Chapter 2

$L$-Duality of a Finsler space with certain $(\alpha, \beta)$-metrics

The geometry of higher order Finsler spaces were investigated in ([4], [49], [59]). The study of higher order Lagrange and Hamilton spaces were discussed in ([57], [58], [60]). Many fundamental problems concerning the $L$-duality and classes of Finsler spaces were studied in ([34], [72]). Various geometers have studied the $L$-duals of Randers, Kropina and Matsumoto space in ([33], [34], [45]). In this chapter, we have obtained the $L$-dual of a fundamental $(\alpha, \beta)$-metric $\frac{(\alpha+\beta)^2}{\alpha}$.

2.1 Introduction

The concept of $L$-duality between Lagrange and Finsler space was initiated by R. Miron [56] in 1987. Since then, many Finsler geometers have studied this topic.

One of the remarkable results obtained are the concrete $L$-duals of Randers and Kropina metrics ([33], [34]). The importance of $L$-duality is by far not limited to computing the dual of some Finsler fundamental func-
tions but there are so many problems which have been solved by taking the $\mathcal{L}$-duals of Finsler spaces. The $\mathcal{L}$-duality between Finsler and Cartan spaces is used to theory of the geometry of a Cartan space.

### 2.2 The Legendre transformation

A Finsler space with the basic function:

$$F(x, y) = \frac{(\alpha(x, y) + \beta(x, y))^2}{\alpha(x, y)}$$

(2.2.1)

is called a Finsler space with quadratic metric.

**Definition 2.2.1.** "[4]. A Cartan space $C^n$ is a pair $(M, H)$ which consists of a real $n$-dimensional $C^\infty$-manifold $M$ and a Hamiltonian function $H : T^*M \setminus \{0\} \to \mathbb{R}$, where $(T^*M, \pi^*, M)$ is the cotangent bundle of $M$ such that $H(x, p)$ has the following properties:

1. It is two homogeneous with respect to $p_i$ $(i, j, k, \ldots = 1, 2, \ldots, n)$.
2. The tensor field $g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate".

Let us denote an $n$-dimensional Cartan space with the fundamental function $K(x, p)$ by $C^n = (M, K)$. One can also describe Cartan spaces with the metric functions of the forms given below:

$$K(x, p) = \sqrt{a^{ij}(x)p_ip_j + b^i(x)p_i}$$

and

$$K(x, p) = \frac{a^{ij}p_ip_j}{b^i(x)p_i}$$
and call these spaces Randers and Kropina spaces respectively on the cotangent bundle $T^*M$.

**Definition 2.2.2.** “[4]. A regular Lagrangian $L(x, y)$ on a domain $D \subset TM$ is a real smooth function $L : D \to \mathbb{R}$ and a regular Hamiltonian $H(x, p)$ on a domain $D^* \subset T^*M$ is a real smooth function $H : D^* \to \mathbb{R}$. Thus, the matrices with entries

$$g_{ab}(x, y) = \dot{\partial}_a \dot{\partial}_b L(x, y)$$

and

$$g^{ab}(x, p) = \dot{\partial}^a \dot{\partial}^b H(x, p)$$

are overall nondegenerate on $D$ and $D^*$ respectively.”

**Example.** (a) Each Finsler space $F^n$ is a Lagrange manifold with $L = \frac{1}{2} F^2$.

(b) Each Cartan space $C^n = (M, \bar{F}(x, p))$ is a Hamilton manifold with $H = \frac{1}{2} \bar{F}^2$. (Here $\bar{F}$ is positively one-homogeneous in $p_i$ and the tensor $\bar{g}^{ab} = \frac{1}{2} \bar{\partial}^a \bar{\partial}^b \bar{F}^2$ is nondegenerate).

