Chapter 5

Minus Partial Order and Rank

Additivity

The notion of dimension of a vector space over a field is well defined and it plays a significant role in the proof of Lemma 1.2.2. But, in dealing with the same lemma over a commutative ring, we feel the absence of a notion analogous to the dimension and look for different tools. Initially in this chapter, the results proved in the Chapter 2 on the regular matrices are used in extending some important results which are well established for the matrices over a field. In fact, we notice that the Lemma 1.2.2 does not remain valid for matrices over a general commutative ring, as we see that (i) ⇒ (iv) of the lemma does not hold in the case of the following example.

Example 5.0.1. Let \( A \) be any commutative ring which has an idempotent \( e \) other than 0 and 1. Then consider any \( n \times n \) diagonal matrix \( E \) with entries \( e \) on the diagonal. Clearly, \( E \) and \( (I_n - E) \) are idempotent matrices and are with determinantal rank equals to \( n \). Further, note that \( E \leq I_n \) and satisfies the condition (i) of Lemma 1.2.2, but not the condition (iv) of the same lemma.

Our concern in this chapter is to generalize the Lemma 1.2.2 to the extent possible for the class of regular matrices over commutative ring and probe when the rank additive property \( \rho(A) + \rho(B) = \rho(A +
$B$) holds whenever $A \leq^- (A + B)$.

The contents of this chapter are drawn from the article [19, Prasad, Mohana and Shenoy].

### 5.1 Minus Partial Order: Characterizations

The following theorem extends the Lemma 1.2.2 for the class of matrices over a commutative ring, except for proving the equivalence of statement (iv) with the rest.

**Theorem 5.1.1.** Let $A \in \mathbb{R}^{m \times n}$ and $B, C \in \mathbb{A}^{m \times n}$ such that $A = B + C$. Then the following statements are equivalent.

1. $B \in \mathbb{R}$ and $B \leq^- A$.
2. $B \in \mathbb{R}$ and $\{A^-\} \subseteq \{B^-\}$.
3. $B, C \in \mathbb{R}$ and both $B, C \leq^- A$.
4. $A = B \oplus C$.
5. $C(A) = C(B) \oplus C(C)$ (‘Range Summability’ condition).
6. $R(A) = R(B) \oplus R(C)$ (‘Row Space Summability’ condition).

**Proof.** (i) $\Rightarrow$ (ii). Let $B \in \mathbb{R}$ and $B \leq^- A$ satisfying (i). Let $G_B \in \{B^-\}$ such that $BG_B = AG_B$ and $G_BB = G_BA$. Therefore

\[
B = BG_BBG_B
= BG_BAG_BB
= BG_BAA^-AG_BB
= BG_BBA^-BG_BB
= BA^-B
\]
for every $A^-$. This proves (ii).

(ii) $\Rightarrow$ (iii). Let $B \in \mathfrak{R}$ and $\{A^-\} \subseteq \{B^-\}$ satisfying (ii). Since $B$ is regular over $\mathcal{A}$, from (2.2) and (2.3), we have that $\langle B \rangle$ is an ideal generated by an idempotent, say $e_B$. Note that $e_B = 0$ if and only if $B = 0$, and (ii) $\Rightarrow$ (iii) holds trivially when $B = 0$. Now, we shall consider the nontrivial case in which $e_B$ is nonzero. $\{A^-\} \subseteq \{B^-\}$ implies that $BA^-B$ is invariant under the choices of $A^-$ and equals $B$. So, from (i) $\Rightarrow$ (ii) of Theorem 2.2.2, we get that

$$\mathcal{C}(e_BB) \subseteq \mathcal{C}(A)$$

and

$$\mathcal{R}(e_BB) \subseteq \mathcal{R}(A).$$

Since $\langle e_B \rangle = \langle B \rangle$ and $e_B$ is an idempotent, we obtain $e_BB = B$ and therefore

$$\mathcal{C}(B) \subseteq \mathcal{C}(A) \quad \text{and} \quad \mathcal{R}(B) \subseteq \mathcal{R}(A). \quad (5.1)$$

