Chapter 1

A Brief Introduction to Finsler Geometry

1.1 Introduction

The subject matter of this work has its genesis in Riemann’s 1854 “habilitation” address: “Über die Hypothesen, welche der Geometrie zu Grunde liegen” (On the Hypotheses, which lie at the Foundations of Geometry). Riemann, undoubtedly the greatest mathematician of the 19th century, has introduced the notion of a manifold and its structures. The problem involves great difficulties. But, with hypotheses on the smoothness of the functions in question, the issues can be settled satisfactorily and there is now a complete treatment. Traditionally, the structure being focused on is the Riemannian metric, which is a quadratic differential form. Put another way, it is a smoothly varying family of inner products, one on each tangent space. The resulting geometry-Riemannian geometry has undergone tremendous development in this century. Areas in which it has had the significant impact include Einstein’s theory of general relativity, and global differential geometry.
In the context of Riemann’s lecture, this restriction to a quadratic differential form constituted only a special case. Nevertheless, Riemann saw the great merit of this special case, so much so that he introduced for it the curvature tensor and the notion of sectional curvature. Such was done through a Taylor expansion of the Riemannian metric. The Riemann curvature tensor posed a fundamental problem. Namely: how did one decide, in principle, whether two given Riemannian structures differed only by a coordinate transformation? This was solved in 1870, independently by Christoffel and Lipschitz, using different methods and without the benefit of tensor calculus. It was almost 50 years later, in 1917, that Levi-Civita introduced his notion of parallelism (equivalent to a connection), thereby giving the solution a simple geometrical interpretation. Riemann saw the difference between the quadratic case and the general case. However, the latter had no choice but to lay dormant when he remarked that “The study of the metric which is the fourth root of a quartic differential form is quite time-consuming and does not throw new light on the problem.” Happily, interest in the general case was revived in 1918 [24] by Paul Finsler’s thesis, written under the direction of Caratheodory [19]. For this reason, the general case was named as Riemann-Finsler geometry or Finsler geometry for short.

Finsler geometry is closely related to the calculus of variations with specific reference to the original geometrical history which was discovered by Caratheodory in connection with problems in parametric form. The kernel of these systems is the supposed indicatrix, at the time the property of convexity is of basic value with interest to the necessary conditions for a minimal in the calculus of variations. The significant similarity within
some conditions of differential geometry and the calculus of variations have been observed few years preceding to the publication of Finsler’s thesis, in specific by Bliss [16], Landsberg [38] and Blaschke [15]. Bliss and Landsberg have discovered definitions of angle in particulars of invariants of a parametric issue in the calculus of variation. However the geometrical theories of Bliss and Landsberg have been discovered opposite to an Euclidean background and cannot, consequently, be regarded as to conclude the accurate purpose of the generalization of Riemann’s idea. Definitely, Finsler’s thesis is regarded as the beginning stride in this direction. As such its deeper study goes back at least to Jacobi and Adolf Kneser.

In his Paris address in 1900, Hilbert formulated 23 unsolved problems. The last one was devoted to the geometry of the calculus of variations. It is the only problem for which he did not formulate a specific question/conjecture. He provided an account of the invariant integral, and emphasized the importance of the problem of multiple integrals.

The Hilbert invariant integral plays an important role in all modern treatments of the subject. The geometrical data in Finsler geometry consists of a smoothly varying family of Minkowski norms, rather than a family of inner products. This family of Minkowski norms is known as a Finsler structure. Just like Riemannian geometry, there is the equivalence problem: how can one decide (in principle) whether two given Finsler structures differ only by a transformation induced from a coordinate change? It is not unreasonable to expect that the solution of the equivalence problem will again involve a connection and its curvature, together with the proper space on which these objects live.
In Riemannian geometry, the connection of choice is that constructed by Levi-Civita, using the Christoffel symbols. It has two remarkable attributes: metric-compatibility and torsion-freeness. Although we now know that in Finsler geometry proper, these both cannot be present in the same connection, such was perhaps not common knowledge during the turn of the century. Even after reaching this realization, one still faces the daunting task of writing down viable structural equations for the connection. Furthermore, the Levi-Civita (Christoffel) connection operates on the tangent bundle $TM$ of our underlying manifold $M$. But the same cannot be said of its Finslerian counterpart.

