Chapter 3

Nonholonomic Frames for Finsler spaces with certain \((\alpha, \beta)\)-metrics

The main aim of this chapter is to find the nonholonomic frames for Finsler spaces with special \((\alpha, \beta)\)-metrics \(F = \frac{(\alpha + \beta)^2}{\alpha}\) and \(F = \alpha + \frac{\beta^2}{\alpha}\). Firstly, we study the nonholonomic Finsler frames for a class of Generalized Lagrange spaces with \((\alpha, \beta)\)-metric and then find the two Finsler deformations for the aforesaid metric. Consequently, we obtain the nonholonomic Finsler frames for aforesaid Finsler space.

3.1 Introduction

A Finsler space equipped with \((\alpha, \beta)\)-metric comprises of two metrics which are, Riemannian metric \(a_{ij}\) and the Finsler metric \(g_{ij}\). The above mentioned metrics are co-related by a formula given by M. Matsumoto [51]:

\[
g_{ij}(x, \dot{x}) = \rho a_{ij}(x) + \rho_0 b_i(x) b_j(x) + \rho_{-1} (b_i(x) \dot{x}_j + b_j(x) \dot{x}_i) + \rho_{-2} \dot{x}_i \dot{x}_j, \quad (3.1.1)
\]
where Finsler invariants are notified by, \( \rho, \rho_0, \rho_{-1} \) and \( \rho_{-2} \). From (3.1.1) it can not be proved whether there are some properties of Riemannian metric \( a_{ij} \) that are also related with the Finsler metric \( g_{ij} \). It was P. R. Holland [31], a physicist who constructed an advanced relationship between Finsler metric \( g_{ij} \) and the Riemannian metric \( a_{ij} \) for a special case of a Rander space, during the study of unified formalism that uses a nonholonomic frame on space-time originated by assuming a moving charged particle moving in an external electromagnetic field. For this special case, Lorentz force law could be written as geodesic equations on a Finsler space \( F^n \), called a Randers space [70]. This fact was pointed out by R. S. Ingarden. P. R. Holland studied the Lagrangian per unit mass of a charged particle moving in 4-dimensional manifold \( M_4 \) in an external field \( A_i \) with a Minkowski metric \( \eta_{ij} \) as:

\[
L(x, \dot{x}) = (\eta_{ij} \dot{x}^i \dot{x}^j)^{\frac{1}{2}} + kA_i \dot{x}^i. \tag{3.1.2}
\]

Accordingly, the elementary metric tensor of the Finsler space \( F^n \) are can be obtained as:

\[
ge_{ij}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 L^2}{\partial \dot{x}^i \partial \dot{x}^j} \tag{3.1.3}
\]

The Finsler metric \( g_{ij} \) and the Minkowski metric \( \eta_{ij} \) are connected by the formula:

\[
g_{ij} = Y^k_i Y^l_j \eta_{kl}. \tag{3.1.4}
\]

where \( Y^k_i(x, \dot{x}) \) represents a nonholonomic frame on \( TM \). In ([3], [5]), P. L. Antonelli and I. Bucataru have generalized these outcomes by dealing with general Randers space and Kropina space.

From ([12], [13]) we obtain a generalized Lagrange metric, given by the
physicist R. G. Beil as:

\[ g_{ij} = \eta_{ij} + k B_i B_j, \]  

(3.1.5)

where \( B_i(x)dx^i \) represents a 1-form on the base manifold. The equation (3.1.4) provides us the relation between the Generalized Lagrange metric \( g_{ij} \) and the Lorentz metric \( \eta_{ij} \) and a nonholonomic frame appearing in (3.1.4) is called a gauge transformation by R. G. Beil, which is used to develop unifield field theories.

M. Anastasiei and H. Shimada [2] have dealt with the class of generalized Lagrange metrics that further generalizes the metric given in (3.1.5) and the resulting metric is termed as Beil metric. In this direction some work has been done in ([17], [18]) with Generalized Lagrange spaces with \((\alpha, \beta)\)-metrics and Finsler spaces with \((\alpha, \beta)\)-metrics. The concept which we can be extracted from the aforesaid theory is to consider this Beil metric as a Finsler deformation of the Riemannian metric, which in turn can help us to discuss a nonholonomic frame that generalizes the frame used by R. G. Beil in [12] for our research work.

