CHAPTER - 2

RELIABILITY BASED DESIGN USING FEM

2.1 INTRODUCTION

A mechanical or structural component is considered to have failed when it ceases to function properly for its intended use. Reliability analysis in structural engineering recognizes that both loads and strengths have statistical frequency distributions that must be considered in evaluating safety. In designs based on reliability, a structure is designed for a specified probability of failure depending on the type of structure and consequences of failure. Through the probability of failure, the possibility of failure of the structure is admitted and quantified. The uncertainties in the design parameters are taken into account through probabilistic descriptions of the loads and system parameters.

In the present chapter, deterministic design (conventional design) and Classical Reliability Based design (CRBD) are presented. A computational procedure for the reliability based design (for structural element/ machine component) using Finite Element Method (FEM) is proposed and presented in this chapter. Two structural elements (Cantilever beam and simply supported beam) and a machine component (circular shaft) are considered for the design.
Results obtained using the above three methods for the cross-section dimensions are presented and compared. Variation in cross-section area with the variation in reliability index (β) and co-efficient of variation of random variables are studied.

2.1.1 Stress—strength interference method

The stress-strength interference method is one of the methods of structural reliability analysis and is a popular method of reliability analysis among practicing engineers in many industries. The attractiveness of the method lies in its simplicity, ease and economy. Within the context of stress-strength interference method, failure is said to have occurred if the stress (load) exceeds the strength (capacity). Failure probability or unreliability is the probability that the stress is greater than the strength. The stress-strength interference method may be used in conjunction with a variety of failure modes such as yielding, buckling, fracture and fatigue. The term "stress" should be considered in a broader sense as any applied load or load-induced response quantity that has the potential to cause failure. The term "strength" should be considered in a broader sense as the capacity of the component or system to withstand the applied load ("Stress"). The method is also termed as "load-capacity interference method", instead of "Stress-strength interference method", to indicate the broader scope of the method.
Computation of the reliability of a structural component/machine element requires knowledge of random nature of the strength (R) or resistance and the stress (S) or load. Let the strength (R) and the stress (S) be known to follow normal distribution with density functions as

\[ f_R(r) = \frac{1}{\sigma_R \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{r - \mu_R}{\sigma_R} \right)^2} \]

and

\[ f_S(s) = \frac{1}{\sigma_S \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{s - \mu_S}{\sigma_S} \right)^2} \] (2.1)

Where \( \mu_R \) and \( \mu_S \) denote the mean values, and \( \sigma_R \) and \( \sigma_S \) represent the standard deviations of variables R and S respectively.

A structural component/machine element is said to fail when the resistance of the structural component/machine element is less than the load (stress). Then the probability of failure of the structural component/machine element can be expressed as

\[ P_f = P(R < S) \]

\[ = P(R - S < 0) \] (2.2)

Where \( P_f \) is the probability of failure of the structural component/machine element. (Reliability of a component is one minus the probability of failure).

By introducing a new random variable \( U \), as

\[ U = R - S \] (2.3)

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When $R$ and $S$ are independent and normally distributed, then $U$ is also normally distributed with mean value $\mu_U$ and standard deviation $\sigma_U$ given by

$$\mu_U = \mu_R - \mu_S$$

and

$$\sigma_U = \left(\sigma_R^2 + \sigma_S^2\right)^{1/2}$$  \hspace{1cm} (2.4)

The failure probability is given by

$$P_f = P(U < 0)$$

$$= \int_{-\infty}^{0} f_U(u) \, du = F_U(0)$$  \hspace{1cm} (2.5)

where $f_U(u)$ is the probability density function of $U$, and is known as the difference distribution. $F_U(0)$ is the cumulative distribution function of $U$ at $U = 0$.

Then

$$P_f = \Phi\left(\frac{0 - \mu_U}{\sigma_U}\right)$$

$$= \Phi\left(\frac{0 - (\mu_R - \mu_S)}{\left(\sigma_R^2 + \sigma_S^2\right)^{1/2}}\right)$$  \hspace{1cm} (2.6)

Let

$$\beta = \frac{\mu_U}{\sigma_U} = \frac{\mu_R - \mu_S}{\left(\sigma_R^2 + \sigma_S^2\right)^{1/2}}$$  \hspace{1cm} (2.7)

then the value of $P_f$ corresponding to $\beta$ is given by

$$P_f = \Phi(-\beta)$$  \hspace{1cm} (2.8)

and the value $\beta$ corresponding to a given $P_f$ is

$$\beta = -\Phi^{-1}(P_f)$$  \hspace{1cm} (2.9)

the parameter $\beta$ is known as the reliability index and $\Phi(-\beta)$ is the cumulative distribution function of the standard normal variable. Value
of this function $\phi (x)$ for various values of $x$ can be obtained from the standard normal tables.