(c) $(M, L)$ and $(M, H)$ with

$$L(x, y) = \frac{1}{2} a_{ij}(x)y^i y^j + b_i(x) y^i + c(x)$$

and

$$H(x, p) = \frac{1}{2} \bar{a}^{ij}(x)p_ip_j + \bar{b}^i(x)p_i + \bar{c}(x)$$

are Lagrange and Hamilton manifolds respectively. (Here $a_{ij}(x)$, $\bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, $b_i$ are components of
covector field, $\tilde{b}^i$ are the components of a vector field, $C$ and $\tilde{C}$ are the smooth functions on $M$).

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If $L \in F(D)$ is a differential map, we can describe the fiber derivative of $L$, locally defined by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$

$$\psi(x, y) = (x^i, \partial_a L(x, y)),$$

which will be called the Legendre transformation.

It is smoothly identified that $L$ is a regular Lagrangian if and only if $\psi$ is a local diffeomorphism.

In the identical aspect if $H \in F(D^*)$, the fiber derivative is defined locally by

$$\varphi(x, y) = (x^i, \partial^a H(x, y)),$$

which is a local diffeomorphism if and only if $H$ is regular.

Let us describe a regular Lagrangian $L$. Next, $\psi$ is a diffeomorphism between the open sets $U \subset D$ and $U^* \subset D^*$. We can describe in this case the function:

$$H : U^* \to R, \ H(x, y) = p_a y^a - L(x, y), \quad (2.2.2)$$

where $y = (y^a)$ is the solution of the equations $y_a = \partial_a L(x, y)$.

Also, if $H$ is a regular Hamiltonian on $M$, $\psi$ is a diffeomorphism between respective open sets $U^* \subset D^* \subset U \subset D$, we can describe the function

$$L : U \to R, \ L(x, y) = p_a y^a - H(x, p), \quad (2.2.3)$$
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\[ y = (y_a) \text{ is the solution of the equations} \]

\[ y^a = \dot{y}^a H(x, p). \]

The Hamiltonian given by (2.2.2) is the Legendre transformation of the Lagrangian \( L \) and the Lagrangian written in (2.2.3) is said to be the Legendre transformation of the Hamiltonian \( H \).

In case, \((M, K)\) is a Cartan space, then \((M, H)\) is a Hamilton manifold ([58], [61]), where \( H(x, p) = \frac{1}{2}K^2(x, p) \) is 2-homogenous on a domain of \( T^*M \). So we get the following transformation of \( H \) on \( U \):

\[ L(x, y) = p_ay^a - H(x, p) = H(x, p). \quad (2.2.4) \]

**Theorem 2.2.1.** [61] The scalar field given by (2.2.4) is a positively 2-homogeneous regular Lagrangian on \( U \).

Therefore, we get Finsler metric \( F \) of \( U \), so that

\[ L = \frac{1}{2}F^2. \]

Thus, for the Cartan space \((M, K)\) we always can locally combine a Finsler space \((M, F)\) which will be called the \( L \)-dual of a Cartan space \((M, C_{\mathcal{U}})\) vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the \( L \)-dual of a Finsler space \((M, F_{\mathcal{U}})\).
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2.3 The $L$-dual of a special Finsler space with metric $\frac{(\alpha+\beta)^2}{\alpha}$

In this case, we put $\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$. we have $F = \frac{(\alpha+\beta)^2}{\alpha}$ and

\[ p_i = \frac{1}{2} \dot{\partial}_i F^2 = F \left[ \frac{\partial}{\partial y^i} \left( \frac{(\alpha + \beta)^2}{\alpha} \right) \right], \]

\[ p_i = \frac{(\alpha + \beta)}{\alpha^2} \left[ (1 - \frac{\beta}{\alpha}) y_i + 2ab_i \right] F. \quad (2.3.1) \]

Contracting (2.3.1) with $p^i$ and $b^i$ respectively, we get

\[ \alpha^{*2} = \frac{(\alpha + \beta)}{\alpha^2} \left[ (1 - \frac{\beta}{\alpha}) F^2 + 2\alpha \beta^* \right] F. \quad (2.3.2) \]

and

\[ \beta^* = \frac{(\alpha + \beta)}{\alpha^2} \left[ (1 - \frac{\beta}{\alpha}) \beta + 2ab^2 \right] F. \quad (2.3.3) \]

In [72], for a Finsler $(\alpha, \beta)$-metric $F$ on a Manifold $M$, one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ and $\beta = b_i y^i$ with $||\beta||_x < b_0, \forall x \in M$.