The nonholonomic Finsler frames have also been studied by R. Miron and H. Izumi. With it they have also investigated the concept of induced Finsler connection [62] for the famous strongly non-Riemannian Finsler spaces. M. Matsumoto has also worked upon these nonholonomic frames, in [52], where he has mentioned these frames as the Miron frames of a strongly non-Riemannian Finsler space \( F^n \). The Miron frame can be considered as a generalization Berwald frame of 2-dimensional Finsler space \( F^2 \) or of the the Moor frame for a 3-dimensional Finsler space \( F^3 \).
The basic tensor field for an \((\alpha, \beta)\)-metric Finsler space \(F^n\), is considered as the outcome of two Finsler deformation. Thus associated frame for each of these two Finsler deformations can be obtained. Therefore a nonholonomic frame for a Finsler space \(F^n\) equipped with \((\alpha, \beta)\)-metric will emerge out as the output of two Finsler frames already obtained.

### 3.2 Nonholonomic frames for Beil-metric

Let \(M\) represents a real smooth base manifold of \(n\)-dimension. Let \((TM, \pi, M)\) represents the tangent bundle of the base manifold \(M\) and \((\tilde{T}M, \pi, M)\), be the tangent bundle with the null cross-section removed. Local coordinates on the manifold \(M\) are denoted by \((x^i)\), while the induced local coordinates on \(TM\) are denoted by \((x^i, y^j)\).

Let us assume \(\pi_*\) to be the linear map induced by the canonical submersion \(\pi : TM \rightarrow M\). In consideration for all \(u \in TM\), \(\pi_{*,u} : T_uTM \rightarrow T_{\pi(u)}M\) is an epimorphism, leading its kernel to determine on \(n\)-dimensional distribution \(V : u \in TM \mapsto V_uTM = Ker\pi_{*,u} \subset T_uTM\), which is called as the vertical distribution of the tangent bundle. In case, the constant basis of \(T_uTM\) is denoted by \(\{\frac{\partial}{\partial x^i} |_u, \frac{\partial}{\partial y^j} |_u\}\), then \(\{\frac{\partial}{\partial y^j} |_u\}\) represents a basis of \(V_uTM\).

**Definition 3.2.1.** “A generalized Lagrange metric (a GL-metric) is defined as a metric \(g\) described on the vertical subbundle \(VTM\) of the tangent space \(TM\), concluding that for each \(u \in TM\), \(g_u : V_uTM \times V_uTM \rightarrow \mathbb{R}\) is bilinear, symmetric, of rank \(n\) and of constant signature. A pair \(GL^n = (M, g)\), equipped with generalized Lagrange metric is said to be the generalized Lagrange space”.

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As per the local coordinates, for each \(u \in TM\) we denote \(g_{ij}(u) = g_u(\frac{\partial}{\partial x^i} | u, \frac{\partial}{\partial y^j} | u)\). Then a GL-metric is described by a collection of functions \(g_{ij}(x, y)\) in a way that we have:

1. \(\text{rank}(g_{ij}) = n\), \(g_{ij}(x, y) = g_{ji}(x, y)\);
2. the quadratic form \(g_{ij}(x, y)\xi^i\xi^j\) has the constant signature on TM;
3. if w.r.t. other system of local coordinates \((\tilde{x}^i, \tilde{y}^i)\) at \(u \in TM\),

\[
g_{ij} = \partial \tilde{x}^k \partial \tilde{x}^l \tilde{g}_{kl}.
\] (3.2.1)

Now considering \(a_{ij}(x)\) as the components of a Riemannian metric defined on the base manifold \(M\) with \(a(x, y) > 0\) and \(b(x, y) \geq 0\) as the two Finsler scalars and \(B_i(x, y)dx^i\) a Finsler 1-form on TM, we have:

\[
g_{ij}(x, y) = a(x, y)a_{ij}(x) + b(x, y)B_i(x, y)B_j(x, y) \tag{3.2.2}
\]

as a generalized Lagrange metric [2], which is called the Beil-metric. It’s also been said that the metric tensor \(g_{ij}\) represents a Finsler deformation of the Riemannian metric \(a_{ij}\), which has been researched upon by R. Miron and R. K. Tavakol and applied [63] in General Relativity for \(a(x, y) = \exp(2\sigma(x, y))\) and \(b = 0\). The case \(a(x, y) = 1\) with many choices of \(b\) and \(B_i\) was brought into and investigated by R. G. Beil for designing a different unified field theory in [13]. We shall keep rising and lowering indices throughout the chapter only with the Riemannian metric \(a_{ij}\), i.e. \(y_i = a_{ij}y^j, b^i = a^{ij}b_j\), and so on.