Since $R$ and $S$ generally depend on several other random design parameters, one has to determine $\mu_R$, $\mu_S$, $\sigma_x$ and $\sigma_y$ in terms of the means and standard deviations of the random design parameters. In general if $Y$ is a nonlinear function of several random variables $X_1, X_2, \ldots, X_n$ the approximate values of the mean and variance of $Y$ can be found by linearizing $Y$ about the mean values of $X_1, X_2, \ldots, X_n$ using Taylor's series expansions. The equations for $\mu_Y$ and $\sigma_Y$ are given by

$$
\mu_Y \approx Y(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n})
$$

and

$$
\sigma_Y \approx \left\{ \sum_{i=1}^{n} \left[ \frac{\partial Y}{\partial X_i} \right]_{(\mu_{X_1}, \mu_{X_2}, \ldots, \mu_{X_n})}^2 \sigma_{X_i}^2 \right\}^{\frac{1}{2}}
$$

(2.10)

Where the partial derivatives are to be evaluated at the mean values of random variables and are assumed to have no correlation.

Representing mean and standard deviation of a random variable $x$ as $\bar{x}$ and $\sigma_x$ respectively, the reliability index ($\beta$) is expressed as

$$
\beta = \frac{\bar{R} - \bar{S}}{\left[ \sigma_R^2 + \sigma_S^2 \right]^{\frac{1}{2}}}
$$

(2.11)

In engineering practice reliability index $\beta$ (safety index), is used to represent the reliability level. The selection of $\beta$ is problem dependent.
A commonly used value of $\beta = 3.0$, corresponding to a reliability of 0.99865 (a probability of failure of 0.00135) for a normally distributed stress and strength. In general higher values of $\beta$ correspond to higher reliability levels.

For the specified reliability (using the corresponding value of $\beta$), by expressing the mean and standard deviation of stress in terms of design variables, it is possible to determine the cross-section dimensions of a structural element/machine component.

2.1.2 Deterministic Design (Conventional Design) (DD)

Mean values of the random variables are considered for the deterministic design. Analytical expressions are used to express induced stress in terms of design variables. As it is possible to determine only one design variable using deterministic design, a ratio between the design variables is to be considered. By using an appropriate factor of safety, design variables are determined using deterministic design procedure. It is assumed that there is no change in the cross-section area of the structural element/machine component along its length in all the three design procedures.

2.1.3 Classical Reliability Based Design (CRBD)

In the present work, probability of failure of the structural element/machine component is defined as the probability that maximum induced bending stress in the structural element/machine
component exceeds the strength of the material. Hence reliability is a function of material strength and the external loads acting on the structural element/machine component. The external load, material strength and induced stresses are assumed as normally distributed random variables. It is assumed that stress is independent of the strength. The design parameters are cross-section dimensions of the structural element/machine component. It is also assumed that manufacturing tolerances on cross-section dimensions are very small and are treated as deterministic design variables. Material properties like young’s modulus (E), Poisson’s ratio (ν) and the length of the structural element/machine component are considered as deterministic.

Analytical expressions in terms of design variables are used to express induced stresses. Mean and standard deviation of induced stress can be obtained using the algebra of independent normal distribution functions [4, 91] or by using partial derivative rule [90].

The reliability index (β) in terms of the expected values and the standard deviation of random variables Ys and Sb is

$$
\beta = \frac{Y_s - S_b}{\left(\sigma_{ys}^2 + \sigma_{sb}^2\right)^{1/2}}
$$

(2.12)

where $Y_s, \sigma_{ys}$ and $S_b, \sigma_{sb}$ are the mean and standard deviation of yield strength and induced stress respectively. Equation (2.12) can be solved
for the specified reliability by substituting the values of $\bar{S}_x$ and $\sigma_s$ in terms of design variables and reducing it to a single variable equation.

In the present work co-efficient of variation of random variables is considered as 0.1. The co-efficient of variation is a measure of dispersion in non-dimensional form and is defined as the ratio of standard deviation to mean value.

The Co-efficient of variation of random variable $x = C_x$

and $C_x = (\text{Standard deviation of } x / \text{Mean of } x) = \frac{\sigma_x}{\bar{x}}$

### 2.1.4 Reliability Based Design using Finite Element Method (RBDF)

The analysis and design of large realistic structures requires computer based numerical procedures, such as Finite Element Analysis. Finite Element Method is used widely in the field of structural analysis. With this approach it is easy to analyze complicated structures considering their behavior as realistically as possible.

Assuming that the induced stresses and the strength of the material follow normal distribution, the reliability index using stress-strength interference method is

$$
\beta = \frac{\bar{Y}_r - \bar{S}_b}{(\sigma^2_{\text{ass}} + \sigma^2_{\text{sb}})^{\frac{1}{2}}}
$$

(2.12)

Mean value and standard deviation of induced stress in the above equation are found using partial derivative rule. Induced stresses are
calculated using finite element method. Partial derivatives are evaluated at the mean values of random variables using finite difference method. The formulated problem is solved for the cross-section dimensions of the structural element/machine component using Newton Raphson method for the specified reliability using single design variable. Steps followed in the RBDF procedure are presented in a flow chart shown in Fig.2.14.

