CHAPTER -II

ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A
GENERALISED DIFFERENTIAL OPERATOR

2.1. In this chapter, we introduce certain new subclasses \( f(z) \in T^m S(\gamma, k) \) and \( G(A, B) \) of analytic functions by using a generalized differential operator. Also we find the various results including coefficient estimates, growth and distortion theorems, radius of starlikeness, convexity and close-to-convexity, integral means inequalities for the function.

2.2. Let \( f \in A \), then \( f \) is of the form

\[
f(z) = z + \sum_{i=2}^{\infty} a_i z^i
\]

which are analytic in the open unit disk \( E = \{z \in \mathbb{C} : |z| < 1\} \).

Let \( f \in T \), where \( T \) is the subclass of \( A \), then \( f \) is in the form

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).
\]

This subclass was introduced and extensively studied by Silverman [71].

Let \( f \) be a function in the class \( A \). We define the following differential operator introduced by Raducanu and Orhan [59].

\[
\begin{align*}
\mathcal{D} f(z) &= f(z) \\
\mathcal{D}^2 f(z) &= \alpha \beta z^2 f''(z) + (\alpha - \beta)zf'(z) + (1 - \alpha + \beta) f(z)
\end{align*}
\]
\[ D f(z) = D( \frac{m^3}{\alpha^3} f(z)), \]  
(2.5)

where \( 0 \leq \beta \leq \alpha \leq 1 \) and \( m \in \mathbb{N} \leq \{1, 2, 3, \ldots\} \).

If \( f \) is given by (2.1) then from the definition of the operator \( D f(z) \) it is to see that

\[ m \frac{D}{\alpha^\beta} f(z) = z + \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^n \]  
(2.6)

where

\[ A_n(\alpha, \beta, m) = [1 + (\alpha \beta n + \alpha - \beta)(n - 1)]^m \]  
(2.7)

When \( \alpha = 1 \) and \( \beta = 0 \) we get Salagean differential operator [68]. When \( \beta = 0 \), we obtain the differential operator defined by Al-Oboudi [5].

If \( f \in T \) is given by (2.2) then we have

\[ m \frac{D}{\alpha^\beta} f(z) = z - \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^n \]  
(2.8)

where \( A_n(\alpha, \beta, m) \) is given by (2.7)

In this chapter, using the operator \( m \frac{D}{\alpha^\beta} f(z) \), we define the following new subclass motivated by Murugusunderamoorthy and Magesh [49].

**Definition 2.2.1.** If \( f(z) \in S_{\alpha^\beta}(\gamma, k) \), where \( f \) is in the form (2.1), then
for \(0 \leq \gamma \leq 1, \ k \geq 0\).

Further we define \(T_{m} S_{\alpha}(\gamma, k) = S_{\alpha}^{m}(\gamma, k) \cap T\).

**Theorem 2.2.2.** If \(f(z) \in S_{\alpha}^{m}(\gamma, k)\), where \(f\) is in the form (2.1), then

\[
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)] A_{n}(\alpha, \beta, m)|a_n| \leq 1 - \gamma
\]  

(2.9)

where \(0 \leq \gamma < 1, \ k \geq 0\) and \(A_{n}(\alpha, \beta, m)\) is given by (2.7)

**Proof:** It is enough to show that

\[
\Re \left\{ \frac{z^{m} D f(z)'}{D f(z)} - 1 \right\} \leq 1 - \gamma
\]

We have

\[
\Re \left\{ \frac{z^{m} D f(z)'}{D f(z)} - 1 \right\} \leq 1 - \gamma
\]
\[
\leq (1 + k) \left| \frac{\frac{m}{\alpha} D f(z)}{z} \right| - 1
\]

\[
\leq \frac{(1 + k) \sum_{n=2}^{\infty} (n-1)A_n(\alpha, \beta, m) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) |a_n| |z|^{n-1}}
\]

The last expression is bounded above by \((1 - \gamma)\) if

\[
\sum_{n=2}^{\infty} [n(1 + k) - (\gamma + k)] A_n(\alpha, \beta, m) |a_n| \leq 1 - \gamma
\]

And the proof is complete.

**Theorem 2.2.3.** Let \(0 \leq \gamma < 1, k \geq 0\) then \(f \in T_{m, (\alpha) \gamma} S(\gamma, k)\), where \(f\) is in the form (2.2), Iff

\[
\sum_{n=2}^{\infty} [n(1 + k) - (\gamma + k)] A_n(\alpha, \beta, m) \leq 1 - \gamma
\]

(2.10)

where \(A_n(\alpha, \beta, m)\) is given by (2.7)

**Proof:** In view of the above Theorem, it is enough to prove the necessity. If \(f \in T_{m, (\alpha) \gamma} S(\gamma, k)\) and \(z\) is real then
\[
\frac{1 - \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} n A_n(\alpha, \beta, m) a_n z^{n-1}} - \gamma \geq k \frac{\sum_{n=2}^{\infty} (n-1) A_n(\alpha, \beta, m) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^{n-1}}
\]

Along the real axis, \( z \to 1 \) we get the desired inequality

\[
\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)] A_n(\alpha, \beta, m) |a_n| \leq 1 - \gamma,
\]

where \( 0 \leq \gamma < 1, \ k \geq 0 \) and \( A_n(\alpha, \beta, m) \) are given by (2.7).