The function $\phi$ satisfies $\phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, (|s| \leq b_0)$. This metric is an $(\alpha, \beta)$-metric with $\phi = (1 + s)^2$.

Using Shen’s notation [77], $s = \frac{\beta}{\alpha}$ in (2.3.2) and (2.3.3), we get

\[ \alpha^{*2} = \left[ (1 - s^2)(1 + s)^2 F + 2(1 + s)\beta^* \right] F; \quad (2.3.4) \]
and
\[ \beta^* = \left[(1 - s^2)s + 2(1 + s)b^2\right]F, \quad (2.3.5) \]

Putting \(1 + s = t\), i.e, \(s = (t - 1)\) in equations (2.3.4) and (2.3.5), we get
\[ \alpha^{*2} = t[(2 - t)F + 2\beta^*]F, \quad (2.3.6) \]

and
\[ \beta^* = t[(2 - t)(t - 1) + 2b^2]F. \quad (2.3.7) \]

We consider two cases:
(1) \(b^2 = 1\).
(2) \(b^2 \neq 1\).

**Case 1.** For \(b^2 = 1\) from (2.3.7), we get
\[ \beta^* = t[(2 - t)(t - 1) + 2]F, \]

or
\[ F = \frac{\beta^*}{t^2(3 - t)}. \quad (2.3.8) \]

and by substitution of \(F\) in (2.3.6), after some computations, we get a cubic equation
\[ t^3 - 6t^2 + 3t(3 + k) - 8k = 0, \]

where
\[ k = \frac{\beta^{*2}}{\alpha^{*2}}. \]
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Using Mathematica for solving the above cubic equation, we get following roots of the above equation

\[
t = 2 - \frac{-9 + 9k}{9(-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}}} + (-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}},
\]

\[
t = 2 + \frac{(1 \pm i\sqrt{3})(-9 + 9k)}{18(-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}}} - \frac{1}{2}(1 \pm i\sqrt{3})(-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}},
\]

As our Finsler fundamental function is real, the dual Hamilton function is also real. So we choose real root

\[
t = 2 - \frac{-9 + 9k}{9(-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}}} + (-1 + k + \sqrt{k - 2k^2 + k^3})^{\frac{1}{3}},
\]

or

\[
t = 2 - \frac{l}{(l + m)^{\frac{1}{3}}} + (l + m)^{\frac{1}{3}}, \tag{2.3.9}
\]

where

\[
l = -1 + k = \frac{-\alpha^* + \beta^*}{\alpha^*}, m = \sqrt{k - 2k^2 + k^3} = \frac{\beta^*(\alpha^* - \beta^*)}{\alpha^3}. \tag{2.3.10}
\]

From (2.3.8) and (2.3.9), we get

\[
F = \frac{\beta^*}{\left(1 + \frac{l}{(l + m)^{\frac{1}{3}}} - (l + m)^{\frac{1}{3}}\right)\left(2 - \frac{l}{(l + m)^{\frac{1}{3}}} + (l + m)^{\frac{1}{3}}\right)^{\frac{1}{2}}},
\]

As we know that \(H(x, p) = \frac{1}{2}F^2\), hence we get

\[
H(x, p) = \frac{\beta^{*2}}{2\left(1 + \frac{l}{(l + m)^{\frac{1}{3}}} - (l + m)^{\frac{1}{3}}\right)^2\left(2 - \frac{l}{(l + m)^{\frac{1}{3}}} + (l + m)^{\frac{1}{3}}\right)^{\frac{1}{2}}}.
\]

\[
\tag{2.3.11}
\]