2.2. DESIGN OF A CANTILEVER BEAM

A cantilever beam with rectangular cross-section subjected to a vertical point load $P_y$ at the tip as shown in Fig.2.1 is considered for the design. It is assumed that there are variations in vertical load $P_y$ and yield strength of the material $Y_s$, and the variations in them follow normal distribution. Statistics of load and yield strength are given in Table2.1. Material properties, length of the beam and other details are given in Table 2.2.

![Diagram of Cantilever Beam](image)

*Fig. 2.1. Idealized Cantilever Beam*
2.2.1 Deterministic Design

When a cantilever beam of rectangular cross section of length ‘L’ as shown in Fig. 2.1 is subjected to an end point load $P_y$, then the maximum bending stress ($S_b$) at the fixed end is

$$S_b = \frac{M}{(I_x/y)}$$  \hspace{1cm} (2.13)

Where

Bending moment at the fixed end $M = P_y \cdot L$

Moment of inertia about x-axis $I_x = \frac{wb^3}{12}$

Distance of top layer from the neutral axis $y = \frac{b}{2}$

substituting these values in equation (2.13)

$$S_b = \frac{6L}{wb^3} P_y$$

substituting $w = 0.75b$ in the above equation.

$$S_b = \frac{8L}{b^3} P_y$$  \hspace{1cm} (2.14)

by introducing a factor of safety (n) equation (2.14) can be used to determine 'b' as

$$b = \left( \frac{8L}{P_y} \left( \frac{n}{Y_s} \right) \right)^{1.3}$$  \hspace{1cm} (2.15)

where $Y_s = $ Yield strength in bending

Considering an appropriate value for the factor of safety (n) we can determine the cross-sectional dimensions of the beam using equation (2.15). Deterministic design results obtained by using mean values of
random variables and a factor of safety of 1.5 are presented in Table 2.3.

2.2.2 Classical Reliability Based Design (CRBD)

In the classical reliability based design, analytical expressions in terms of design variables are used to express induced bending stress. Analytical equation for the bending stress at the fixed end of the beam in terms of a single design variable is

\[ S_b = \frac{8L}{b^3} P_r \]  

(2.14)

Mean and standard deviations of bending stress can be obtained using the algebra of independent normal distribution functions as

\[ \bar{S}_b = \frac{8L}{b^3} \bar{P}_r \]

and

\[ \sigma_{ab} = \frac{8L}{b^3} \sigma_{P_r} \]  

(2.16)

Assuming that both the induced stress and strength distributions follow normal distribution and are independent of each other, the reliability index (\( \beta \)) in terms of the expected values and the standard deviations of random variables \( Y_s \) and \( S_b \) is

\[ \beta = \frac{\bar{Y}_s - \bar{S}_b}{\left(\sigma_{Y_s} + \sigma_{S_b}\right)^{\frac{1}{2}}} \]  

(2.12)

Substituting the values of \( \bar{S}_b \) and \( \sigma_{S_b} \) in the equation (2.12) and by rearranging the equation we get
Equation (2.17) is a quadratic in $b'$ and is solved for the dimensions of the beam.

\[ \left( \beta^2 \sigma_{y}^2 - \overline{Y_s}^2 \right) b' + 16\overline{Y_s}L \overline{P} b' + 64L^2 \left( \beta^2 \sigma_{y}^2 - \overline{P}^2 \right) = 0 \]  

(2.17)

2.2.3 Reliability Based Design using Finite Element Method

(RBDF)

Cantilever beam is idealized into five beam elements with six degrees of freedom at each node as shown in Fig.2.1. Encircled numbers indicate the element number and the free numbers indicate the node numbers. A vertical point load is applied at node 1. To fix the beam at the end all the degrees of freedom at the node 6 are arrested. Elimination approach is used to impose the boundary conditions. Formulated simultaneous equations are solved using Gaussian elimination method for the nodal displacements. Elemental stresses are found using nodal displacements.

Because of the uncertainties in load $P_y$ and yield strength $Y_s$, maximum induced bending stress $S_b$ is also a random variable. Assuming that the bending stress induced in the beam follows normal distribution, the reliability index using stress-strength interference criteria for a normally distributed stress and strength is

\[ \beta = \frac{\overline{Y_s} - \overline{S_b}}{\left( \sigma_{y}^2 + \sigma_{b}^2 \right)^{1/2}} \]  

(2.12)
Where $Y_s$, $\sigma_Y$, and $S_b$, $\sigma_{S_b}$ are the mean and standard deviation of yield strength and bending stress respectively. The standard deviation of $S_b$ can be found from equation (2.10) using the partial derivative rule as

$$
\sigma_{S_b} = \left( \frac{\partial S_b}{\partial P} \right)^2 \sigma_P^2
$$

(2.18)

Where the partial derivatives are to be evaluated at the mean values of random variables. Maximum bending stress ($S_b$) at the fixed end is calculated using finite element method and the partial derivatives for the calculation of standard deviation of bending stress $\sigma_{S_b}$ are found using finite difference method. The problem is solved for the cross section dimensions of the beam using Newton-Raphson method for the specified reliability.