**Corollary 2.2.4.** If \( f(z) \in T^{m}_{\alpha \beta} S(\gamma, k) \), then

\[
|a_n| \leq \frac{1 - \gamma}{[n(1+k) - \lambda(\gamma + k)] A_n(\alpha, \beta, m)}
\]

where \( 0 \leq \gamma < 1, \ k \geq 0 \) and \( A_n(\alpha, \beta, m) \) are given by (2.7). Equality holds for the function

\[
f(z) = z - \frac{1 - \gamma}{[n(1+k) - (\gamma + k)] A_n(\alpha, \beta, m)} z^n
\]

**Theorem 2.2.5.** Let \( f_i(z) = z \) and

\[
f_n(z) = z - \frac{1 - \gamma}{[n(1+k) - (\gamma + k)] A_n(\alpha, \beta, m)} z^n, \ n \geq 2.
\]

Then \( f(z) \in T^{m}_{\alpha \beta} S(\gamma, k) \), Iff it can be in the form
\[ f(z) = \sum_{n=1}^{\infty} w_n f_n(z) , \quad w_n \geq 0, \quad \sum_{n=1}^{\infty} w_n = 1 \]  
(2.14)

Proof. Suppose that \( f(z) \) can be written as in (2.14). Then

\[ f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1 - \gamma}{n(1+k) - (\gamma + k)} A_n(\alpha, \beta, m) z^n \]

Now,

\[ \sum_{n=2}^{\infty} w_n \frac{(1 - \gamma)(n(1+k) - (\gamma + k)) A_n(\alpha, \beta, m)}{(1 - \gamma) (n(1+k) - (\gamma + k)) A_n(\alpha, \beta, m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1. \]

Thus \( f(z) \in T^m S(\gamma, k) \).

Conversely, let us have \( f(z) \in T^m S(\gamma, k) \). Then by using (2.11), we get

\[ w_n = \frac{n(1+k) - (\gamma + k) A_n(\alpha, \beta, m)}{(1 - \gamma)} a_n, \quad n \geq 2 \]

and \( w_1 = 1 - \sum_{n=2}^{\infty} w_n \). Then we have \( f(z) = \sum_{n=1}^{\infty} w_n f_n(z) \) and hence this completes the proof of Theorem.

**Theorem 2.2.6.** The class \( T^m S(\gamma, k) \) is a convex set.

**Proof.** Let the function

\[ f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad j=1,2 \]  
(2.15)
be in the class $T^m_{(a, b)}(\gamma, k)$. It is enough to show that the function $h(z)$ defined by

$$h(z) = \xi f_1(z) + (1-\xi) f_2(z), \quad 0 \leq \xi < 1,$$

is in the class $T^m_{(a, b)}(\gamma, k)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} \left[ \xi a_{n,1} + (1-\xi) a_{n,2} \right] z^n,$$

with the help of Theorem 2.2.3, and by an easy computation, we get

$$\sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)] \xi A_n(\alpha, \beta, m)a_{n,1} + \sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)] (1-\xi) A_n(\alpha, \beta, m)a_{n,2}$$

$$\leq \xi (1-\gamma) + (1-\xi)(1-\gamma)$$

$$\leq (1-\gamma),$$

which implies that $h \in S(\gamma, k)$.

Hence $T^m_{(a, b)}(\gamma, k)$ is convex.

**Theorem 2.2.7.** If $f \in T^m_{(a, b)}(\gamma, k)$, where $f(z)$ is in the form (2.2) Then

is close-to-convex of order $\delta$ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 = \inf_{n \geq 2} \left[ \frac{(1-\delta) \sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)] A_n(\alpha, \beta, m) n(1-\gamma)}{n(1-\gamma)} \right]^{1/\gamma_{n-1}}, \quad n \geq 2. \quad (2.16)$$

The outcome is sharp, with the extremal function $f(z)$ by (2.13)
**Proof.** Given $f \in T$, and $f$ is close-to-convex of order $\delta$, we have

$$|f'(z) - 1| < 1 - \delta \quad (2.17)$$

For the L.H.S. of (2.17) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

The R.H.S. of the above inequality is less than $1 - \delta$

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n |z|^{n-1} \leq 1.$$

We have $f(z) \in T_{\alpha}^{m}(\gamma, k)$ iff

$$\sum_{n=2}^{\infty} \frac{[n(1+k) - \lambda(\gamma + k)]A_n(\alpha, \beta, m)}{(1-\gamma)} a_n \leq 1,$$

We can (2.17) is true if

$$\frac{n}{1-\delta} |z|^{n-1} \leq \frac{[n(1+k) - (\gamma + k)]A_n(\alpha, \beta, m)}{(1-\gamma)}$$

or, equivalently,

$$|z| \leq \left( \frac{(1-\delta)[n(1+k) - (\gamma + k)]A_n(\alpha, \beta, m)}{n(1-\gamma)} \right)^{\frac{1}{n-1}}$$

the proof is completed.