It was not until 1926 that a significant progress has been made by Ludwig Berwald (1883-1942), from an analytical perspective. One could see the poignant and informative obituary by Max Pinl in Scripta Math. 27 (1965), 193-203. Berwald’s work stems from the study of systems of differential equations, and is very much rooted in the calculus of variations. He has introduced a connection and two curvature tensors, all rightfully bearing his name, referred in “A History of Finsler Geometry” in Proceedings of the 33rd Symposium on Finsler Geometry, 1998, Lake Yamanaka. The Berwald connection is torsion-free, but is (necessarily) not metric-compatible. The Berwald curvature tensors are two types: an $hh$- one not unlike the Riemann curvature tensor, and an $hv$- one which automatically vanishes in the Riemannian setting. Berwald’s constructions have, since their inception, been indispensable to the geometry of path spaces.

Enthusiasts of metric-compatibility are not to be outdone. It is an amusing irony that although Finsler geometry starts with only a norm in any given tangent space, it regains an entire family of inner products, one for each
direction in that tangent space. This is why one can still make sense of metric-compatibility in the Finsler setting. In 1934, Elie Cartan [20, 21] introduced a connection that is metric-compatible but has torsion. The Cartan connection remains, to this day, immensely popular with the Matsumoto and the Miron schools of Finsler geometry. Besides the curvature tensors of hh- and hv-type, there is a third curvature tensor associated with the Cartan connection. It is of vv-type. In Cartan’s theory of Finsler spaces we have three curvature tensors $R_{hijk}, P_{hijk}, S_{hijk}$ and three torsion tensors $R^i_{jk} (= y^h R^i_{hjk}), P^i_{jk} (= y^h P^i_{hjk}), C^i_{jk}$. L. Berwald has defined a Finsler connection from the fundamental function in the viewpoint of the so-called geometry of paths. His method is very simple, but his connection is not metrical.

The aforementioned theories make the value of a positive tool which fundamentally contains the discussion of a space whose components are not the points of the elementary manifold, however the line-elements of the next, which form a $(2n - 1)$-dimensional variety. This simplifies the preface of what Cartan calls the “Euclidean connection”, which, by means of satisfied proposes, may be copied individually from the essential metric function $F(x, dx)$. The system also depends on the introduction of a purported “element of support”, particularly, that at every point a formerly authorize guidance fundamental given, which then assist as directional exchange in all functions depending on direction including position. Hence, for particular, the length of a vector and the vector collected from it by an insignificant parallel movement depends on the approximate special of the basic of foundation. It is this tool which drive to the advancement of Finsler geometry in phrase of explicit generalizations of the system of Riemannian geometry. It has been manipulated, on the other hand, that the
introduction of the component of support is annoying from a geometrical point of prospect, in the time the general connective with the calculus of variations is determined unsteadily. This prospect has been disclosed individually by various authors, in specific by Vagner, Busemann and H. Rund. It has been emphasized that the natural local metric of a Finsler space is a Minkowskian one, and that the approximate deception of a Euclidean metric would to some expansion involved some of the largest impressive virtues of the Finsler space. Hence at the basis of the current decade, further ideas have been establishing ahead. The exclusion of the purpose of the fundamental of base, yet fetching from a geometrical point of aspect, has led to different complications: for particular, the essential orthogonality in the middle of two vectors is not in regular symmetric, although the analytical complications are adequately enhanced, especially after all Ricci’s lemma cannot be generalized as aforetime. Auspiciously, from the point of the aspect of differential invariants, there exist marked affinity in the middle of all these ideas, which is a perfectly essential object and could have been familiarized. It is in the utilization and in the explanation of these constants that the two points of aspect emerge to be inconsistent.