Now, (5.1) implies that

$$AA^-B = BA^-A = B,$$

for every choice of $A^-$. So, substituting $A = B + C$ in $AA^-A = A$, we get

$$B + C = AA^-(B + C) = AA^-B + AA^-C.$$

Now, $AA^-B = B$ implies that

$$AA^-C = C.$$
Similarly, by considering \( BA^{-}A = B \) in

\[
B + C = (B + C)A^{-}A \\
= BA^{-}A + CA^{-}A,
\]
we get that

\[
CA^{-}A = C.
\]

Now using \( BA^{-}B = B \) from (ii) and substitute the same in \( AA^{-}B = BA^{-}A = B \), we get that

\[
CA^{-}B = BA^{-}C = 0 \quad \text{for all} \quad A^{-}. \tag{5.2}
\]

Substituting this in \( AA^{-}C = CA^{-}A = C \) we get

\[
CA^{-}C = C \quad \text{for all} \quad A^{-}.
\]

Therefore, \( C \in \mathbb{R}^{m \times n} \) and in fact, \( \{A^{-}\} \subseteq \{B^{-}\} \cap \{C^{-}\} \). Now, from (5.2) the matrix \( G_B = G_ABG_A \), where \( G_A \in \{A^{-}\} \), satisfies \( BG_BB = BG_ABG_A = B \) and \( GB_B = CG_B = 0 \). In fact, \( G_B \in \{B^{-}\} \) and satisfies the conditions

\[
G_BB = G_BA, \quad BG_B = AG_B,
\]

in other words,

\[
B \leq A.
\]

Similarly, for \( G_C = G_ACG_A \), we get

\[
G_CC = G_CA, \quad CG_C = AG_C,
\]
or

$$C \leq A,$$

thus proving (iii).

(iii) $\Rightarrow$ (iv). Given $C \leq A$, by definition, there exists a $G_C \in \{C^-\}$ such that

$$G_C C = G_C A \quad \text{and} \quad CG_C = AG_C.$$

Now, $CG_CC = C$ together with $G_CC = G_CA$ implies $CG_CB = 0$, which in turn implies that $\mathcal{C}(B)$ is in the kernel of $CG_C$ and therefore $\mathcal{C}(B) \cap \mathcal{C}(C) = (0)$. Similarly, $\mathcal{R}(B) \cap \mathcal{R}(C) = (0)$ is proved by using $CG_CC = C$ and $BG_CC = 0$. So, $A = B \oplus C$.

(iv) $\Rightarrow$ (v). Since $A$ is regular and $A = B + C$, we get that $AA^-(B + C) = B + C$ and therefore

$$AA^-B + AA^-C = B + C \quad (5.3)$$

for every choice of $A^-$. But from (iv), we have that row spaces of $B$ and $C$ are virtually disjoint and therefore (5.3) implies that $AA^-B = B$ and $AA^-C = C$. In other words, both $\mathcal{C}(B), \mathcal{C}(C) \subseteq \mathcal{C}(A)$. Since $\mathcal{C}(B)$ and $\mathcal{C}(C)$ are virtually disjoint, now we get that

$$\mathcal{C}(A) = \mathcal{C}(B) \oplus \mathcal{C}(C).$$

(v) $\Rightarrow$ (ii). From (v), we notice that $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and for each $x \in A^n$ we can find a $y \in A^n$ such that $Bx = Ay = By + Cy$. But $\mathcal{C}(B)$ and $\mathcal{C}(C)$ are virtually disjoint, and therefore $Cy = 0$ and

$$Bx = Ay = By \quad (5.4)$$
For the same reason, $AA^{-}A = A$ implies that

$$By = BA^{-}Ay$$

(5.5)

for every choice of $A^{-}$. So, from equations (5.4) and (5.5), we get that $Bx = BA^{-}Bx$ for every $x$. Therefore $BA^{-}B = B$ for every choice $A^{-}$ and this proves (ii).

The proof of (ii) $\Rightarrow$ (i) is built in the proof of (ii) $\Rightarrow$ (iii). The equivalence of (vi) with the rest follows from the fact that $B \leq^{-} A$ if and only if $B^{T} \leq^{-} A^{T}$, which is immediate from the definition of $\leq^{-}$.

Remark 5.1.2. The above Theorem 5.1.1 is well known for the matrices over a field. In general, when we consider the matrices over a commutative ring, the regularity of the matrices involved play a crucial role in the proof. For example, if we drop the condition of regularity of $A$ over a commutative ring, (iv) $\Rightarrow$ (v) need not hold always. We see the same in the following example.