**Theorem 2.2.8.** If $f \in T_{\alpha}^{m}(\gamma, k)$ Then $f(z)$ is starlike of order $\delta$ $(0 \leq \delta < 1)$ in the disc $|z| < r_2$, where
\[
    r_2 = \inf_{n \geq 2} \left[ \frac{(1 - \delta) \sum_{n=2}^{\infty} n(1+k) - (\gamma + k) A_n(\alpha, \beta, m)}{(n-\delta)(1-\gamma)} \right]^{\gamma/(\alpha-1)}
\]  

(2.18)

The result is sharp, with the extremal function given by (2.13).

**Proof.** Given \( f \in T \), and is starlike of order \( \delta \), we have

\[
    \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta
\]  

(2.19)

For the L.H.S. of (2.19), we have

\[
    \left| \frac{zf'(z)}{f(z)} \right| \leq \sum_{n=2}^{\infty} \frac{n(1-\delta) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}
\]

R.H.S. of the above is less than \( 1 - \delta \) if

\[
    \sum_{n=2}^{\infty} \frac{n(1-\delta) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} < 1
\]

We have \( f(z) \in T_{m}^{S(\gamma,k)} \) iff

\[
    \sum_{n=2}^{\infty} \frac{n(1+k) - (\gamma + k) A_n(\alpha, \beta, m)}{(1-\gamma)} a_n \leq 1
\]

(2.19) is true if

\[
    \sum_{n=2}^{\infty} \frac{n(1-\delta) |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq \frac{n(1+k) - (\gamma + k) A_n(\alpha, \beta, m)}{(1-\gamma)}
\]

Or equivalently
\[ |z|^{n-1} \leq \frac{(1-\delta)[n(1+k)-(\gamma +k)]A_{k}(\alpha, \beta, m)}{(n-\delta)(1-\gamma)} \]

It yields starlikeness of the family.

In [72], Silverman found that the function \( f_{2}(z) = z - \frac{z^2}{2} \) is often extremal over the family \( T \). He applied this function to resolve his integral means inequality conjectured [72] and settled in [73], that

\[
\int_{0}^{2\pi} \| f(re^{i\varphi}) \|^{n} d\varphi \leq \int_{0}^{2\pi} \| f_{2}(re^{i\varphi}) \|^{n} d\varphi ,
\]

for all \( f \in T \), \( \eta > 0 \) and \( 0 < r < 1 \). In [73], he also proved his conjecture for the subclasses \( T^{*}(\alpha) \) and \( C(\alpha) \) of \( T \).

Now, we prove Silverman’s conjecture for the class of functions \( T^{m}_{\gamma,k}S(\alpha) \).

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [38].

Two functions \( f \) and \( g \), which are analytic in \( E \), the function \( f \) is said to be subordinate to \( g \) in \( E \) if there exists a function \( w \) analytic in \( E \) with \( w(0) = 0 \), \( |w(z)| < 1 \), \( (z \in E) \) such that \( f(z) = g(w(z)) \), \( (z \in E) \). We denote this subordination by \( f(z) \prec g(z) \).

**Lemma 2.2.9.** [38] If the functions \( f \) and \( g \) are analytic in \( E \) with \( f(z) \prec g(z) \), then for \( \eta > 0 \) and \( z = re^{i\varphi} \), \( 0 < r < 1 \)
Now, we discuss the integral means inequalities for functions $f \in T^m(\gamma, k)$.

\[
\left[ \int_0^{2\pi} \left| g(re^{i\theta}) \right|^i d\phi \right]^{\frac{1}{i}} \leq \left[ \int_0^{2\pi} \left| f(re^{i\theta}) \right|^i d\phi \right]^{\frac{1}{i}}
\]

**Theorem 2.2.10.** Let $f(z) \in T^m(\gamma, k)$, $0 \leq \gamma < 1$, $k \geq 0$ and $f_2(z)$ be defined by

\[
f_2(z) = z - \frac{1-\gamma}{\phi_2(\gamma, k)} z^2 \quad (2.20)
\]

**Proof.** For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, (2.20) is equivalent to

\[
\left[ \int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^n \right|^i d\phi \right]^{\frac{1}{i}} \leq \left[ \int_0^{2\pi} \left| 1 - \frac{1-\gamma}{\phi_2(\gamma, k)} z \right|^i d\phi \right]^{\frac{1}{i}}
\]

By Lemma 2.2.9, it is enough to prove that

\[1 - \sum_{n=2}^{\infty} a_n z^{n-1} < 1 - \frac{1-\gamma}{\phi_2(\gamma, k)} z\]

Assuming

\[1 - \sum_{n=2}^{\infty} a_n z^{n-1} < 1 - \frac{1-\gamma}{\phi_2(\gamma, k)} w(z),\]

and using (2.10) we obtain
\[ |w(z)| = \left| \sum_{n=2}^{\infty} \frac{\varphi_n(\gamma, k)}{1-\gamma} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{\varphi_n(\gamma, k)}{1-\gamma} a_n \leq |z| \]

where \( \varphi_n(\gamma, k) = [n(1+k) - (\gamma+k)]A_n(\alpha, \beta, m) \)

Hence the proof is completed.