2.3 DESIGN OF A SIMPLY SUPPORTED BEAM

A simply supported beam with hollow circular cross-section subjected to a central point load $P_y$ as shown in Fig 2.2 is considered for the design. It is assumed that there are variations in vertical load $P_y$ and yield strength of the material $Y_s$, and the variations in them follow normal distribution. Statistics of load and yield strength are given in Table 2.1. Material properties, length of the beam, and other details are given in Table 2.2. Cross-section dimensions of the beam ($d_0$ & $d_1$) are considered as deterministic design variables.
2.3.1 Deterministic Design

When a simply supported beam of hollow circular cross-section with a span length \( L \) as shown in Fig.2.2 is subjected to a central point load \( P_y \) then the maximum bending stress \( S_{y0} \) in the extreme fiber at the middle of the span is

\[
S_{y0} = \frac{M_{y}}{I_{w}}
\]  

(2.19)

where

- Bending moment at the center of the span \( M = \frac{P_y L}{4} \)
- Moment of inertia about x-axis \( I_{w} = \frac{\pi}{64} (d_0^4 - d_1^4) \)
- Distance of top layer from the neutral axis \( y = \frac{d_h}{2} \)

Substituting these values in equation (2.19)

\[
S_{y0} = \frac{8P_y L}{\pi d_0^3 (1 - k^4)} \left( \frac{C P_y}{d_0^4} \right)
\]  

(2.20)

where

\[
k = \frac{d_h}{d_0} \text{ and } C = \frac{8L}{\pi (1 - k^4)}
\]
by introducing a factor of safety \( n \) equation (2.20) can be used to determine '\( d_0 \)'

\[
d_0 = \left( \frac{CP}{\frac{n}{Y_s}} \right)^{1/3}
\]

(2.21)

where \( Y_s \) = Yield strength in bending

Deterministic design results obtained by using mean values of random variables and a factor of safety of 1.5 are presented in Table 2.3.

### 2.3.2 Classical Reliability Based Design (CRBD)

Analytical expression for the maximum bending stress in the extreme fiber at the middle of the span is

\[
S_b = \frac{8PL}{\pi d_0^2(1-k^4)} = \frac{CP}{d_0^3}\]

(2.22)

where \( k = \frac{d}{d_0} \) and \( C = \frac{8L}{\pi (1-k^4)} \)

Mean and standard deviation of bending stress can be found by using the algebra of independent normal distribution functions as

\[
\overline{S}_b = \frac{C}{d_0^3} \overline{P}_s
\]

and

\[
\sigma_{S_b} = \frac{C}{d_0^3} \sigma_{P_s}
\]

(2.23)

Assuming that both the induced stress and strength distributions follow normal distribution and are independent of each other the reliability index \( \beta \) in terms of the expected values and the standard deviations of random variables \( Y_s \) and \( S_b \) is
\begin{equation}
\beta = \frac{\bar{Y}_S - \bar{S}_b}{\left(\sigma_{sb}^2 + \sigma_{as}^2\right)^{1/2}} \tag{2.12}
\end{equation}

substituting the values of \( \bar{S}_b \) and \( \sigma_{sb} \) in the above equation and by rearranging the equation we get

\begin{equation}
\left[ \bar{Y}_S - \beta^2 \sigma_{sb} \right] d_i^4 - \left( 2\bar{Y}_S \bar{S}_b - \beta^2 \sigma_{sb} \right) d_i^3 + \left( C^2 \sigma_{sb}^2 - \beta^2 \sigma_{sb}^2 \right) = 0 \tag{2.24}
\end{equation}

Equation (2.24) is a quadratic in \( d_i \) and is solved for the dimensions of the beam.

2.3.3 Reliability Based Design using Finite Element Method (RBDF)

Simply supported beam is idealized into six beam elements with six degrees of freedom at each node. A vertical point load is applied at node 4. Boundary conditions are imposed at the nodes 1 and 7. Elimination approach is used to impose the boundary conditions. Formulated simultaneous equations are solved using Gaussian elimination method for the nodal displacements. Stresses are found using nodal displacements.

Assuming that the bending stress induced in the beam follows normal distribution, the reliability index using stress-strength interference criteria for a normally distributed stress and strength is

\begin{equation}
\beta = \frac{\bar{Y}_S - \bar{S}_b}{\left(\sigma_{sb}^2 + \sigma_{as}^2\right)^{1/2}} \tag{2.12}
\end{equation}

where \( \bar{Y}_s, \sigma_{sb} \) and \( \bar{S}_b, \sigma_{as} \) are the mean and standard deviation of yield strength and bending stress respectively. The standard deviation
of $S_b$ can be found from equation (2.10) by using the partial derivative rule

$$
\sigma_{rb} = \left[ \left( \frac{\partial S_b}{\partial R} \right)^2 \right]^{1/2} \sigma_r
$$

(2.25)

Maximum bending stress is calculated using Finite Element Method (FEM) and the partial derivatives for the calculation of standard deviation of bending stress ($\sigma_{rb}$) are found by using finite difference method. The problem is solved for the cross-section dimensions of the beam using Newton – Raphson method for the specified reliability.