Back in the torsion-free camp, the next progress could be said to come in 1948, when the Chern connection was discovered. Its formula differs from that of Berwald’s by an \( \dot{A} \) term. In natural coordinates on the slit tangent bundle \( TM_0 \), the Chern connection coefficients are given by

\[
g^{is} \left( \frac{\delta g_{is}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s} + \frac{\delta g_{ks}}{\delta x^j} \right).
\]

To get those for the Berwald connection, one simply adds on the tensor \( \dot{A}^{ik} \). More importantly, replacing the operator \( \frac{\delta}{\delta x} \) by \( \frac{\partial}{\partial x} \) gives the familiar Levi-Civita (Christoffel) connection of Riemannian metrics. The connections

\begin{center}
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of Berwald and Chern are both torsion-free. They also fail, slightly but
expectedly, to be metric-compatible. Of the two, the Chern connection is
simpler in form, while the Berwald connection effects a leaner hh-curvature
for spaces of constant flag curvature. These connections coincide when
the underlying Finsler structure is of Landsberg type. They further re-
duce to a linear connection on $M$, one which operates on $TM$, when the
Finsler structure is of Berwald type. In the generic Finslerian case, none
of the connections we mentioned operates directly on the tangent bundle
$TM$ over $M$. Chern must have realized in his solution of the equivalence
problem that, by pulling back $TM$ so that it sits over the manifold of rays
$SM$ rather than $M$, one provides a natural vector bundle on which these
connections may operate. It is within this geometrized setting that the
equivalence problem and its solution admit a sound conceptual interpreta-
tion.

The $n$-dimension Finsler space $F^n$ is a differentiable manifold such that
the length $s$ of a curve $x^i(t)$ of $F^n$ is given by the integral

$$s = \int L(x, \frac{dx}{dt}) dt$$

The so-called fundamental function $L(x, y) = L(x^1, \ldots, x^n, y^1, \ldots, y^n)$ is sup-
posed to be differentiable for $y \neq (0)$ and to satisfy the usual regularity
conditions in the variation calculus:
1. Positively homogeneous: $L(x, py) = pL(x, y)$, $p > 0$.
2. Positive: $L(x, y) > 0$, $y \neq (0)$.
3. $g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}$ is positive-definite.

Riemann himself further supposed the symmetry condition:
4. $L(x, -y) = L(x, y)$.

In case of Riemannian space, the basic function $L(x, y)$ is defined by $L(x, y) =$
\[ \sqrt{g_{ij}(x)y^iy^j} \] so that (1.) and (4.) hold good. Further, \( g_{ij}(x) \) are nothing but the ones in (3.). Therefore, \( g_{ij}(x, y) \) in (3.) is said to be the basic tensor of the Finsler space \( F^n \). It is well-known that (3.) is too restrictive in application to physics in the case of Riemannian geometry, so that (3.) as well as (2.) are usually not supposed. On the other hand, (4.) means the independence of the length of curves on their orientation and is neglected in Finsler geometry.

Since 1934 many mathematicians had been studying Finsler geometry along Cartan’s line and we have various interesting results by authors of many countries for instance, E. T. Davis [22](England), M. Haimovici (Romania) [25, 26], H. Hombu (Japan) [32], O. Varga (Hungary) [79], V. V. Wagner (U. S. R.) [80] etc. On the other hand, G. Randers (U. S. A) [70], introduced a special Finsler space to treat the so-called unified field theory of gravitation and electromagnetism, and many physicists have paid attention to Finsler geometry. But the progress of studying of Finsler geometry had lost its spread and finally arrived at the end, when Berwald died at a Jew Camp of Lodz in Poland in 1942. One of his posthumous papers is the greatest paper in Finsler geometry. In 1951, a young German H. Rund [71] introduced a new parallelism from the standpoint of the so-called Minkowskian geometry, Cartan did the parallelism from the standpoint of Euclidean geometry. But the reviewers of Rund’s paper, E. T. Davies (Math. Rev.) and A. Deicke (Zentralblatt.) indicated unfortunately that Rund’s parallelism was the same with Cartan’s. By the stimulus of Rund, however, young German W. Barthel [11], A. Deicke [23], D. Laugwitz [39] and R. Sulanke [78] have very actively studied again Finsler geometry.

In 1974, Radu Miron, build a field of orthonormal frames essentially re-
lated to an \( n \)-dimensional Finsler space. This field has certain cases, the field of frames framed by L. Berwald and A. Moor in the two and three dimensions respectively. It was called Miron frame by M. Matsumoto in his monograph [52] dedicated to Finsler spaces. In 1980, on his lecture at the University of Brasov, Miron introduced the Finsler connection, for the first time, as linear connections in the total space \( TM \) tangent to a manifold \( M \), consistent with the virtually complex structure essentially related to nonlinear connections on \( TM \). He pointed out the significant role of the nonlinear connections, likewise that of the geometric phenomenon’s of the Finsler type, called next well-known geometric phenomenon’s. The most significant uniqueness in his lecture was the initiation and study of the space with metrical Finsler structures, next named as generalized Lagrange spaces. This supplement absolutely converted the structure of the Finsler geometry and led to different generalizations and different points of aspect.

