Changes in any of these parameters can induce variations of temperature at the skin surface. As the surface temperature can be easily measured in clinical diagnosis as non-invasive thermometry, it is possible to detect the temperature change reflecting the physiological state of the human body. The mathematical model for the distribution of temperature in the human dermal layers can be represented as:

$$k \frac{d^2 T}{dr^2} + \frac{2k}{r} \frac{dT}{dr} + Q = 0$$  \hspace{1cm} (3.2.1)

and the associated boundary conditions are

$$\lim_{r \to 0^+} \frac{\partial T}{\partial r} = 0, \hspace{1cm} -k \left( \frac{\partial T}{\partial r} \right)_{r=R} = E(T_H - T_a)$$  \hspace{1cm} (3.2.2)

where $r$ is the radial distance from the origin, $R$ is the radius of the domain, $E$ is the ambient cooling constant, $T_a$ ambient temperature, $T_H$ periphery temperature, $Q$
- the heat production per unit volume and \( k \) is the thermal conductivity inside the dermal region.

To study the effect of environmental temperatures on human dermal regions various mathematical models were formulated. Earlier experimental investigations were made by Patterson\[69\] to obtain temperature profiles in the SST region. Some theoretical investigations have been carried out during the last few decades by Cooper and Trezek \[10\]. Chao et al. \[7\] discussed temperature distribution in SST region under normal environmental and physiological conditions. Song et. al \[87\] established models describing the macro and micro vascular level heat transfer in limbs and Jas \[29\] studied the thermal behaviour of human organs in malignancies. Richardson and Whitelaw \[77\] predicted the temperature profiles and the heat conduction and skin surface as functions of surface temperature. Flesch \[18\] estimated the temperature distribution using the heat equation (3.2.1) by assuming the heat generation rate as an explicit function of the radial distance and an implicit function of the environmental temperature. Khanday and Saxena \[38\],\[39\] calculated the mass and temperature distribution at multilayered skin and sub-dermal tissues by using variational finite element method with respect to various environmental temperatures. Also, they studied the conditions under which the brain maintains thermostat and also estimated the cold effect with respect to ambient temperatures. Thron \[88\] studied the above model to estimate the temperature distribution in human head and suggested that if there is no singularity in the differential equation (3.2.1), then the solution is given by

\[
T(r) = T_a + \frac{QR^2}{6k} \left[1 + \frac{2k}{ER} - \left(\frac{r}{R}\right)^2\right] \quad (3.2.3)
\]

In addition, he calculated the temperature distribution by assuming additional heat sources as

\[
Q = Q_0 + Q_b \quad (3.2.4)
\]

where \( Q_b = Vs(T_1 - T) \), \( Q_0 \) is the heat production of tissue, \( V \) is volume of the flow of blood in unit time, and \( s = 0.9\text{cal}^\circ\text{C}cm^3 \).

The present chapter is an attempt to study the temperature distribution at deep dermal layers of the human body with thermal conductivity being a function of temperature.

**Mathematical Formulation**

Estimation of temperature distribution in human body using mathematical techniques has gained interest among many researchers. The heat transfer in biological tissues was studied initially by Pennes' \[70\] and later on by Perl \[71\]. The existing models for heat transfer in dermal regions mostly assumed thermal conductivity term \( k \) either as constant or function of displacement. The thermal conductivity of the material may also depend on the temperature, thus it is meaningful to assume thermal conductivity of the material \( k \) as temperature dependent. Assuming \( k \) as a function of temperature
as \( k(T) = k_0 (T - T_H)^n \). The mathematical model of the heat transfer in the human tissue is given as follows

\[
\rho c \frac{\partial T}{\partial t} = k_0 \frac{d^2 T}{dr^2} + 2k_0 \frac{d T}{dr} + \frac{nk_0}{(T - T_H)} \left( \frac{dT}{dr} \right)^2 + \frac{Q}{(T - T_H)^n} = 0
\]  \hspace{1cm} (3.2.5)

where \( \rho, c \) and \( k_0 \) represent the density, specific heat of the tissue and thermal conductivity respectively. The steady state case of the above equation is given as

\[
k_0 \frac{d^2 T}{dr^2} + 2k_0 \frac{d T}{r} + \frac{nk_0}{(T - T_H)} \left( \frac{dT}{dr} \right)^2 + \frac{Q}{(T - T_H)^n} = 0
\]  \hspace{1cm} (3.2.6)

Assuming the heat generation term \( Q \) as a function of temperature \( T \), \( Q = q_1 (37 - T) \) for some positive constant \( q_1 \), we have

\[
k_0 \frac{d^2 T}{dr^2} + 2k_0 \frac{d T}{r} + \frac{nk_0}{(T - T_H)} \left( \frac{dT}{dr} \right)^2 + \frac{q_1 (37 - T)}{(T - T_H)^n} = 0
\]  \hspace{1cm} (3.2.7)

The boundary conditions associated with the system are

\[
\left( \frac{dT}{dr} \right)_{r=0} = 0, \hspace{1cm} T(r) = T_H
\]  \hspace{1cm} (3.2.8)

Using transformations

\[
y = T - T_H, \hspace{1cm} t = r/R
\]  \hspace{1cm} (3.2.9)