Example 5.1.3. Suppose $A = \mathbb{Z}$, the ring of integers. The matrix

$$A = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}$$

is not a regular matrix. But, for

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix},$$

we have $A = B + C$ and the matrices $A, B$ and $C$ satisfy the condition (iv) of the Theorem 5.1.1. Clearly, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in $C(B)$, but not in $C(A)$. Therefore, the matrices $A, B$ and $C$ do not satisfy the condition (v) of the Theorem 5.1.1.

In fact, the above matrices $A, B$ and $C$ are typical examples which counter many arguments. Note that $B$ and $C$ are regular matrices over the ring of integers, as it may be observed that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \{B^{-}\}$ and $\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \in \{C^{-}\}$. 

\[ \]
\{C^{-}\}, but still $A$ is not regular, though, the rank additivity $\rho(B) + \rho(C) = \rho(A)$ holds.

The following theorem provides yet another characterization of matrices $B \in \mathcal{A}^{m \times n}$ such that $B \preceq A$, where $A \in \mathcal{R}^{m \times n}$.

**Theorem 5.1.4.** Let $A \in \mathcal{R}^{m \times n}$ and $B, C \in \mathcal{A}^{m \times n}$ such that $A = B + C$ and $\langle B \rangle, \langle C \rangle \subseteq \langle A \rangle$. Then $BA^{-}C = CA^{-}B = 0$ for every $A^{-}$ if and only if $B$ is regular and $B \preceq A$ (or equivalently, $C$ is regular and $C \preceq A$).

**Proof.** From Theorem 2.2.4, $BA^{-}C = CA^{-}B = 0$ for every choice of $A^{-}$ it follows that

\[
\langle C \rangle \subseteq \text{Ann}[B(I - A^{-}A)]
\]

and

\[
\langle C \rangle \subseteq \text{Ann}[(I - AA^{-})B].
\]

Therefore, for every $c \in \langle C \rangle$, we have

\[
cB(I - A^{-}A) = c(I - AA^{-})B = 0
\]

and further $CA^{-}B = 0$ implies

\[
cB = cBA^{-}A = cAA^{-}B = cBA^{-}B \tag{5.6}
\]

for every $c \in \langle C \rangle$. Similarly, we have

\[
\langle B \rangle \subseteq \text{Ann}[C(I - A^{-}A)]
\]

and

\[
\langle B \rangle \subseteq \text{Ann}[(I - AA^{-})C],
\]
and therefore \( BA^{-}C = 0 \) implies

\[
bC = bCA^{-}A = bAA^{-}C = bCA^{-}C \quad (5.7)
\]

for every \( b \in \langle B \rangle \). The equation (5.7) together with

\[
bB + bC = bA = bAA^{-}A = bBA^{-}B + bCA^{-}C \quad (\because \; BA^{-}C = CA^{-}B = 0)
\]

yields \( bBA^{-}B = bB \), which together with (5.6) gives us

\[
(b + c)BA^{-}B = (b + c)B \quad (5.8)
\]

for all \( b \in \langle B \rangle, c \in \langle C \rangle \). Note that \( A = B + C \) implies \( \langle A \rangle \subseteq \langle B \rangle + \langle C \rangle \), and \( A \in \mathfrak{H}^{m \times n} \) implies that there exists an idempotent \( e_A \) such that \( \langle A \rangle = \langle e_A \rangle \). Now, \( \langle B \rangle, \langle C \rangle \subseteq \langle A \rangle \) implies \( \langle B \rangle + \langle C \rangle = \langle A \rangle = \langle e_A \rangle \).

Therefore \( e_A = b_1 + c_1 \) for some \( b_1 \in \langle B \rangle, c_1 \in \langle C \rangle \) and further \( e_A B = B \). So, by substituting \( b = b_1 \) and \( c = c_1 \) in (5.8), we get \( BA^{-}B = B \).

Now, the result \( B \leq^{-} A \) follows from Theorem 5.1.1.