### 2.4 DESIGN OF A HOLLOW CIRCULAR SHAFT

A shaft is a rotating member, usually of circular cross-section and is used to transmit power or motion. Shafts are one of the most important elements of machines. Shafts support rotating parts like gears and pulleys and in turn are themselves supported in bearings which rest in rigid machine housings. Shafts are subjected to torque due to power transmission and bending moment due to reactions on the members that are supported by them. Straight shafts are commonest to be used for power transmission. Shafts are usually of solid or hollow circular cross-sections. A hollow shaft has greater strength and stiffness than solid shaft of equal weight.

Failure of a shaft usually necessitates a costly and time-consuming major overhaul. Stress analysis at a specific point on a shaft can be made using only the shaft geometry in the vicinity of that point. Shafts
are generally made of ductile materials. The maximum shear stress theory which gives results on the safe side and is simple to apply is widely used to determine the diameter of the shaft.

A hollow circular shaft of length 'L' subjected to combined bending and torsion as shown in Fig. 2.3 is considered for the design. It is assumed that maximum bending moment occurs at the middle of the shaft and a torque is applied on the shaft. It is also assumed that bending moment is introduced in the shaft due to a central point load $(P_y)$ and the shear stress is introduced due to the application of torque at extreme ends of the shaft. It is assumed that there are variations in vertical load $P_y$, torsion moment $T$, and yield strength in shear $Y_{sh}$. These variations follow normal distribution. Statistics of load, torque and yield strength are given in Table 2.1. Material properties, length of shaft and other details are given in Table 2.2. Cross-section dimensions of the beam ($d_0$ & $d_1$) are considered as deterministic design variables.

![Fig.2.3 Idealized hollow circular shaft](image)
2.4.1 Deterministic Design

When a hollow circular shaft of length ‘L’ as shown in Fig.2.3 is subjected to combined central point load \( P \), and a torque \( T \), then the induced bending and shear stresses are

Induced bending stress
\[
S_b = \frac{M_y}{I_u}
\]  (2.26)

where

Maximum bending moment at the center of the shaft \( M = \frac{P \cdot L}{4} \)

Moment of inertia about x-axis
\( I_u = \frac{\pi}{64} (d_e^4 - d_i^4) \)

Distance of top layer from the neutral axis \( y = \frac{d_e}{2} \)

Substituting these values in equation (2.26)

\[
S_b = \frac{8P \cdot L}{\pi d_e^5 (1 - k^4)} = \frac{C_1 \cdot P}{d_i^4}
\]  (2.27)

where
\[
k = \frac{d_i}{d_e} \quad \text{and} \quad C_1 = \frac{8I}{\pi (1 - k^4)}
\]

Induced shear stress \( (\tau) = \frac{T}{J \cdot r} \)  (2.28)

where

\( T = \) Applied torque

\( J = \) Polar moment of inertia
\[
J = \frac{\pi}{32} \cdot d_e^4 (1 - k^4)
\]

\( r = \) Outside radius of the shaft \( = \frac{d_e}{2} \)

substituting these values in equation (2.28)
\[ \tau = \frac{16T}{\pi d_0^2 (1 - k^4)} = \frac{C_2 I}{d_0^3} \]  

(2.29)

where \( k = \frac{d_l}{d_0} \) and \( C_2 = \frac{16}{\pi (1 - k^4)} \)

According to maximum shear stress theory, maximum shear stress induced in the shaft subjected to combined bending and torsion is

\[ \tau_{\text{max}} = \left[ \left( \frac{S_b}{2} \right)^2 + \tau^2 \right]^\frac{1}{2} \]  

(2.30)

substituting the value of \( S_b \) and \( \tau \) in terms of design variables in equation (2.30)

\[ \tau_{\text{max}} = \frac{1}{d_0^3} \left[ \frac{C_1^2 P_l^2}{4} + C_2^2 T^2 \right]^\frac{1}{2} \]  

(2.31)

by introducing a factor of safety \( n \), equation (2.31) can be used to determine \( d_0 \) as

\[ d_0 = \left\{ \left( \frac{C_1^2 P_l^2}{4} + C_2^2 T^2 \right) \left( \frac{n}{Y_{sh}} \right)^2 \right\}^{\frac{1}{6}} \]  

(2.32)

where \( Y_{sh} \) - Yield strength in shear

Deterministic design results obtained by using mean values of random variables and a factor of safety of 1.5 are presented in Table 2.3.