In 1989, M. Matsumoto proved that a slope of a mountain is a Finsler surface w.r.t. a time measure [53]. As was also pointed out by P. Finsler himself in his letter to Matsumoto a time measure is thought to be a typical model of a Finsler metric. Moreover, it is noted that \( \frac{\alpha^2}{\alpha - \beta} \) is an \((\alpha, \beta)\)-metric. The notion of \((\alpha, \beta)\)-metric has been discovered by Matsumoto [54] and has been worked upon in detail. The well-known examples, of \((\alpha, \beta)\)-metric are Randers metric \( F(\alpha, \beta) = (\alpha + \beta) \) [70] and a Kropina metric \( F(\alpha, \beta) = \frac{\alpha^2}{\beta} \) [37], whose studies have greatly contributed to the growth of Finsler geometry. So the metric of type \( F(\alpha, \beta) = \frac{\alpha^2}{\alpha - \beta} \) seems to be interesting as a new example of \((\alpha, \beta)\)-metrics.

There have been advancements in Finsler geometry in current years which
justified consideration. It has been exposed that recent differential geometry supports the tools to respond an analysis of Riemannian geometry, beyond the quadratic restriction, in an explicit and elegant way so that all outcomes, local and global, are combined. This not only provides the better consideration of the geometry but accessible a vista comparable to the advancements of algebraic geometry from quadrics to general algebraic variations. Currently the work that is being done in Finsler Geometry is on Finsler Spaces with \((\alpha, \beta)\)-metric and on its applications in Physics and Biology.

\section*{1.2 Preliminaries}

First, we discuss some basic definitions and concepts of Finsler geometry which has been used frequently in subsequent chapters.

\subsection*{1.2.1 Tangent space}

We consider a change of local coordinates as represented by

\[ x'^i = x'^i(x^1, x^2, ..., x^n), \quad (i = 1, 2, ... n), \]

along the curve \(x^i = x^i(t)\) referred to an invariant parameter \(t\), the new components of the tangent vector \((y^i = dx^i/dt)\) are obtained by differentiating the relation

\[ x'^i = x'^i(x^i(t)), \quad (1.2.1) \]
with respect to $t$, which gives

$$y' = \frac{\partial x'}{\partial x^i} y^i, \quad (1.2.2)$$

or in terms of differentials

$$d x' = \frac{\partial x'}{\partial x^i} dx^i. \quad (1.2.3)$$

Here $dx^i$ is interpreted as the components of the displacement in $M$ from a point $P(x^i)$ to a point $Q(x^i + dx^i)$.

If the point $P(x^i)$ is fixed i.e., the coefficient $\frac{\partial x'}{\partial x^i}$ of the transformation (1.2.3) are fixed, this relation represents a linear transformation of the $dx^i$ on the $dx'$'. The same is true for the variables $y^i$ and $y'$ in the transformation (1.2.2). Therefore, the n-entities of this kind may be taken to define the elements of an n-dimentional linear vector space.

A system of n quantities $X^i$ where transformation law under $x^i = x^i(x^1, x^2, ..., x^n)$, ($i = 1, 2, ..n$) is equivalent to that of the $y^i$ is called a contravariant vector attached to the point $P(x^i)$ of $M$. Such contravariant vector constitutes the elements of vector space. The totality of all contravariant vectors attached to $P(x^i)$ of $M$ is the tangent space denoted by $T_n(p)$ or $T_n(x^i)$. 
1.2.2 Tangent Bundle