Conversely, let \( BA^{-}B = B \) for every choice of \( A^{-} \). Now referring to (i) \( \Rightarrow \) (v) of Theorem 5.1.1, we get that \( \mathcal{C}(B) \subseteq \mathcal{C}(A) \). Similarly from (i) \( \Rightarrow \) (vi) of the same theorem, we have \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \). Therefore \( B = BA^{-}B = AA^{-}B = BA^{-}A \). Hence \( BA^{-}C = CA^{-}B = 0 \), proving the theorem.

In the following theorem, we give a more elegant necessary and sufficient condition for \( B \leq^{-} A \) when it is known that \( B \in \mathfrak{H}^{m \times n} \).

**Theorem 5.1.5.** Let \( A, B, C \in \mathfrak{H}^{m \times n} \) and \( A = B + C \). Then the following statements are equivalent.

(i) \( BA^{-}C = 0 \) for every \( A^{-} \).

(ii) \( B \leq^{-} A \).
(iii) $CA^{-}B = 0$ for every $A^{-}$.

(iv) $C \preceq A$.

Proof. (i) $\Rightarrow$ (ii). First, we shall prove that $\langle B \rangle \subseteq \langle A \rangle$. Taking the contrary view let us assume that $\langle B \rangle$ is not a subset of $\langle A \rangle$. Since $A$ and $B$ are regular, there exist idempotents $e_A$ and $e_B$ such that $\langle A \rangle = \langle e_A \rangle$ and $\langle B \rangle = \langle e_B \rangle$. Then $e_1 = e_B(1 - e_A)$ is a nonzero idempotent, otherwise, we would get $e_B = e_A e_B$ which suggests

$$\langle B \rangle = \langle e_B \rangle \subseteq \langle e_A \rangle = \langle A \rangle,$$

a contradiction. Clearly, $e_1 B$ is a nonzero matrix, otherwise,

$$e_1 B = e_B (1 - e_A) B = (1 - e_A) B = 0$$

which implies $B = e_A B$, in other words, $\langle e_B \rangle \subseteq \langle e_A \rangle$, again a contradiction. But at the same time, we have $e_1 A = e_B (1 - e_A) A = 0$. So, we have

$$e_1 B = -e_1 C 
eq 0. \quad (5.9)$$

If $G_1$ is any reflexive $g$-inverse of $A$ then $A G_1 A = A$ and $G_1 A G_1 = G_1$, and therefore $\langle G_1 \rangle = \langle A \rangle = \langle e_A \rangle$. From (iii) of Theorem 2.1.5, we get that $e_1 B = -e_1 C$ is a regular matrix and $\langle e_1 B \rangle = \langle e_1 C \rangle = \langle e_1 \rangle$. Now, there exists a matrix $Z \in A^{n \times m}$ such that the entries $(Z)_{ij} \in \langle e_1 \rangle$ and $B Z C$ is a nonnull matrix, otherwise, $(B)_{ki} (C)_{jl} = 0$ for all $i, j, k, l$ which contradicts the fact that $e_1 \in \langle B \rangle \cap \langle C \rangle$. Since $(Z)_{ij} \in \langle e_1 \rangle$, where $e_1 = e_B (1 - e_A)$, we get $Z A = 0$. So, $G = G_1 + Z$ is a $g$-inverse of $A$ but $B G C = B G_1 C + B Z C \neq B G_1 C$. This contradicts the invariance of $B A^{-} C$. So, $\langle B \rangle \subseteq \langle A \rangle$ and hence $e_B e_A = e_B$. 
Now, let $e_C$ be an idempotent such that $\langle e_C \rangle = \langle C \rangle$ and define $e = e_B e_C, E = eB, F = eC$ and $D = eA$. From the definition of $e$, it is clear that $\langle E \rangle = \langle F \rangle = \langle e \rangle$. Therefore from Theorem 2.2.2, invariance of $BA^\sim C$ gives that

$$\mathcal{R}(E) \subseteq \mathcal{R}(A). \quad (5.10)$$

Noting that $(1 - e_C)A = (1 - e_C)B$, we get $\mathcal{R}((1 - e_C)B) \subseteq \mathcal{R}(A)$. Therefore (5.10) implies that $C(B') \subseteq C(A')$, which in turn gives $BA^\sim A = B$ for every $A^\sim$. From (i) we have $BA^\sim C = 0$, and therefore $BA^\sim A = B$ implies $BA^\sim B = B$ for all $A^\sim$. In other words, $B \leq^\sim A$.

(ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are in the similar lines of the ‘if part’ of Theorem 5.1.4 and (iii) $\Rightarrow$ (iv) is similar to (i) $\Rightarrow$ (ii).

**Remark 5.1.6.** Hartwig in his paper [9, Hartwig], presented an open problem:

*If $a = b + c$ in a regular ring $R$ satisfies the condition $bR \cap cR = (0) = Rb \cap Rc$, then does there exists a $b^-$ in $R$ such that $b^-c = 0$ and $cb^-$?*

We generalize the above problem in context of matrices over a commutative ring, ignoring the regularity of ring, as given below:

*If $A = B + C$ is a regular matrix over a commutative ring satisfies the condition $A = B \oplus C (C(B) \cap C(C) = (0))$ and $\mathcal{R}(B) \cap \mathcal{R}(C) = (0)$, then does there exists a $B^-$ such that $B^-C = 0$ and $CB^- = 0$.*

We have an affirmative answer to the above question, in the context of Theorem 5.1.1, Theorem 5.1.5 and by choosing $B^- = A^\sim BA^\sim$ for any choice of $A^\sim$. 
5.2 Rank Additivity

As we have already seen in the Example 5.0.1 that the matrices \( B \) and 
\((A - B)\) need not satisfy the ‘rank condition’, even though \( B \preceq\ A \).  
This observation demands a suitable exploration of extension for the  
known rank condition, when we consider the regular matrices over  
a commutative ring. In the following Theorem 5.2.1, we will address  
this problem.

**Theorem 5.2.1.** Let \( A \in \mathbb{R}^{m \times n} \) and \( B, C \in \mathbb{R}^{m \times n} \) such that \( A = B + C \). Then the following statements are equivalent.

(i) \( B \in \mathbb{R} \) and \( B \preceq\ A \).

(ii) There exist orthogonal idempotents \( e_1, \ldots, e_k \) such that \( \sum e_i = 1 \) and for \( 1 \leq i \leq k \), the matrices defined by \( A_i = e_i A, B_i = e_i B, \) and \( C_i = e_i C \) satisfy \( B_i \preceq\ A_i \) and each of \( A_i, B_i, C_i \) is a Rao–regular matrix with Rao–idempotent \( e_i \) or 0.

(iii) There exist orthogonal idempotents \( e_1, \ldots, e_k \) such that \( \sum e_i = 1 \) and for \( 1 \leq i \leq k \), the matrices defined by \( A_i = e_i A, B_i = e_i B \) and \( C_i = e_i C \) satisfy rank additive condition \( \rho(A_i) = \rho(B_i) + \rho(C_i) \) and \( A_i \) is a Rao–regular matrix with \( I(A_i) = e_i \) or 0.

We require the following lemmas to address the above theorem.

**Lemma 5.2.2.** Let \( A, B, C \in \mathbb{R}^{m \times n} \). Then there exist orthogonal idempotents \( e_1, e_2, \ldots, e_r \) such that

(i) \( e_1 + \cdots + e_r = 1 \) and

(ii) each of \( e_i A, e_i B \) and \( e_i C \) is a Rao–regular matrix with Rao–idempotent equals zero or \( e_i \).

*Proof.* Let \( A, B, C \) be regular matrices. Referring Theorem 1.1.8, we observe that there exists a set of orthogonal idempotents \( \{f_0, f_1, \ldots, f_r\} \)
such that \( \sum f_i = 1 \) and \( f_i A \) is a Rao–regular matrix of rank \( i \) and with \( \mathcal{I}(f_i A) = 0 \) or \( f_i \) for every \( i, 1 \leq i \leq r_1 \). Similarly, consider the sets of orthogonal idempotents \( \{g_0, g_1, \ldots, g_{r_2}\} \) and \( \{h_0, h_1, \ldots, h_{r_3}\} \) for \( B \) and \( C \), respectively. Now observe that \( \{f_i g_j h_k\} \) is a finite set of orthogonal idempotents, where \( 0 \leq i \leq r_1, 0 \leq j \leq r_2 \) and \( 0 \leq k \leq r_3 \), and from (i) of Theorem 2.1.5 we get that each of \( (f_i g_j h_k) A, (f_i g_j h_k) B \) and \( (f_i g_j h_k) C \) is a Rao–regular matrix with Rao–idempotent equals either 0 or \( (f_i g_j h_k) \). Here, \( r = r_1 r_2 r_3 \).