2.3. In this section we study the coefficient bounds, radius of close-to-convex and starlikeness, convex linear combinations for the class \( T_{\alpha\beta}^m(\lambda, \gamma) \).

Also, we obtained integral means inequalities for the function \( f(z) \in T_{\alpha\beta}^m(\lambda, \gamma) \).

**Definition 2.3.1.** If \( f(z) \in S_{\alpha\beta}^m(\lambda, \gamma) \) where \( f \) is in the form (2.1), then it satisfies the inequality

\[
\text{Re} \left\{ \frac{z \left( \frac{m}{\alpha\beta} \mathcal{D} f(z) \right)'}{(1-\lambda)z + \lambda \frac{m}{\alpha\beta} \mathcal{D} f(z)} - \alpha \right\} > \left| \frac{z \left( \frac{m}{\alpha\beta} \mathcal{D} f(z) \right)'}{(1-\lambda)z + \lambda \frac{m}{\alpha\beta} \mathcal{D} f(z)} - 1 \right|
\]

for \( 0 \leq \lambda \leq 1, \ 0 \leq \gamma \leq 1, \) and \( \frac{m}{\alpha\beta} \mathcal{D} f(z) \) is defined in (2.6).

Further we define \( T_{\alpha\beta}^m(\lambda, \gamma) = S_{\alpha\beta}^m(\lambda, \gamma) \cap T \).

**Theorem 2.3.2.** If \( f(z) \in S_{\alpha\beta}^m(\lambda, \gamma) \) where \( f \) is in the form (2.1), then
\[
\sum_{n=2}^{\infty} [2n-\lambda(\gamma+1)] A_n(\alpha, \beta, m) |a_n| \leq 1 - \gamma \tag{2.21}
\]

where \(0 \leq \lambda \leq 1,\ 0 \leq \gamma < 1\), and \(A_n(\alpha, \beta, m)\) is given by (2.7).

**Proof:** It suffices to show that

\[
\left| \frac{z^{m f(z)}}{(1-\lambda)z + \lambda D f(z)} \right| - 1 \leq 1 - \gamma \leq \left| \frac{z^{m f(z)}}{(1-\lambda)z + \lambda D f(z)} \right| - 1
\]

We have

\[
\left| \frac{z^{m f(z)}}{(1-\lambda)z + \lambda D f(z)} \right| - 1 \leq 2 \left| \frac{z^{m f(z)}}{(1-\lambda)z + \lambda D f(z)} \right| - 1
\]

\[
\leq 2 \sum_{n=2}^{\infty} (n-\lambda) A_n(\alpha, \beta, m) |a_n| z^{n-1}
\]

\[
\leq 1 - \sum_{n=2}^{\infty} \lambda A_n(\alpha, \beta, m) |a_n| z^{n-1}
\]
\[
\frac{2 \sum_{n=2}^{\infty} (n - \lambda)A_n(\alpha, \beta, m)a_n}{1 - \sum_{n=2}^{\infty} \lambda A_n(\alpha, \beta, m)a_n}
\]

The last expression is bounded above by \((1 - \gamma)\) if

\[
\sum_{n=2}^{\infty} [2n - \lambda(\gamma + 1)] A_n(\alpha, \beta, m)a_n \leq 1 - \gamma
\]

and the proof is complete.

**Theorem 2.3.3.** Let \(0 \leq \lambda \leq 1, \ 0 \leq \gamma < 1, f(z) \in T^m_{S(\lambda, \gamma)}, \) where \(f\) is in the form (2.2) Iff

\[
\sum_{n=2}^{\infty} [2n - \lambda(\gamma + 1)] A_n(\alpha, \beta, m) \leq 1 - \gamma
\]

(2.22)

where \(A_n(\alpha, \beta, m)\) are given by (2.7)

**Proof:** In view of Theorem 2.3.2, it is enough to prove the necessity.

If \(f \in T^m_{S(\lambda, \gamma)}\) and \(z\) is real then

\[
\Re \left\{ \frac{1 - \sum_{n=2}^{\infty} nA_n(\alpha, \beta, m)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \lambda A_n(\alpha, \beta, m)a_n z^{n-1}} - \gamma \right\} > \frac{\sum_{n=2}^{\infty} (n - \lambda)A_n(\alpha, \beta, m)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \lambda A_n(\alpha, \beta, m)a_n z^{n-1}}
\]

Along the real axis, \(z \to 1\) we get the desired inequality

\[
\sum_{n=2}^{\infty} [2n - \lambda(\gamma + 1)] A_n(\alpha, \beta, m)a_n \leq 1 - \gamma
\]
where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, and $A_n(\alpha, \beta, m)$ are given by (2.7).