2.4.2 Classical Reliability Based Design (CRBD)

Analytical expression for the maximum shear stress according to maximum shear stress theory is

\[ \tau_{\text{max}} = \frac{1}{d_0^3} \left[ \frac{C_1^2 P_l^2}{4} + C_2^2 T^2 \right]^\frac{1}{2} \]  

(2.33)
Mean and standard deviation of maximum shear stress ($\tau_{\text{max}}$) can be found by using the partial derivative rule as

$$\tau_{\text{max}} = \frac{1}{d_0^2} \left[ \frac{C_1^2 \bar{P}}{4} + C_1^2 \frac{\bar{T}^2}{4} \right]$$

and

$$\sigma_{\tau_{\text{max}}} = \frac{1}{d_0^3} \left\{ \frac{C_1^2 \bar{P}^2 \sigma_i^4}{16} + C_1^2 \frac{\bar{T}^2}{4} \sigma_i^2 \right\}^{1/2}$$

(2.34)

Assuming that both the induced shear stress and shear strength distributions follow normal distribution and are independent of each other, the reliability index ($\beta$) in terms of the expected values and the standard deviation of $Y_{\text{sh}}$ and $\tau_{\text{max}}$ is

$$\beta = \frac{\bar{Y}_{\text{sh}} - \bar{\tau}_{\text{max}}}{(\sigma_{Y_{\text{sh}}}^2 + \sigma_{\tau_{\text{max}}}^2)^{1/2}}$$

(2.35)

substituting the values of $\bar{\tau}_{\text{max}}$ and $\sigma_{\tau_{\text{max}}}$ in the above equation and by rearranging the equation we get

$$\left( \bar{Y}_{\text{sh}} - \beta^2 \sigma_{Y_{\text{sh}}}^2 \right) d_0^4 - 2 \bar{Y}_{\text{sh}} (B) \frac{1}{d_0^4} + B - \beta^2 \left( \frac{A}{B} \right) = 0$$

(2.36)

where

$$A = \frac{C_1^2 \bar{P}_0^2 \sigma_i^2}{16} + C_1^2 \frac{\bar{T}^2}{4} \sigma_i^2$$

and

$$B = \frac{C_1^2 \bar{P}_0^2}{4} + C_1^2 \frac{\bar{T}^2}{4}$$

equation (2.36) is a quadratic in $d_0^4$ and is solved for the dimensions of the shaft.
2.4.3 Reliability Based Design using Finite Element Method (RBDF)

Hollow circular shaft is idealized into six beam elements with six degrees of freedom at each node. A vertical point load is applied at node 4. Assuming that the shaft is supported in short bearings, it can be modeled as a simply supported beam. At the nodes 1 and 7 boundary conditions are imposed and torque is also applied. Elimination approach is used to impose the boundary conditions. Formulated simultaneous equations are solved using Gaussian elimination method for the nodal displacements. Stresses are found using nodal displacements.

Assuming that the induced shear stress in the shaft follow normal distribution, the reliability index using stress-strength interference criterion for a normally distributed stress and strength is

$$\beta = \frac{\bar{Y}_{sk} - \bar{r}_{\text{max}}}{\left(\sigma_{sk}^2 + \sigma_{\text{max}}^2\right)^{1/2}}$$

(2.35)

where $\bar{Y}_{sk}, \sigma_{sk}$ and $\bar{r}_{\text{max}}, \sigma_{\text{max}}$ are the mean and standard deviation of yield strength in shear and maximum induced shear stress respectively. The standard deviation of $\bar{r}_{\text{max}}$ can be found using the partial derivative rule as

$$\sigma_{\text{max}} = \left\{\left(\frac{\partial \bar{r}_{\text{max}}}{\partial \sigma_i}\right)^2 \sigma_i^2 + \left(\frac{\partial \bar{r}_{\text{max}}}{\partial T}\right)^2 \sigma_T^2\right\}^{1/2}$$

(2.37)

Maximum induced shear stress in the shaft is calculated using Finite Element Method (FEM) and the partial derivatives for the calculation of standard deviation of shear stress ($\sigma_{\text{max}}$) are found using finite difference
method. The problem is solved for the cross-section dimensions of the shaft using Newton-Raphson method for the specified reliability.

2.5 RESULTS AND DISCUSSIONS

Results obtained using deterministic design procedure for the cantilever beam, simply supported beam and circular shaft are presented in Table 2.3. Results obtained using CRBD and RBDF procedures are presented for the specified (reliability index) in Table 2.4, 2.7 & 2.10 for the cantilever beam, simply supported beam and circular shaft respectively. Results obtained by varying the co-efficient of variation of random variables are presented in Table 2.5 & 2.6 for the cantilever beam, in Table 2.8 & 2.9 for the simply supported beam and in Table 2.11, 2.12 & 2.13 for the circular shaft. Co-efficient of variation of random variable is varied from 0.025 to 0.15 in steps of 0.025.

Fig 2.4, 2.7 & 2.10 shows the variation in cross-section area of cantilever beam, simply supported beam and circular shaft with the variation in reliability index(β) variation in cross-section area with the variation in co-efficient of variation of random variables are shown in Fig 2.5 & 2.6 for cantilever beam (\( C_{py} & C_{ys} \)), in Fig 2.8 & 2.9 for the simply supported beam (\( C_{py} & C_{ys} \)) and in Fig 2.11, 2.12 & 2.13 for the circular shaft (\( C_{py}, C_{ysh} & C_T \)).