**Lemma 5.2.3.** Let \( A \in R^{m \times n} \) and \( B, C \) be Rao–regular matrices such that \( A = B + C, \mathcal{I}(B) = \mathcal{I}(C) \) and \( B, C \leq A \). Then \( \rho(A) = \rho(B) + \rho(C) \).

In fact, in such a case \( A \) is also Rao–regular with \( \mathcal{I}(A) = \mathcal{I}(B) = \mathcal{I}(C) \).

**Proof.** Note that \( A = B + C = (B C) \begin{pmatrix} I \\ I \end{pmatrix} \) and therefore

\[
\rho(A) \leq \rho(B C) \leq \rho(B) + \rho(C). \tag{5.11}
\]

From Theorems 5.1.1 and 5.1.5, \( B \leq A \) implies that \( BA^\perp B = B, CA^\perp C = C, BA^\perp C = 0 \) and \( CA^\perp B = 0 \). So, for \( P = \begin{pmatrix} B & C \end{pmatrix} \)

and \( Q = \begin{pmatrix} A^\perp \\ A^\perp \end{pmatrix} \) we get that \( PQP = P \) for every choice of \( A^\perp \).

By choosing a reflexive g-inverse of \( A^\perp \) to define \( Q \), we get that \( \rho \left( \begin{pmatrix} B & C \end{pmatrix} \right) \leq \rho(Q) = \rho(A) \) and therefore \( \rho(A) = \rho \left( \begin{pmatrix} B & C \end{pmatrix} \right) \). Also, note that

\[
\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -BA^\perp & I \end{pmatrix} \begin{pmatrix} B & C \\ B & C \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},
\]

where the first two factors on the LHS are invertible. Let the ranks of \( B \) and \( C \) be \( r_1 \) and \( r_2 \), respectively. Since \( B \) and \( C \) are Rao–regular
matrices with same Rao–idempotent, say $e \neq 0$, then $\mathcal{I}[C_{r_1}(B)] = \mathcal{I}[C_{r_2}(C)] = e$ for some idempotent $e$. So, $e$ can be represented by a linear combination of $r_1 \times r_1$ minors of $B$, and similarly it can be represented by a linear combination of $r_2 \times r_2$ minors of $C$. So, $eC_{r_1}(B) = C_{r_1}(B)$ and $eC_{r_2}(C) = C_{r_2}(C)$ implies that there exist $|B_{\beta \delta}^\alpha|$ and $|C_{\delta \beta}^\gamma|$, a pair of $r_1 \times r_1$ minor of $B$ and $r_2 \times r_2$ minor of $C$ respectively, such that $|B_{\beta \delta}^\alpha||C_{\delta \beta}^\gamma| \neq 0$. This implies that $\rho\left(\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\right) = \rho(B) + \rho(C)$, and hence

$$\rho(A) = \rho(BC) = \rho\left(\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\right) = \rho(B) + \rho(C).$$

The second part of the lemma is easily proved by observing that

$$\rho(eA) = \rho(eB) + \rho(eC), \quad \rho(eB) = \rho(B), \quad \rho(eC) = \rho(C)$$

for every idempotent $e$ such that $e\mathcal{I}(B) = e\mathcal{I}(C) \neq 0$, and

$$fa = fB = fC = 0$$

for every idempotent $f$ such that $f\mathcal{I}(B) = f\mathcal{I}(C) = 0$.

**Lemma 5.2.4.** Let $A \in \mathcal{R}_{m \times n}$ be a Rao–regular matrix with $\mathcal{I}(A) = e$ and $B, C \in \mathcal{A}_{m \times n}$ such that $A = B + C$, $\langle B \rangle, \langle C \rangle \subseteq \langle e \rangle$ and $\rho(A) = \rho(B) + \rho(C) = r$. Then the following statements are true.

(i) Both $B$ and $C$ are Rao–regular with Rao–idempotent equals to either $e$ or $0$.