“A tangent bundle on an $n$-dimension smooth manifold $M$ is the disjoint union of the tangent spaces of $M$. It is denoted by $TM$, and written by

$$TM = \bigcup_{p \in M} T_p M = \bigcup_{p \in M} \{p\} \times T_p M,$$

where $T_p M$ indicates the tangent space to $M$ at $p$. So, an element of $TM$ can be thought of as a pair $(p, v)$, where $p$ is a point in $M$ and $v$ is a tangent vector to $M$ at $p$. There is an essential projection

$$\pi : TM \rightarrow M.$$ 

given by $\pi(p, v) = p$. The projection maps every tangent space $T_p M$ to the particular point $p$. The tangent bundle of a manifold is the typical object of a vector bundle. The fundamental part of the tangent bundle is to give a domain and range for the derivative of a smooth function. If $f : M \rightarrow N$ is a smooth function, with $M$ and $N$ smooth manifold, then the derivative is a smooth function $Df : TM \rightarrow TN$.”

1.2.3 Indicatrix

We consider the function $L(x^i, y^i)$ defined for all the line elements $(x^i, y^i)$ over the region $R$ of $M$. The equation

$$L(x^i, y^i) = 1 \{x^i \text{ fixed, } y^i \text{ variable}\}$$

represents an $(n - 1)$ dimensional locus in $T_n(p)$ i.e., a hypersurface. This hypersurface plays the role of the unit sphere in the geometry of the vector space $T_n(p)$ and is called indicatrix [71].
1.2.4 Minkowski space

“In mathematics and physics, Minkowski space or Minkowski space time (named after the mathematician Hermann Minkowski) is the mathematical context in which Einstein’s idea of fundamental relativity is most smoothly defined. In this setting, the three general dimensions of space are connected with a particular dimension of time to from a 4-dimensional manifold for defining a spacetime.

In theoretical physics, Minkowski space is usually diverged with Euclidean space. Although a Euclidean space has only spacelike dimensions, a Minkowski space also has one timelike dimension. Thus, the symmetry group of a Euclidean space becomes a Poincare group for a Minkowski space”.

1.3 Finsler Connections

Any quantities in a Finsler space is function of line element \((x, y)\). If \(S(x, y)\) is a scalar field in a Finsler space then \(\frac{\partial S}{\partial x^i}\) are not components of a covariant vector. If we have a non-linear connection \(N(x, y)\), we can collect the covariant vector field of the components

\[
S = \frac{\delta S}{\delta x^i}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}.
\]

Further, if we have quantities \(F_{jk}(x, y)\) which obey the transformation rule similar to Christoffel symbol, the covariant derivatives \(K^i_{jk}\) of a Finsler
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tensor field of (1,1)-type are defined by

\[ K_{jkl}^i = \frac{\delta K_i^j}{\delta x^k} + K_j^r F_{rk}^i - K_r^i F_{jk}^r. \] (1.3.1)

On the other hand, the partial derivatives of components of a tensor field \( K^i_j \) w.r.t. \( y^i \) gives a new tensor field, but we shall modify them as,

\[ K_{jkl}^i = \frac{\partial K_i^j}{\partial y^k} + K_j^r C_{rk}^i - K_r^i C_{jk}^r, \] (1.3.2)

where \( C_{jk}^i(x,y) \) are components of a tensor field of (1,2)-type. The collection \( (F_{jk}^i, N^i_j, C_{jk}^i) \), constitute a Finsler connection, and covariant derivatives given by (1.3.1) and (1.3.2) are called \( h \) and \( v \)-covariant derivatives of \( K^i_j \) respectively.

Now for any Finsler connection \( (F_{jk}^i, N^i_j, C_{jk}^i) \), we have five torsion tensors and three curvature tensors which are given by

\begin{align*}
(h) h - \text{torsion} : \quad & T_{jk}^i = F_{jk}^i - F_{kj}^i, \\
(v) v - \text{torsion} : \quad & S_{jk}^i = C_{jk}^i - C_{kj}^i, \\
(h) hv - \text{torsion} : \quad & C_{jk}^i \text{ as the vertical connection } C_{jk}^i, \\
(v) h - \text{torsion} : \quad & R_{jk}^i = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}, \\
(v) hv - \text{torsion} : \quad & P_{jk}^i = \frac{\delta F_{jk}^i}{\delta y^k} - F_{kj}^i, \\
h - \text{curvature} : \quad & R_{hjk}^i = \frac{\delta F_{hj}^i}{\delta x^k} - \frac{\delta F_{hk}^i}{\delta x^j} + F_{hk}^m F_{mj}^i + C_{hm}^i R_{jk}^m, \\
hv - \text{curvature} : \quad & P_{hjk}^i = \frac{\delta F_{hj}^i}{\delta y^k} - C_{hkij}^i + C_{hm}^i P_{jk}^m, \\
v - \text{curvature} : \quad & S_{hjk}^i = \frac{\delta C_{hj}^i}{\delta y^k} - \frac{\delta C_{hk}^i}{\delta y^j} + C_{hk}^m C_{mk}^i - C_{hk}^i C_{mj}^i.
\end{align*}
The deflection tensor field $D^i_j$ of a Finsler connection $F\Gamma$ is given by