Let us consider \(U\) to be an open set of the tangent space \(TM\) and

\[
V_i : u \in U \mapsto V_i(u) \in V_uTM, i = 1, ..., n
\]
to be a vertical frame over $U$. If $V_i(u) = V_i^j(u)\frac{\partial}{\partial y^j}|_u$, then $V_i^j(u)$ denotes the entries of an invertible matrix for every $u \in U$, and $\tilde{V}_k^j(u)$, is denoted by the inverse of this matrix, implying:

$$V_j^i\tilde{V}_k^j = \delta_k^i, \quad \tilde{V}_j^i V_k^j = \delta_k^i.$$  

Here $V_j^i$ is called as a nonholonomic Finsler frame.

**Theorem 3.2.1.** Consider a generalized Lagrange space $GL^n = (M, g)$, equipped with $GL$-metric $g$, Beil metric (3.2.2) and $B^2(x, y) = a_{ij}(x)B^i(x, y)B^j(x, y)$, then we have:

$$V_j^i = \sqrt{a}\delta_j^i - \frac{1}{B^2}(\sqrt{a} \pm \sqrt{a + bB^2})B^iB_j$$

(3.2.3)

as a nonholonomic Finsler frame. Thus giving

$$g_{ij}(x, y) = V_i^k(x, y)V_j^l(x, y)a_{kl}(x),$$

(3.2.4)

which is a relation between a Beil metric (3.2.2) and a Riemannian metric $a_{ij}(x)$.

**Proof.** We have

$$\tilde{V}_k^j = \frac{1}{\sqrt{a}}\delta_k^j - \frac{1}{B^2}\left(\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{a + bB^2}}\right)B^iB_k.$$  

(3.2.5)

It can be easily proved that $\tilde{V}_k^j$ is the inverse of $V_j^i$, implying $V_j^i$ is a nonholonomic frame. Then we have that $V_i^kV_j^l a_{kl} = aa_{ij} + bB_iB_j = g_{ij}$, so the formula (3.2.5) holds good.
### 3.3 Nonholonomic frames for Finsler spaces with $(\alpha, \beta)$-metrics

The fundamental theory of Finsler spaces consists of a class of Finsler spaces with $(\alpha, \beta)$-metrics [52]. In [70] G. Randers was the one who brought into the concept of first Finsler space equipped with $(\alpha, \beta)$-metric which is known as the Randers space. The other notable Finsler spaces equipped with $(\alpha, \beta)$-metrics are Kropina space, Generalized Kropina space and Matsumoto space which were introduced later on.

**Definition 3.3.1.** “A Finsler metric $F(x, y)$ [52] is said to be an $(\alpha, \beta)$-metric, when $F$ is a positively homogeneous function $F(\alpha, \beta)$ of first degree in two variables $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$.

**Examples:**

a. **Randers metric:** If $F(\alpha, \beta) = (\alpha + \beta)^2$, then the Finsler metric $F(x, y) = (a_{ij}(x)y^iy^j)^{\frac{1}{2}} + b_i(x)y^i$ is said to be a Randers metric and the space equipped with this metric is called Randers space.

b. **Kropina metric:** If $F(\alpha, \beta) = \frac{\alpha^4}{\beta^2}$, then the Finsler metric $F(x, y) = \frac{a_{ij}(x)y^iy^j}{|b_i(x)y^i|}$ is said to be a Kropina metric and the space equipped with this metric is called Kropina space.

c. **Generalized Kropina metric:** If $F(\alpha, \beta) = \frac{\alpha^2m+2}{\beta^2m}$, $(m \neq 0, -1)$, then the Finsler metric $F(x, y) = \frac{(a_{ij}(x)y^iy^j)^{\frac{1}{2}m+\frac{1}{2}}}{(b_i(x)y^i)^m}$ is said to be a generalized Kropina metric and the space equipped with this metric is called Generalized Kropina space.

d. **Matsumoto metric:** If $F(\alpha, \beta) = \frac{\alpha^4}{(\alpha-\beta)^2}$, then the Finsler metric $F(x, y) = \frac{a_{ij}(x)y^iy^j}{[(a_{ij}(x)y^iy^j)^{\frac{1}{2}} - b_i(x)y^i]}$, is said to be a Matsumoto metric and the space equipped with this metric is called Matsumoto space.

All these classes of Finsler spaces have a meaningful existence in Finsler...
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geometry. The spaces in first two examples are dual in the sense of [33]. The \(L\)-duals of Generalized Kropina space and Matsumoto space have been also found in ([73], [45]) respectively.