(ii) Let $p, q, s, t$ be any nonnegative integers such that $p + s = q + t = r + 1$, $\alpha \in \mathcal{Q}_{p,m}$, $\beta \in \mathcal{Q}_{q,n}$, $\gamma \in \mathcal{Q}_{s,m}$ and $\delta \in \mathcal{Q}_{t,n}$. Define $T = \begin{bmatrix} B_{\beta \delta}^\alpha & C_{\delta \beta}^\gamma \\ B_{\beta \delta}^\gamma & 0 \end{bmatrix}$ and $S = \begin{bmatrix} B_{\beta \delta}^\alpha & B_{\delta \beta}^\gamma \\ C_{\delta \beta}^\gamma & 0 \end{bmatrix}$. Then $|T| = |S| = 0$.  

(iii) Let $I \in \mathbb{Q}_{r,m}$ and $J \in \mathbb{Q}_{r,n}$. For any $i, j, 1 \leq i \leq m, 1 \leq j \leq n$, define $P_1 = \begin{pmatrix} A_{i}^{I} & A_{j}^{I} \\ B_{i,j}^{(i)} & B_{i,j} \end{pmatrix}$, $P_2 = \begin{pmatrix} A_{i}^{J} & B_{j}^{(i)} \\ A_{j}^{(i)} & B_{i,j} \end{pmatrix}$, $P_3 = \begin{pmatrix} A_{i}^{(i)} & B_{j}^{(i)} \\ B_{i,j}^{(i)} & B_{i,j} \end{pmatrix}$, $Q_1 = \begin{pmatrix} A_{i}^{I} & C_{j}^{I} \\ B_{i,j}^{(i)} & b_{i,j} \end{pmatrix}$, $Q_2 = \begin{pmatrix} A_{i}^{J} & B_{j}^{I} \\ C_{j}^{(i)} & b_{i,j} \end{pmatrix}$. Then $|P_1| = |P_2| = |P_3| = |Q_1| = |Q_2| = 0$.

(iv) $B \leq -A$.

Proof. (i). If any of $B$ and $C$ is a null matrix, then (i) holds trivially.

For nontriviality, let us assume that $A, B, C$ are nonzero matrices and, let $B$ and $C$ be matrices of rank $r_1$ and $r_2$ respectively. From the definition of Rao–idempotent, $I(A)$ is a linear combination of $r \times r$ minors of $A$. Since $\rho(A) = \rho(B) + \rho(C)$, noting that determinant is a multilinear function on columns (rows) of a square matrix and $A = B + C$, we get that every $r \times r$ minor of $A$ is a linear combination of determinant of a $r \times r$ matrix with $r_1$ columns from $B$ and $r_2$ columns of $C$. So by using Laplace expansion, we get that $r \times r$ minor of $A$ is a linear combination of $r_1 \times r_1$ minors of $B$ as well as a linear combination of $r_2 \times r_2$ minors of $C$. Since $\langle B \rangle \subseteq \langle e \rangle$, now we get that $\langle B \rangle = \langle C_{r_1}(B) \rangle = \langle e \rangle$ and therefore $B$ is a Rao–regular with $I(B) = e$. Similarly, $C$ is a Rao–regular with $I(C) = e$.

(ii). From the definition of $T$ and $S$, it is clear that $\rho(T), \rho(S) \leq \rho(B) + \rho(C) = r$. Since $T$ and $S$ are matrices of size $(r + 1) \times (r + 1)$, trivially $|T| = |S| = 0$.

(iii). Proof follows immediately from the fact that $\rho(B) + \rho(C) = r$ and by writing $A_{i}^{T} = B_{i}^{T} + C_{i}^{T}, A_{j}^{I} = B_{j}^{I} + C_{j}^{I}$ and $A_{i}^{(i)} = B_{i}^{(i)} + C_{i}^{(i)}$. 