Cross-section dimensions obtained using deterministic design are shown as a straight line in Fig. 2.4, 2.7, & 2.10 for the cantilever,
simply supported beam and circular shaft respectively. Since deterministic design will not consider the variations in random variables, the results obtained using deterministic design (cross-section dimensions) will not change with reliability index \( \beta \).

It is observed from the Tables 2.8 to 2.17 and Fig 2.4 to 2.12 that the results obtained using RBDF are very close to the results obtained using CRBD. It is observed that as the reliability or reliability index increases cross-section area of the cantilever/simply supported beam/circular shaft also increase. It is also observed that as the co-efficient of variation of random variable increases cross-section area also increases.

Further it is observed from the Fig 2.4, 2.7 & 2.10 that the cross-section dimensions obtained using deterministic design with a factor of safety of 1.5 are less than the cross-section dimensions obtained for the reliability index of 3, which means a reliability of 0.99865, for the problems considered.

Next chapter deals with the deterministic optimum design and reliability based optimum design of structural element/machine component.
### Table 2.1 Statistics of random variables

<table>
<thead>
<tr>
<th>Structural element /machine component</th>
<th>Random variable (x)</th>
<th>Mean (x̄)</th>
<th>Standard deviation (σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cantilever beam</td>
<td>Vertical load (P_y)</td>
<td>4000N</td>
<td>400N</td>
</tr>
<tr>
<td></td>
<td>Yield strength in bending (Y_s)</td>
<td>250N/mm²</td>
<td>25 N/mm²</td>
</tr>
<tr>
<td>Simply supported beam</td>
<td>Vertical load (P_y)</td>
<td>4000N</td>
<td>400N</td>
</tr>
<tr>
<td></td>
<td>Yield strength in bending (Y_s)</td>
<td>250N/mm²</td>
<td>25 N/mm²</td>
</tr>
<tr>
<td>Hollow circular shaft</td>
<td>Vertical load (P_y)</td>
<td>4000N</td>
<td>400N</td>
</tr>
<tr>
<td></td>
<td>Torsion moment (T)</td>
<td>0.4×10⁶N-mm</td>
<td>0.4×10⁶N-mm</td>
</tr>
<tr>
<td></td>
<td>Yield strength in shear (Y_{sh})</td>
<td>150N/mm²</td>
<td>15 N/mm²</td>
</tr>
<tr>
<td>S. No.</td>
<td>Structural element /machine component</td>
<td>Material properties and other details</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>--------------------------------------</td>
<td>--------------------------------------</td>
<td></td>
</tr>
</tbody>
</table>
| 1     | Cantilever beam                      | Young’s modulus (E) = 2.1 x 10^5 N/mm²  
Poisson’s ratio (ν) = 0.3  
Length of the beam (L) = 500 mm  
Factor of safety (n) = 1.5  
**Design variables**  
Width of cross-section = w  
Breadth of cross-section = b  
Ratio of width to breadth = w/b = 0.75 |
| 2     | Simply supported beam                | Young’s modulus (E) = 2.1 x 10^5 N/mm²  
Poisson’s ratio (ν) = 0.3  
Length of the span (L) = 500 mm  
Factor of safety (n) = 1.5  
**Design variables**  
Inside diameter = d_i  
Outside diameter = d_o  
Ratio of inside to outside diameter = d_i/d_o = 0.75 |
| 3     | Hollow circular shaft                | Young’s modulus (E) = 2.1 x 10^5 N/mm²  
Poisson’s ratio (ν) = 0.3  
Length of the shaft (L) = 500 mm  
Factor of safety (n) = 1.5  
**Design variables**  
Inside diameter = d_i  
Outside diameter = d_o  
Ratio of inside to outside diameter = d_i/d_o = 0.75 |
### Table 2.3 Deterministic Design Results

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Structural element / machine component</th>
<th>Cross-section dimensions</th>
</tr>
</thead>
</table>
| 1      | Cantilever beam                        | Width of cross-section (w)= 34.341 mm  
Breadth of cross-section (b)= 45.788 mm  
Cross-section area (A) = w.b= 1572.444 mm² |
| 2      | Simply supported beam                  | Inside diameter (d₁)=26.614 mm  
Outer diameter (d₀) =35.485 mm  
Cross-section area (A) = \( \pi (d₀^2 - d₁^2) \)= 432.854 mm² |
| 3      | Hollow circular shaft                  | Inside diameter (d₁)= 27.197 mm  
Outer diameter (d₀) =36.262 mm  
Cross-section area (A) = \( \pi (d₀^2 - d₁^2) \)= 452.031 mm² |

### Cantilever Beam Results:

**Table 2.4 Comparison of results obtained using CRBD and RBDF**

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Reliability Index β</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b(mm)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>41.938</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>44.025</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>46.336</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>48.983</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>52.143</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>56.129</td>
</tr>
</tbody>
</table>