$$D^i_j = y^k F^i_{jk} - N^i_j.$$ 

When a Finsler metric is given, various Finsler connections are constructed from the metric. The well-known examples are Cartan’s connection, Rund’s connection, Berwald’s connection and Hashiguchi’s connection.

### 1.3.1 Cartan’s Connection

E. Cartan (1934) introduced a system of axioms to give uniquely a Finsler connection from the basic function $L(x, y)$. It seems that his principles are rather artificial and introduced after foreseeing the desirable results. According to M. Matsumoto (1966) the Cartan’s axioms are equivalent to the following natural and elegant ones:

1. $g_{ij|k} = 0$,
2. $g_{ij|k} = 0$,
3. $F^i_{jk} = F^i_{kj}$,
4. $C^i_{jk} = C^i_{kj}$,
5. $D^i_j = 0$.

Cartan connection is denoted by $CT = (\Gamma^i_{jk}, \Gamma^i_{0k}, C^i_{jk})$ and is given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left( \frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h} \right),$$

$$C^i_{jk} = \gamma^i_{jk} y^k - 2C^i_{jm} G^m,$$

$$C^i_{jk} = C^i_{kj} = \frac{1}{2} g^{ih} \frac{\delta g_{jh}}{\delta y^k}.$$

Cartan’s $F^i_{jk}$ are of complicated form but $C^i_{jk}$ are equal to $C_{jkh} y^h$, where $C_{jkh}$ can be obtained from the above relations. Cartan showed $G^i_{jk} (= Berwald’s F^i_{jk}) = \Gamma^i_{jk} (= Cartan’s F^i_{jk}) + C^i_{jk|h} y^h$. 

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1.3.2 Rund’s Connection

The Rund’s connection is separately built from the basic function $L(x, y)$ with the following properties:

\begin{align*}
(i) \quad & g_{ij|k} = 0, \\
(ii) \quad & F^i_{jk} = F^i_{kj}, \\
(iii) \quad & C^i_{jk} = 0, \\
(iv) \quad & D^i_j = 0.
\end{align*}

Thus, Rund’s connection of a Finsler space is a Finsler connection which is obtained from Cartan’s connection $C\Gamma$ by the $C$-process. The $C$-process is characterized by expelling the torsion tensor $C^i_{jk}$. Thus, the first two connection coefficients of the Rund’s connection $R\Gamma$ are the same with those of the Cartan’s connection $C\Gamma$, while the third is equal to zero. Thus, the Rund’s connection $R\Gamma$ of the Finsler space $F^n$ is given by

\[ R\Gamma = (\Gamma^i_{jk}, \Gamma^i_{0k}, 0). \]

1.3.3 Berwald’s Connection

The Berwald’s connection is separately built from the basic function $L(x, y)$ and it has following properties:

\begin{align*}
(i) \quad & L_{|ii} = 0, \\
(ii) \quad & F^i_{jk} = F^i_{kj}, \\
(iii) \quad & C^i_{jk} = 0, \\
(iv) \quad & D^i_j = 0, \\
(v) \quad & P^i_{jk} = \partial_k N^i_j - F^i_{kj} = 0.
\end{align*}

Thus, the Berwald’s connection of a Finsler space is a Finsler connection which is obtained from Rund’s connection $R\Gamma$ by the $P^1$-process. The $P^1$-process is characterized by expelling the torsion tensor $P^i_{jk}$. The Berwald’s connection of Finsler space $F^n$ is denoted by $B\Gamma = (G^i_{jk}, G^i_j, 0)$, where $G^i_{jk} = \partial_j G^i_k$. 