(iv). Since $A$ is a Rao–regular matrix with $\mathcal{I}(A) = e$, there exist $c_I^j \in A$ such that $\sum_{I,J} c_I^j |A_J^I| = e$ and a matrix $G = (g_{kl})$ obtained by

$$g_{kl} = \sum_{I,J} c_I^j \frac{\partial}{\partial a_{ik}} |A_J^I|$$

(5.12)

is a g-inverse of $A$. First, we shall prove that the $G$ obtained by (5.12) is a g-inverse of $B$ as well. From the determinant expansion of $|P_3|$ which equals zero, we get that

$$b_{ij} |A_J^I| = \sum_{l \in I, k \in J} b_{lj} b_{ik} \frac{\partial}{\partial a_{ik}} |A_J^I|.$$  

(5.13)

Multiplying both the sides of (5.13) by $|c_I^j|$, and taking sum over $I$ and $J$, we get

$$eb_{ij} = \sum_{I,J} c_I^j \sum_{l \in I, k \in J} b_{lj} b_{ik} \frac{\partial}{\partial a_{ik}} |A_J^I|$$

(5.14)

$$= \sum_{l,k} b_{lk} b_{ij} \sum_{I \in I, J \in J} c_I^j \frac{\partial}{\partial a_{ik}} |A_J^I|.$$  

(5.15)

Hence from (5.15) we get that $BGB = eB = B$ and $G$ is a g-inverse of $B$.

Similarly, from the determinant expansion of $|P_1|, |P_2|, |R_1|$ and $|R_2|$ we get that $BGA = B$, $AGB = B$, $BGC = 0$ and $CGB = 0$ respectively. Note that

$$BGA = A \quad BG = 0 \Rightarrow BGB = B.$$  

Further,

$$AGB = BGA = B \Rightarrow \mathcal{C}(B) \subseteq \mathcal{C}(A) \text{ and } \mathcal{R}(B) \subseteq \mathcal{R}(A).$$
Now, referring to Theorem 2.2.2, we obtain that $BA^{-}B$ is invariant under the choice of $A^{-}$ and equals $BGB = B$. Hence $B \preceq A$.

Now we shall prove Theorem 5.2.1.

Proof. [Theorem 5.2.1.] (i) $\Rightarrow$ (ii) of the theorem is immediate from Lemma 5.2.2 and ‘only if’ part of Theorem 2.1.5 (vi). The (ii) $\Rightarrow$ (i) part of the theorem is a consequence of ‘if part’ of the Theorem 2.1.5 (vi). The part (ii) $\Rightarrow$ (iii) is immediate from the Lemma 5.2.3. Lemma 5.2.4 (iv) proves the (iii) $\Rightarrow$ (ii) of the theorem.

Remark 5.2.5. Suppose $B \preceq A$. Over a field, it is well known that $A = B$ if and only if $\rho(A) = \rho(B)$. But over an arbitrary commutative ring with nontrivial idempotent this is not necessarily true. From Theorem 5.2.1, it can be observed that $A = B$ if and only if $\rho(eA) = \rho(eB)$ for every idempotent $e \in A$.

Consider the matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$ over $\mathbb{Z}_{12}$. All the three matrices $A$, $B$ and $C$ are idempotent matrices and also Rao–regular, but with different Rao–idempotents, where $\mathcal{I}(A)$, $\mathcal{I}(B)$ and $\mathcal{I}(C)$ are 1, 4 and 9 respectively. We have already seen that these matrices do not satisfy the condition

$$\rho(A) = \rho(B) + \rho(C),$$

though we have that

$$A = B + C$$

and

$$B, C \preceq A.$$
Consider the idempotents $4, 9 \in \mathbb{Z}_{12}$ such that

$$4 + 9 \equiv 1.$$ 

Now, for $e_1 = 4, e_2 = 9$ we have that $e_1 A = B$ and $e_2 A = C$. Further, the nonzero matrices among $e_1 A, e_1 B$ and $e_1 C$ are with same Rao–idempotents and

$$\rho(e_i A) = \rho(e_i B) + \rho(e_i C).$$

Hence the matrices $A, B$ and $C$ satisfy the condition (ii) of Theorem 5.2.1.

5.3 Conclusion

Noticing that the well-known ‘rank-additive’ property fails to hold good for the matrices over a commutative ring satisfying $A = B + C$ and $B, C \leq A$, we obtain an alternative rank addition property. In fact, we have proved that the well known rank additive property by considering determinantal rank if the matrices $A, B$ and $C$ are Rao–regular with same Rao–idempotents.

In the process, we extend several characterizations of such matrices $A, B$ and $C$ which generally hold when the commutative ring is field, by providing some algebraic proof which are independent of notions such as column rank and dimension of subspace. We also resolve an open problem proposed by Hartwig in [9, Hartwig] in the more general context where the elements are the matrices and not every matrix is regular.