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Putting $\beta^* = b^i p_i$, in (2.3.11), we get

$$H(x, p) = \frac{(b^i p_i)^2}{2 \left( \frac{l}{(l+m)^{\frac{1}{2}}} - (l + m)^{\frac{1}{2}} \right)^2 \left( \frac{l}{(l+m)^{\frac{1}{2}}} + (l + m)^{\frac{1}{2}} \right)^4}.$$  

**Case 2.** Next, we find $H(x, p)$ for $b^2 \neq 1$. From (2.3.7), we have

$$F = \frac{\beta^*}{t[(2-t)(t-1) + 2b^2]}.$$  

(2.3.12)

Using (2.3.12) in (2.3.6), we get

$$t^4 - 6t^3 + [9 - 4(b^2 - 1) + 3k]t^2 + [12(b^2 - 1) - 8k]t + [4(b^2 - 1)^2 - 4k(b^2 - 1)] = 0$$

Using Mathematica for solving the above quartic equation, we get four real roots, given in the following equation

$$t = \left( \frac{3}{2} \pm \delta_i \right), \ (i = 1, 2)$$  

(2.3.13)

where

$$\delta_1 = \frac{a_1}{2} + \frac{1}{2} \sqrt{a_2 - \frac{a_3}{4a_1}},$$

$$\delta_2 = \frac{a_1}{2} - \frac{1}{2} \sqrt{a_2 - \frac{a_3}{4a_1}},$$

$$a_1 = \sqrt{\left[ -4 + 4b^2 - 3k + \frac{1}{3} b_1 + \frac{2 \sqrt{b_2}}{3(b_3 + \sqrt{b_4})} \right] + \frac{[b_3 + \sqrt{b_4}]^\frac{1}{2}}{3.2 \frac{1}{2} \frac{1}{2}}},$$

$$a_2 = \left[ 5 + 4b^2 + \frac{b_1}{3} - 3k - \frac{2 \sqrt{b_2}}{3(b_3 + \sqrt{b_4})} - \frac{[b_3 + \sqrt{b_4}]^\frac{1}{2}}{3.2 \frac{1}{2} \frac{1}{2}} \right],$$
\[a_3 = (216 - 32b_5 - 24b_1),\]
\[b_1 = (13 - 4b^2 + 3k),\]
\[b_2 = (1 + 16b^2 + 64b^4 - 18k - 72b^2k + 9k^2),\]
\[b_3 = (2 + 48b^2 + 384b^4 + 1024b^6 - 54k - 648b^2k - 1728b^4k + 270k^2 + 648b^2k^2 + 54k^3),\]
\[b_4 = (1728b^2k^2 + 25920b^4k^2 + 82944b^6k^2 - 110592b^8k^2 - 1728k^3 - 57024b^2k^3 - 114048b^4k^3 + 359424b^6k^3 + 31104k^4 - 15552b^2k^4 - 388800b^4k^4 + 46656k^5 + 139968b^2k^5),\]
\[b_5 = (-3 + 3b^2 - 2k).\]

As our Finsler fundamental function is real, the dual Hamilton function is also real. Putting the value of \(t\) in (2.3.12), we get

\[F = \frac{\beta^*}{\left(\frac{3}{2} \pm \delta_i\right) \left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]},\]

Hence \(H(x, p) = \frac{1}{2} F^2\) is obtained as

\[H(x, p) = \frac{\beta^{*2}}{2 \left(\frac{3}{2} \pm \delta_i\right)^2 \left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]^2},\]  \hspace{1cm} (2.3.14)

Putting \(\beta^* = b^i p_i\), in (2.3.14), we get

\[H(x, p) = \frac{(b^i p_i)^2}{2 \left(\frac{3}{2} \pm \delta_i\right)^2 \left[\left(\frac{1}{2} \pm \delta_i\right)^2 + 2b^2\right]^2},\]  \hspace{1cm} (2.3.15)