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Table 2.5 Effect of Variation in $C_{fy}$ when $\beta = 3.$

<table>
<thead>
<tr>
<th>S. No</th>
<th>$C_{fy}$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>45.146</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>45.420</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>45.831</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>46.336</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>46.900</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>47.498</td>
</tr>
</tbody>
</table>

Table 2.6 Effect of Variation in $C_{fy}$ when $\beta = 3.$

<table>
<thead>
<tr>
<th>S. No</th>
<th>$C_{fy}$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>43.831</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>44.350</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>45.189</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>46.336</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>47.803</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>49.634</td>
</tr>
</tbody>
</table>

Simply Supported Beam Results:

Table 2.7 Comparison of results obtained using CRBD and RBDF

<table>
<thead>
<tr>
<th>S.No</th>
<th>Reliability Index $\beta$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_1$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>32.501</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>34.119</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>35.910</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>37.961</td>
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<tr>
<td>5</td>
<td>5</td>
<td>40.470</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>43.499</td>
</tr>
</tbody>
</table>
Table 2.9 Effect of Variation in $C_y$ when $\beta = 3$.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>$C_y$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_1$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>34.987</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>35.200</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>35.518</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>35.910</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>36.346</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>36.810</td>
</tr>
</tbody>
</table>

Table 2.9 Effect of Variation in $C_y$ when $\beta = 3$.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>$C_y$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_1$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>33.968</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>34.370</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>35.020</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>35.910</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>37.047</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>38.465</td>
</tr>
</tbody>
</table>

Hallow Circular Shaft Results:

Table 2.10 Comparison of results obtained using CRBD and RBDF

<table>
<thead>
<tr>
<th>S.No</th>
<th>Reliability Index $\beta$</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_1$(mm)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>33.040</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>34.547</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>36.259</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>38.261</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>40.696</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>43.814</td>
</tr>
</tbody>
</table>
### Table 2.11 Effect of Variation in C\textsubscript{Py} when \(\beta = 3\).

<table>
<thead>
<tr>
<th>S.No.</th>
<th>C\textsubscript{Py}</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d_1(\text{mm}))</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>35.887</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>35.966</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>36.092</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>36.259</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>36.459</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>36.687</td>
</tr>
</tbody>
</table>

### Table 2.12 Effect of Variation in C\textsubscript{rah} when \(\beta = 3\).

<table>
<thead>
<tr>
<th>S.No.</th>
<th>C\textsubscript{rah}</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d_1(\text{mm}))</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>33.998</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>34.493</td>
</tr>
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<td>0.075</td>
<td>35.257</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>36.259</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>37.500</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>39.016</td>
</tr>
</tbody>
</table>

### Table 2.13 Effect of Variation in C\textsubscript{T} when \(\beta = 3\).

<table>
<thead>
<tr>
<th>S.No.</th>
<th>C\textsubscript{T}</th>
<th>Cross section dimensions of the beam</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using CRBD</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(d_1(\text{mm}))</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>36.113</td>
</tr>
<tr>
<td>2</td>
<td>0.050</td>
<td>36.143</td>
</tr>
<tr>
<td>3</td>
<td>0.075</td>
<td>36.192</td>
</tr>
<tr>
<td>4</td>
<td>0.100</td>
<td>36.259</td>
</tr>
<tr>
<td>5</td>
<td>0.125</td>
<td>36.343</td>
</tr>
<tr>
<td>6</td>
<td>0.150</td>
<td>36.442</td>
</tr>
</tbody>
</table>
CANTILEVER BEAM RESULTS

Fig 2.4 Comparison of Results obtained using CRBD, RBDF and Deterministic Design

Fig 2.5 Effect of Variation in $C_{Py}$ when $\beta = 3$. 
Fig 2.6 Effect of Variation in $C_{Rs}$ when $\beta = 3$.

SIMPLY SUPPORTED BEAM RESULTS

Fig 2.7 Comparison of Results obtained using CRBD, RBDF and Deterministic Design
Fig 2.8 Effect of variation in $C_{py}$ when $\beta = 3$.

Fig 2.9 Effect of variation in $C_{yn}$ when $\beta = 3$. 
HOLLOW CIRCULAR SHAFT RESULTS

Fig 2.10 Comparison of Results obtained using CRBD, RBDF and Deterministic Design

Fig 2.11 Effect of Variation in $C_{p_Y}$ when $\beta = 3$. 

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Fig 2.12 Effect of Variation in $C_{\text{Rsh}}$ when $\beta = 3$.

Fig 2.13 Effect of Variation in $C_T$ when $\beta = 3$.
Define design variables and Random variables

Create Finite Element Model

Perform Finite Element Analysis

Update the Finite Element model

Evaluate Reliability Index ($\beta$)

Solve the problem using Numerical Method (Newton-Raphson method)

Is the Specified Reliability Achieved?

Yes

Stop

No

Fig. 2.14 Flow chart for the reliability based design using FEM