The singular boundary value problem determining the conduction of heat in human dermal layers reduces to the following system of equations

\[
k_0 \frac{d^2 y}{dt^2} + \frac{2k_0}{y} \frac{dy}{dt} + \left( \frac{dy}{dt} \right)^2 + \frac{q_1 (37 - y - T_H)R^2}{(y)^n} = 0, \hspace{1cm} 0 < t < 1
\]  \hspace{1cm} (3.2.10)

\[
\left( \frac{dy}{dt} \right)_{t=0} = 0, \hspace{1cm} y = T_H
\]  \hspace{1cm} (3.2.11)

By using the following substitution

\[
c(t) = t^2
\]

\[
f(t, y, cy') = \frac{nk_0}{y} \left( \frac{dy}{dt} \right)^2 + \frac{q_1 (37 - y - T_H)R^2}{(y)^n}
\]

Equations (3.2.10) and (3.2.11) reduces to

\[
\frac{1}{c(t)} \left[ c(t)y'(t) \right]' + f(t, y, cy') = 0
\]  \hspace{1cm} (3.2.12)

and,

\[
y'(0) = 0, \hspace{1cm} y(1) = 0
\]  \hspace{1cm} (3.2.13)

The solution of this singular non-linear boundary value problem exists and is unique. To compute the approximate solution, finite difference method has been used.
Solution of the Model

The temperature distribution in human dermal regions can be obtained by solving the BVP (3.2.10) and (3.2.11) numerically discussed as

\[ k_0 \frac{d^2 y}{dt^2} + \frac{2k_0}{y} \frac{dy}{dt} + \frac{n k_0}{y} \left( \frac{dy}{dt} \right)^2 + \frac{q_1(37 - y - T_H)R^2}{(y)^n} = 0, \quad 0 < t < 1 \quad (3.2.14) \]

\[ y'(0) = 0, \quad y(1) = 0 \quad (3.2.15) \]

Partitioning the interval \((0, 1)\) into \(p\) subintervals with the length of each subinterval as \(\frac{1}{p}\), then by the central differences, the equation (3.2.10) for \(i = 0\) changes into the following form

\[ 2k_0 y_1 + \frac{q_1(37 - y - T_H)R^2h^2}{(y_0)^n} - 2k_0 y_0 = 0, \quad 0 < t < 1 \quad (3.2.16) \]

and for \(i = 1, 2, 3, \ldots (p - 1)\), we have

\[ \left(1 + \frac{1}{i}\right) y_{i+1} + nk_0 \frac{(y_{i+1} - y_{i-1})^2}{4y_i} + \frac{q_1(37 - y - T_H)R^2}{(y)^n} - 2k_0 y_i + \left(1 - \frac{1}{i}\right) y_{i-1} \quad (3.2.17) \]

where,

\[ p = \frac{1}{3}, \quad q_1 = 0.000002, \quad T_H^2 = 33.03 + 0.14(T_a - 10), \quad k_0 = 0.000009T_H(37 - T_H)^{(1/3)} \]

Making use of numerical technique to solve the resulting non-linear system of equations, the temperature distribution in human dermal regions at various environmental temperatures can be computed.

Discussion and Conclusion

The outcome of this study reflects some innovations in the existing models by means of temperature dependence of various parameters. It is important to mention that the experiments have shown a great role of thermal conductivity on the thermal behaviour of the biological tissue. The numerical solution of the boundary value problem was carried out and the results were compared with the existing solutions as discussed by Thron [88]. The variation of temperature at various dermal layers of the underlying tissue is demonstrated in Figure-(3.4). The curves 1, 2 are due to Thron[88] at two atmospheric temperatures \(T_a = 0^\circ C, 10^\circ C\) while as the curves 3, 4 are our results at the same ambient temperatures respectively. The figure reveals that there are gradual changes of temperature with radial distances. The main reason for such differences is that Thron [88] treated the thermal conductivity as constant.
whereas it is temperature dependent in our case. Therefore, it may be said that the present study is comparatively more realistic. Similarly Figure-(3.5) is described at ambient temperatures $T_a = 21^0C, 25^0C$. It is evident from the Figures-(3.4), (3.5) that the temperature variation in radial distances shows the same tendency with existing study of temperature variations on the human periphery. The value of thermal conductivity gradually increases from outer regions towards core with increase in temperature. The heat generation calculated at the body core (brain and heart) is given in Figure-(3.6). It has been observed from Figure-(3.6) that the heat generation in the regions increases when the environment temperature decreases. For temperature dependent thermal conductivity $k$, the present study shows some realistic values for the estimation of thermoregulation in human dermal layers as compared to various researchers including Khanday and Saxena [40], Thron [88]. Some realistic results were observed in this study while comparing these results with some experimental work carried out by Hodgson [27]. It is worthwhile to mention that the model can be used extensively in medical sciences and biomedical engineering.

Figure 3.4: Temperature distribution in the tissue of human head w.r.t radial distance for various values of environmental temperatures.
Figure 3.5: Temperature distribution in the tissue of human head w.r.t radial distance for various values of environmental temperatures.

Figure 3.6: Heat generation of tissue of the human head w.r.t radial distance for several environmental temperatures.