Hence, we have the following:

**Theorem 2.3.1.** Let \((M, F = (\alpha + \beta)^2/\alpha)\) be a special Finsler space, where
\( \alpha^2 = a(y, y) = a_{ij}(x)g^i y^j \) is Riemannian metric and \( \beta = b_i(x)g^i \) is a 1-form on \( TM \setminus \{0\} \), where \( b^2 \) is the Riemannian length of \( b \). Then, the \( \mathcal{L} \)-dual of \( (M, F = (\alpha + \beta)^2 / \alpha) \) is the space having the fundamental function on \( T^*M \):

1. If \( b^2 = 1 \), the \( \mathcal{L} \)-dual of \( (M, F) \) is the space on \( T^*M \) having the elementary function:

\[
H(x, p) = \frac{(b^i p_i)^2}{2 \left( 1 + \frac{l}{(l+m)^{\frac{3}{2}}} - (l + m)^{\frac{1}{2}} \right)^2 \left( 2 - \frac{l}{(l+m)^{\frac{3}{2}}} + (l + m)^{\frac{1}{2}} \right)^4},
\]

(2.3.16)

where

\[
l = -1 + k = \frac{-\alpha^*^2 + \beta^*^2}{\alpha^*^2}, m = \sqrt{k - 2k^2 + k^3} = \frac{\beta^*(\alpha^*^2 - \beta^*^2)}{\alpha^*^3}.
\]

2. If \( b^2 \neq 1 \), the \( \mathcal{L} \)-dual of \( (M, F) \) is the space on \( T^*M \) having the elementary function:

\[
H(x, p) = \frac{(b^i p_i)^2}{2 \left( \frac{3}{2} \pm \delta_1 \right)^2 \left[ \frac{1}{2} \pm \delta_1 \right]^2 + 2b^2},
\]

(2.3.17)

where

\[
\delta_1 = \frac{a_1}{2} + \frac{1}{2} \sqrt{a_2 - \frac{a_3}{4a_1}}, \quad \delta_2 = \frac{a_1}{2} - \frac{1}{2} \sqrt{a_2 - \frac{a_3}{4a_1}},
\]

\[
a_1 = \sqrt{\left[ -4 + 4b^2 - 3k + \frac{1}{3} \frac{b_1}{b_1} + \frac{2\frac{k}{b_2}}{3(b_3 + \sqrt{b_1})^{\frac{3}{2}}} + \frac{b_3 + \sqrt{b_1}}{(3.2\frac{1}{2})} \right]},
\]

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\[ a_2 = \left[ 5 + 4b^2 + \frac{b_1}{3} - 3k - \frac{2^\frac{1}{2}b_2}{3(b_3 + \sqrt{b_4})^\frac{1}{2}} - \frac{[b_3 + \sqrt{b_4}]^\frac{1}{3}}{(3.2^\frac{1}{2})} \right], \]

\[ a_3 = (216 - 32b_5 - 24b_1), \]

\[ b_1 = (13 - 4b^2 + 3k), \]

\[ b_2 = (1 + 16b^2 + 64b^4 - 18k - 72b^2k + 9k^2), \]

\[ b_3 = (2 + 48b^2 + 384b^4 + 1024b^6 - 54k - 648b^2k - 1728b^4k + 270k^2 \]
\[ + 648b^2k^2 + 54k^3), \]

\[ b_4 = (1728b^2k^2 + 25920b^4k^2 + 82944b^6k^2 - 110592b^8k^2 - 1728k^3 \]
\[ - 57024b^2k^3 - 114048b^4k^3 + 359424b^6k^3 + 31104k^4 - 15552b^2k^4 \]
\[ - 388800b^4k^4 + 46656k^5 + 139968b^2k^5), \]

\[ b_5 = (-3 + 3b^2 - 2k). \]