**Corollary 2.3.4.** If $f(z) \in T^m S^\lambda(\alpha, \gamma)$, then

$$|a_n| \leq \frac{1-\gamma}{[2n-\lambda(\gamma +1)]A_n(\alpha, \beta, m)}$$

(2.23)

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, and $A_n(\alpha, \beta, m)$ are given by (2.7). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[2n-\lambda(\gamma +1)]A_n(\alpha, \beta, m)} z^n$$

(2.24)

**Theorem 2.3.5.** Let $f_i(z) = z$ and

$$f_n(z) = z - \frac{1-\gamma}{[2n-\lambda(\gamma +1)]A_n(\alpha, \beta, m)} z^n, \quad n \geq 2.$$

(2.25)

Then $f(z) \in T^m S^\lambda(\alpha, \gamma)$, iff it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z), \quad w_n \geq 0, \sum_{n=1}^{\infty} w_n = 1$$

(2.26)

**Proof.** Suppose $f(z)$ can be written as in (2.26). Then

$$f(z) = z - \sum_{n=1}^{\infty} w_n \frac{1-\gamma}{[2n-\lambda(\gamma +1)]A_n(\alpha, \beta, m)} z^n$$

Now,
\[
\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)(2n-\lambda(\gamma+1))A_n(\alpha, \beta, m)}{(1-\gamma)(2n-\lambda(\gamma+1))A_n(\alpha, \beta, m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.
\]

Thus \( f(z) \in T^{m}_{\lambda, \gamma}(z) \).

Conversely, let us have \( f(z) \in T^{m}_{\lambda, \gamma}(z) \). The by using (2.23), we get

\[
w_n = \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(1-\gamma)} a_n, \quad n \geq 2
\]

and \( w_i = 1 - \sum_{n=2}^{\infty} w_n \). Then we have \( f(z) = \sum_{n=1}^{\infty} w_n f_n(z) \) and hence this,

Proof is completed

\textbf{Theorem 2.3.6.} The class \( T^{m}_{\lambda, \gamma}(z) \) is a convex set.

\textbf{Proof.} Let the function

\[
f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, j=1,2
\]

be in the class \( T^{m}_{\lambda, \gamma}(z) \). It is enough to show that the function \( h(z) \) defined

by \( h(z) = \xi f_1(z) + (1-\xi) f_2(z), \quad 0 \leq \xi < 1 \), is in the class \( T^{m}_{\lambda, \gamma}(z) \). Since

\[
h(z) = z - \sum_{n=2}^{\infty} \left[ \xi a_{n,1} + (1-\xi) a_{n,2} \right] z^n,
\]

with the help of Theorem 2.3.3, and by easy computation, we get
\[
\sum_{n=2}^{\infty} [2n-\lambda(\gamma+1)] \xi A_n(\alpha, \beta, m)a_{n,1} + \sum_{n=2}^{\infty} [2n-\lambda(\gamma+1)] (1-\xi) A_n(\alpha, \beta, m)a_{n,2} \\
\leq \xi (1-\gamma) + (1-\xi)(1-\gamma) \\
\leq (1-\gamma),
\]

which implies that \( h \in S(\lambda, \gamma) \).

Hence \( T S(\lambda, \gamma) \) is convex.

Next we will get the radius of close-to-convexity, starlikeness and convexity for the class \( T S(\lambda, \gamma) \).

**Theorem 2.3.7.** If \( f(z) \in T S(\lambda, \gamma) \), where \( f(z) \) is in the form (2.2), Then \( f(z) \) is close-to-convex of order \( \delta \) (0 \( \leq \delta < 1 \)) in the disc \( |z| < r_1 \), where

\[
r_1 = \inf_{n \geq 2} \left[ \frac{(1-\delta) \sum_{n=2}^{\infty} [2n-\lambda(\gamma+1)] A_n(\alpha, \beta, m)}{n(1-\gamma)} \right]^{1/n}, \quad n \geq 2. \quad (2.28)
\]

The outcome is sharp, with the extremal function by (2.25)

**Proof.** Given \( f \in T \), and \( f \) is close-to-convex of order \( \delta \), we have

\[
|f''(z) - 1| < 1 - \delta \quad (2.29)
\]

For the L.H.S. of (2.29) we have
\[ |f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1} \]

R.H.S. of the above is less than \( 1 - \delta \)

\[ \sum_{n=2}^{\infty} \frac{n}{1 - \delta} a_n |z|^{n-1} \leq 1. \]

Using the fact, that \( f(z) \in T_{\text{m}} S(\lambda, \gamma) \) iff

\[ \sum_{n=2}^{\infty} \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(1 - \gamma)} a_n \leq 1, \]

(2.29) is true if

\[ \frac{n}{1 - \delta} |z|^{n-1} \leq \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(1 - \gamma)} \]

or, equivalently,

\[ |z| \leq \left( \frac{(1 - \delta)[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{n(1 - \gamma)} \right)^{\frac{1}{n-1}} \]

the proof is completed.