Following are the Finsler invariants [52] in connection with a Finsler space with \((\alpha, \beta)\)-metric \(F^2(x, y) = L(\alpha(x, y), \beta(x, y))\):

\[
\rho = \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha}, \quad \rho_0 = \frac{1}{2} \frac{\partial^2 L}{\partial \beta^2},
\rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta}, \quad \rho_{-2} = \frac{1}{2\alpha^2} \left( \frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha} \frac{\partial L}{\partial \alpha} \right). \tag{3.3.1}
\]

The relation

\[
\rho_{-1} \beta + \rho_{-2} \alpha^2 = 0. \tag{3.3.2}
\]

must be satisfied by a Finsler space equipped with \((\alpha, \beta)\)-metric. As per the above notified equations, the fundamental metric tensor \(g_{ij}\) of a Finsler space equipped with \((\alpha, \beta)\)-metric is given by [52]:

\[
g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)y_j + b_j(x)y_i) + \rho_{-2}y_iy_j. \tag{3.3.3}
\]

Now we can represent the metric tensor \(g_{ij}\) of a Lagrange space with \((\alpha, \beta)\)-metric as:

\[
g_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) + \frac{1}{\rho_{-1}}(\rho_0\rho_{-2} - \rho_{-1}^2) b_i b_j. \tag{3.3.4}
\]

The equation (3.3.4), implies that \(g_{ij}\) is the outcome of two Finsler deformations:

\[
a_{ij} \mapsto h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) \text{ and } \]

\[
h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-1}}(\rho_0\rho_{-2} - \rho_{-1}^2) b_i b_j. \tag{3.3.5}
\]
Taking into account theorem (3.2.1), the first deformation of (3.3.5) gives the nonholonomic Finsler frame given by:

\[ X^i_j = \sqrt{\rho} \delta^i_j - \frac{1}{B^2} \left( \sqrt{\rho} \pm \sqrt{\frac{B^2}{\rho - 2}} \right) (\rho_{-1} b^i \rho_{-2} + \rho_{-2} y^i)(\rho_{-1} b_j \rho_{-2} + \rho_{-2} y_j), \quad (3.3.6) \]

where \( B^2 = a_{ij}(\rho_{-1} b^i + \rho_{-2} y^i)(\rho_{-1} b^j + \rho_{-2} y^j) = \rho_{-1}^2 b^2 + \beta \rho_{-1} \rho_{-2}. \)

The metric tensors \( h_{ij} \) and \( a_{ij} \) are connected by:

\[ h_{ij} = X^k_i X^l_j a_{kl}. \quad (3.3.7) \]

Considering the theorem (3.2.1), the second deformation of (3.3.5) gives the nonholonomic Finsler frame given by:

\[ Y^i_j = \delta^i_j - \frac{1}{C^2} \left( 1 \pm \sqrt{1 + \frac{\rho_{-2}^2 C^2}{\rho_0 \rho_{-2} - \rho_{-1}^2}} \right) b^i b_j, \quad (3.3.8) \]

where

\[ C^2 = h_{ij} b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1} b^2 + \rho_{-2} \beta)^2. \]

The metric tensors \( g_{ij} \) and \( h_{ij} \) are connected by the relation:

\[ g_{mn} = Y^i_m Y^j_n h_{ij}. \quad (3.3.9) \]

From (3.3.7) and (3.3.9), we have that \( V^k_m = X^k_i Y^i_m \), with \( X^k_i \) given by (3.3.6) and \( Y^i_m \) given by (3.3.8), is a nonholonomic Finsler frames of the Finsler space equipped with \((\alpha, \beta)\)-metric.
3.3.1 Nonholonomic frames for Finsler spaces with certain \((\alpha, \beta)\)-metrics

Let us consider a Finsler space with \((\alpha, \beta)\)-metric \(F = \frac{(\alpha + \beta)^2}{\alpha}\). Then the Finsler invariants (3.3.1) are given by

\[
\rho_0 = \frac{6}{\alpha^2}(\alpha + \beta)^2, \\
\rho = \frac{1}{\alpha^4}[\alpha^4 - \beta^4 + 2\alpha^3\beta - 2\alpha\beta^3], \\
\rho_{-1} = \frac{2}{\alpha^4}(\alpha^3 - 2\beta^3 - 3\alpha\beta^2), \\
\rho_{-2} = \frac{-2\beta}{\alpha^6}(\alpha^3 - 2\beta^3 - 3\alpha\beta^2), \\
B^2 = \frac{4(\alpha^3 - 2\beta^3 - 3\alpha\beta^2)^2(b^2\alpha^2 - \beta^2)}{\alpha^{10}}. \tag{3.3.10}
\]