**Theorem 2.3.8** If \( f(z) \in T_{\text{m}} S(\lambda, \gamma) \), where \( f(z) \) is in the form (2.2), then \( f(z) \) is starlike of order of order \( \delta \) \((0 \leq \delta < 1)\) in the disc \(|z| < r_2\), where

\[ r_2 = \inf_{n=2} \left[ \frac{(1 - \delta)[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(n - \delta)(1 - \gamma)} \right]^{\frac{1}{n-1}} \]

(2.30)
The result is sharp, with extremal function \( f(z) \) by (2.25).

**Proof.** Given \( f \in T \), and \( f \) is starlike of order \( \delta \), we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \delta \tag{2.31}
\]

For the L.H.S. of (2.31) we have

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}
\]

R.H.S. of the above is less than \( 1 - \delta \) if

\[
\sum_{n=2}^{\infty} \frac{n - \delta}{1 - \delta} a_n |z|^{n-1} < 1.
\]

Using the fact that \( f(z) \in T^m S(\lambda, \gamma) \) iff

\[
\sum_{n=2}^{\infty} \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(1 - \gamma)} a_n \leq 1,
\]

(2.31) is true if

\[
\sum_{n=2}^{\infty} \frac{n - \delta}{1 - \delta} |z|^{n-1} \leq \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(1 - \gamma)}
\]

or equivalently

\[
|z|^{n-1} \leq (1 - \delta) \frac{[2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m)}{(n - \delta)(1 - \gamma)}
\]

which yields the starlikeness of the family.
Theorem 2.3.9. Let \( f(z) \in T^m \), \( 0 \leq \lambda < 1, \ 0 \leq \gamma < 1 \), and \( f_z(z) \) be defined by

\[
f_z(z) = z - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} z^2
\]

(2.32)

Proof. For \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), (2.32) is equivalent to

\[
\int_0^{2\pi} \left| \sum_{n=2}^{\infty} a_n z^{n-1} \right|^2 d\varphi \leq \int_0^{2\pi} \left| 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} z \right|^2 d\varphi
\]

By Lemma 2.2.9, it is enough to prove that

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} z
\]

Assuming

\[
1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1 - \gamma}{\phi_2(\lambda, \gamma)} w(z),
\]

and using (2.22) we obtain

\[
|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\phi_n(\lambda, \gamma)}{1 - \gamma} a_n z^{n-1} \right| \leq |z| \sum_{n=2}^{\infty} \frac{\phi_n(\lambda, \gamma)}{1 - \gamma} a_n \leq |z|
\]

where \( \phi_n(\lambda, \gamma) = [2n - \lambda(\gamma + 1)]A_n(\alpha, \beta, m) \)

This completes the proof.
2.4. Introduction: In this section we investigate properties like distortion, rotation theorem, coefficient estimates and radius of convexity for functions in the class $G(A, B)$ and showed the results are sharp.

Now we define the class $G(A, B)$ and study some of its properties.

Definition 2.4.1: Let $f$ be analytic in $E$, $f(0) = f'(0) = 1$. Then $f(z) \in G(A, B)$ iff $\exists$ a function $g(z) \in S(a, b)$, s.t. for $z \in E$.

$$\frac{f'(z)}{g'(z)} = \frac{1 + A \ w(z)}{1 + B \ w(z)}, -1 \leq B < A \leq 1,$$

$g(z)$ satisfies the condition

$$|g'(z) - a| < b$$

(2.34)

for $a + b \geq 1$, $b \leq a \leq b+1$, $z \in E$ and $w$ is a Schwartz function analytic in $E$ with $w(0) = 0$ and $|w(z)|<1$ in $E$.

Theorem 2.4.2: (Distortion Theorem)

Let $f(z) \in G(A, B)$. Then for $|z| = r$ and $0 \leq r < 1$

$$\frac{b - (b^2 - a^2 + a)r (1 - Ar)}{b - (1 - a)r (1 - Br)} \leq |f'(z)| \leq \frac{b + (b^2 - a^2 + a)r (1 - Ar)}{b + (1 - a)r (1 - Br)}$$

(2.35)

The outcome is sharp.

Proof: Since $f(z) \in G(A, B)$, then

$$\frac{f'(z)}{g'(z)} = \frac{1 + A \ w(z)}{1 + B \ w(z)}, -1 \leq B < A \leq 1$$

for some $g(z) \in S(a, b)$ and $w(z)$ is a Schwartz function analytic in $E$ with $w(0) = 0$, $|w(z)|<1$. 

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It is known [50] that the images of the closed disk $|z| \leq r$ under the transformation

$$P(z) = \frac{1 + A \frac{w(z)}{1 + B \frac{w(z)}}}{1 - B^2 r^2}$$

are in the closed disk with center ‘C’ and radius ‘d’ where

$$C = \frac{1 - A B r^2}{1 - B^2 r^2}, \quad d = \frac{(A - B) r}{1 - B^2 r^2}.$$

Thus we have

$$\left| \frac{f'(z)}{g'(z)} - \frac{1 - A B r^2}{1 - B^2 r^2} \right| \leq \frac{(A - B) r}{1 - B^2 r^2}$$

so which shows that

$$\frac{1 - A r}{1 - B r} \leq \left| \frac{f'(z)}{g'(z)} \right| \leq \frac{1 + A r}{1 + B r}$$

(2.36)

Since $g(z) \in S(a, b)$, it is known that [51]

$$\frac{b - (b^2 - a^2 + a) r}{b - (1 - a) r} \leq |g'(z)| \leq \frac{b + (b^2 - a^2 + a) r}{b + (1 - a) r}$$

(2.38)

using (2.38) in (2.37) we obtained the result.

This outcome is sharp.

Taking $\frac{f'(z_0)}{g'(z_0)} = \frac{1 + A z}{1 + B z}, \ g'(z_0)$ with $g_0(z_0) \in S(a, b)$

such that $g'_0(z_0) = \frac{b + (b^2 - a^2 + a) z}{b + (1 - a) z}$

**Theorem 2.4.3:** If $f(z) \in G(A, B)$, then

(i) $|a_2| \leq \frac{(A - B) b + (b^2 - (1 - a)^2)}{2 b}$
(ii) \[ |a_3| \leq \frac{(A-B)b + (1+A-B)(b^2-(1-a)^2)}{3b} \]

The estimate (ii) is sharp.

Proof: Let \[ \frac{f'(z)}{g'(z)} = \frac{1+A}{1+B} \frac{w(z)}{w(z)} \] then

\[ w(z) = \frac{f'(z) - g'(z)}{Ag'(z) - Bg'(z)} \] (2.39)

Let \[ w(z) = \sum_{n=1} w_n z^n \] and \[ g(z) = z + \sum_{m=2} b_m z^m. \]

On substituting the power series \( f'(z), g'(z) \) and \( w(z) \) in (2.39) we get

\[ \left[ A \left( 1 + \sum_{m=2}^\infty mb_m z^{m-1} \right) - B \left( 1 + \sum_{m=2}^\infty mb_m z^{m-1} \right) \right] \left( \sum_{m=1}^\infty w_m z^m \right) = \sum_{m=2}^\infty m(a_m - b_m) z^{m-1} \] (2.40)

Equating the coefficients of \( z^2 \) and \( z^3 \) on both sides of (2.40) we get

\[ |a_2| \leq \frac{1}{2} \left[ (A-B)w_1 + 2|b_2| \right] \] (2.41)

and

\[ 3|a_3| \leq 3|b_3| + 2(A-B)\|w_1\| + (A-B)\|w_2 - Bw_1\| \] (2.42)

It is known [23] that

\[ |b_2| \leq \frac{b^2 - (1-a)^2}{2b} \] and \[ |b_3| \leq \frac{b^2 - (1-a)^2}{3b} \] (2.43)

and known [50] that \( |w_1| \leq 1 \) (2.44)

Also we know [34] that for \( S \) any complex

\[ |w_2 - 3w_1| \leq \max \{ |w_3|, |b_3| \} \] (2.45)
On using (2.43), (2.44) and (2.45) we obtain the required results from (2.41) and (2.42) respectively.

The bound is sharp in (i) for the function

\[ f(z) = \int_0^z \frac{b + (b^2 - a^2 + a)z}{b + (1 - a)z} \left( 1 + Az \right) \left( 1 + Bz \right) dz. \]

**Radius of Convexity**

Here we solve the problem of finding the radius of convexity for the class \( G(A, B) \). For this we need the following lemma.

**Lemma 2.4.4:** Let \( p(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \), \(-1 \leq B < A \leq 1\)

where \(|w(z)| < 1\) in \( E \). Then for \(|z| = r < 1\)

\[ \text{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} \geq -\frac{(A - B)r}{(1 - Ar)(1 - Br)} \quad R_1 \leq R_2 \]

and

\[ \text{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} \geq \frac{A + B}{A - B} + \frac{2}{(A - B)(1 - r^2)} \left( L_i K_i \right)^{-\frac{1}{2}} - \left( 1 - ABr^2 \right) \quad \text{if} \quad R_2 \leq R_1 \]

where

\[ R_1 = \left( \frac{L_i}{K_i} \right)^{\frac{1}{2}}, \quad R_2 = \frac{(1 - Ar)}{(1 - Br)}, \quad L_i = (1 - A)(1 + Br^2) \]

and \( K_i = (1 - B)(1 + Br^2) \)