Using equation (3.3.10) in (3.3.6) and (3.3.8), respectively, we get

\[
X^i_j = \sqrt{\frac{(\alpha^4 - \beta^4 - 2\alpha\beta^3 + 2\alpha^3\beta)}{\alpha^4}}\delta^i_j - \frac{\alpha^2}{(b^2\alpha^2 - \beta^2)}\left[\sqrt{\frac{(\alpha^4 - \beta^4 - 2\alpha\beta^3 + 2\alpha^3\beta)}{\alpha^4}}\right]
\pm \sqrt{\frac{\alpha^4\beta + 2(2 + 3b^2)\alpha^3\beta^2 - 8\alpha\beta^4 - 5\beta^5 - 2b^2\alpha^5 + 4b^2\alpha^2\beta^3}{\alpha^4\beta}}
\left(\frac{b^i}{\alpha^2} - \frac{\beta}{\alpha^2}y^i\right)\left(\frac{b_j}{\alpha^2} - \frac{\beta}{\alpha^2}y_j\right), \tag{3.3.11}
\]

\[
Y^i_j = \delta^i_j - \frac{1}{C^2}\left[1 \pm \frac{\alpha^2\beta C^2}{2(\alpha + \beta)^3}\right]b^i b_j, \tag{3.3.12}
\]

where

\[
C^2 = \frac{(\alpha^4 - \beta^4 + 2\alpha^3\beta - 2\alpha\beta^3)b^2}{\alpha^4} - \frac{2(\alpha^3 - 3\alpha\beta^2 - 2\beta^3)}{\alpha^6\beta}(b^2\alpha^2 - \beta^2). \tag{3.3.13}
\]
Hence, we have the following:

**Theorem 3.3.1.** If $F^n = (M, F)$ be a Finsler space equipped with $(\alpha, \beta)$-metric $F = \frac{(\alpha + \beta)^2}{\alpha}$, then its nonholonomic frames is $V^k_m = X^k_i Y^i_m$, where $X^i_j$ and $Y^i_m$ are given by (3.4.11), (3.4.12) and (3.4.13).

Next, we consider a Finsler space equipped with $(\alpha, \beta)$-metric, $F = \alpha + \frac{\beta^2}{\alpha}$. Then the Finsler invariants (3.3.1) are given by

\[
\begin{align*}
\rho &= \frac{(\alpha^4 - \beta^4)}{\alpha^2}, \\
\rho_0 &= \frac{2(\alpha^2 + 3\beta^2)}{\alpha^2}, \\
\rho_{-1} &= \frac{-4\beta^3}{\alpha^4}, \\
\rho_{-2} &= \frac{4\beta^4}{\alpha^6}, \\
B^2 &= \frac{16\beta^6}{\alpha^{10}}(\alpha^2 b^2 - \beta^2). \\
\end{align*}
\]  

(3.3.14)

Using the equation (3.3.14) in (3.3.6) and (3.3.8), respectively, we get

\[
\begin{align*}
X^i_j &= \frac{\sqrt{\alpha^4 - \beta^4}}{\alpha^2} \delta^i_j - \frac{1}{(b^2\alpha^2 - \beta^2)}[\sqrt{\alpha^4 - \beta^4} \pm \sqrt{\alpha^4 - 5\beta^4 + 4\alpha^2\beta^2 b^2}] \\
&\quad \left( b^i - \frac{\beta y^i}{\alpha^2} \right) \left( b_j - \frac{\beta y_j}{\alpha^2} \right), \\
\end{align*}
\]  

(3.3.15)

\[
Y^i_j = \delta^i_j - \frac{1}{C^2} \left[ 1 \pm \sqrt{1 + \frac{\alpha^2 C^2}{2(\alpha^2 + \beta^2)}} \right] b^i b_j, \\
\]

(3.3.16)

where

\[
C^2 = \frac{(\alpha^4 - \beta^4)b^2}{\alpha^4} + \frac{4\beta^2(\beta^2 - \alpha^2 b^2)^2}{\alpha^8}. \\
\]  

(3.3.17)

Hence, we have the following:
Theorem 3.3.2. Let $F^n = (M, F)$ be a Finsler space equipped with $(\alpha, \beta)$-metric $F = \alpha + \frac{\beta^2}{\alpha}$, then its nonholonomic frames is $V^k_m = X^k_i Y^i_m$, where $X^i_j$ and $Y^i_m$ are given by (3.4.15), (3.4.16), (3.4.17).