The outcome is sharp.
**Theorem 2.4.5:** Let \( f \in G(A, B) \). Then

\[
|\arg f'(z)| \leq \arcsin \frac{(A-B)r}{1-ABr^2} + \arcsin \left( \frac{b \{b^2 - (1-a)^2 \} r}{b^2 - (b^2 - a^2 + a)} \right) \frac{1}{(1-ab)^2}
\]

Proof: Since \( f \in G(A, B) \) we may write from (2.36)

\[
\left| \frac{f'(z)}{g'(z)} \right| \geq \frac{(A-B)}{1-ABr^2}, \quad g \in S(a, b).
\]

This implies

\[
\left| \arg \frac{f'(z)}{g'(z)} \right| \leq \arcsin \frac{(A-B)r}{1-ABr^2}
\]

Or

\[
|\arg f'(z)| \leq \arcsin \frac{(A-B)r}{1-ABr^2} + |\arg g'(z)|
\]

(2.46)

For the function \( g(z) \in S(a, b) \), it is known [51] that

\[
|\arg g'(z)| \leq \arcsin \left( \frac{b \{b^2 - (1-a)^2 \} r}{b^2 - (b^2 - a^2 + a)} \right) \frac{1}{(1-ab)^2}
\]

(2.47)

using (2.47) in (2.46) the result follows.

**Theorem 2.4.6:** If \( f \in G(A, B) \) then

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \begin{cases} M_1(r) & \text{for } R_1 \leq R_2 \\ M_2(r) & \text{for } R_2 \leq R_1 \end{cases}
\]

where

\[
M_1(r) = 1 - \frac{b \{b^2 - (1-a)^2 \} r}{(1-r) \{b - (b^2 - a^2 + a)\} r^2} + \frac{(A-B)r}{(1-ar)(1-br)}
\]

and

\[
M_2(r)
\]

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\[ M_z (r) = 1 - \frac{p^2 - (1 - a^2)}{(1 - r)p - (b^2 - a^2 + a)r} + \frac{(A + B)}{(A - B)} + 2 \left( \frac{L K_i^{1}}{A - B} \right) \left( 1 - A B r^2 \right) \]

with \( R_1, R_2, L_1, K_i \) are defined in the above lemma.

Proof: \( f \in G(A, B) \Rightarrow \frac{f'(z)}{g'(z)} = \frac{1 + A w(z)}{1 + B w(z)} \)

and \( g(z) \in s(a, b) \). Now substituting \( p(z) = \frac{f'(z)}{g'(z)} \) then

\[ p(z) = \frac{1 + A w(z)}{1 + B w(z)} \]. Differentiating logarithmically we obtain

\[ \frac{zp'(z)}{p(z)} - \frac{(zf'(z))'}{f(z)} = \frac{(zg'(z))'}{g(z)} \]

So

\[ \text{Re} \left( \frac{zp'(z)}{p(z)} \right) = \text{Re} \left( \frac{(zf'(z))'}{f(z)} - \frac{(zg'(z))'}{g(z)} \right) \]

It is known [51] that for \( g \in s(a, b) \)

\[ \text{Re} \left( \frac{(zg'(z))'}{g'(z)} \right) \geq 1 - \frac{p^2 - (1 - a^2)}{(1 - r)p - (b^2 - a^2 + a)r} \]

(2.48)

Thus using Lemma 2.4.4 and (2.48) we have

\[ \text{Re} \left( \frac{(zf'(z))'}{f(z)} \right) \geq 1 - \frac{p^2 - (1 - a^2)}{(1 - r)p - (b^2 - a^2 + a)r} - \frac{(A - B)r}{(1 - Ar)(1 - Mr)} \]

for \( R_1 \leq R_2 \)

\[ \geq 1 - \frac{p^2 - (1 - a^2)}{(1 - r)p - (b^2 - a^2 + a)r} + \frac{(A + B)}{(A - B)} + 2 \left( \frac{L K_i^{1}}{A - B} \right) \left( 1 - A B r^2 \right) \]

for \( R_2 \leq R_1 \).
Sharpness of the bound when $R_1 \leq R_2$ follows if we take $g_0 \in S(a, b)$ such that

$$P_0(z) = \frac{f_0'(z)}{g_0'(z)} = \frac{1 + Az}{1 + Bz} \text{ and } \left( \frac{zg_0'(z)}{g_0'(z)} \right)' = 1 - \frac{b^2 - (1 - a^2)r}{(1 - r)b - (b^2 - a^2 + a)}.$$

(2.49)

Therefore

$$\frac{zP_0'(z)}{P_0(z)} = \frac{(A - B)z}{(1 + Az)(1 + Bz)} \text{ and at } z = -r$$

$$\text{Re} \frac{zP_0''(z)}{P_0(z)} = -\frac{(A - B)r}{(1 - Ar)(1 - Br)}.$$