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1.3.4 Hashiguchi’s Connection

The Hashiguchi’s connection is separately built from the basic function $L(x, y)$ and it has following properties:

(i) $L_i = 0$,  
(ii) $g_{ij|k} = 0$,  
(iii) $F^{i}_{jk} = F^{i}_{kj}$,  
(iv) $C^{i}_{jk} = C^{i}_{kj}$,  
(v) $D^i_k = 0$,  
(vi) $P^{i}_{jk} = \partial_k N^i_j - F^i_{kj} = 0$.

Thus, the Hashiguchi’s connection of a Finsler space is a Finsler connection which is obtained from Cartan’s connection by the $P^1$-process. The Hashiguchi’s connection of $F^n$ is given by $H\Gamma = (G^{i}_{jk}, G^i_k, C^{i}_{jk})$.

The relations of the above four Finsler connections are illustrated as follows:

$$
\begin{align*}
C\Gamma &= (\Gamma^{si}_{jk}, G^i_j, C^{i}_{jk}) \xrightarrow{C\text{-process}} R\Gamma = (\Gamma^{si}_{jk}, G^i_j, 0) \\
\downarrow P1\text{-process} & \quad \downarrow P1\text{-process} \\
H\Gamma &= (G^i_{jk}, G^i_k, C^{i}_{jk}) \xrightarrow{C\text{-process}} B\Gamma = (G^{si}_{jk}, G^i_j, 0)
\end{align*}
$$

The nonlinear connection $G^i_j$ is common to all these connections.

1.4 Special Finsler Spaces

In Riemannian geometry we have many interesting theorems such that if a Riemannian space is assumed to have special geometrical properties, or to satisfy special tensor equations, or to admit special tensor fields, then the space reduces to one of well-known space forms, for instance, Euclidean space, spheres, topological spheres, projective spaces etc.

On the other hand, in Finsler geometry we have special Finsler spaces,
namely, Riemannian spaces and Minkowskian spaces, but there are various kinds of Riemannian spaces and Minkowskian spaces. As a consequence, we have an important problem to classify all the Minkowskian spaces. It is easy to write down concrete forms of fundamental functions $L(x, y)$ which are interesting as a function, for instance, a Randers metric, Kropina metric, generalized Kropina metric, Matsumoto metric.

It is essential for the progress of Finsler geometry to find Finsler spaces, which are quite related to Riemannian spaces, but not Riemannian and Minkowskian spaces, which are related to flat spaces, but not flat. In the following, we give definitions of some special Finsler spaces along with some important properties.

1.4.1 Riemannian space

A Finsler space $F^n$ on a manifold $M$ is said to be a Riemannian space, if its basic function $L(x, y)$ is written as,

$$L(x, y) = g_{ij}(x)y^i y^j$$

Among Finsler spaces, the class of all the Riemannian spaces is characterized by $C_{ijk} = 0$ i.e. vertical connection $\Gamma^i$ of the Cartan’s connection $\nabla$ is flat.

1.4.2 Locally Minkowskian space

A Finsler space $F^n$ on a manifold $M$ is called a Locally Minkowskian space, if there exists a co-ordinate system $x^i$ in which $L$ is a function of $y^i$ only [52].

A Finsler space is locally Minkowskian if any only if
(a) For $C\Gamma : R^h_{ijk} = C^h_{ijkl} = 0,$
(b) For $R\Gamma : R^h_{ijk} = F^h_{ijk} = 0,$
(c) For $B\Gamma : H^h_{ijk} = G^h_{ijk} = 0.$

1.4.3 Berwald space

If the connection coefficient $G^i_{jk}$ of the Berwald’s connection $B\Gamma$ given by,

$$G^i_{jk} = \dot{\partial}_j G^i_k$$

are function of position alone, the space is said to be a Berwald space [52]. A Finsler space is Berwald space if and only if

(a) For $C\Gamma : C^h_{ijkl} = 0,$
(b) For $R\Gamma : F^h_{ijk} = 0,$
(c) For $B\Gamma : G^h_{ijk} = 0.$

1.4.4 Landsberg space

A Finsler space $F^n$ on a manifold $M$ is said to be Landsberg space [52] if the Berwald connection $B\Gamma$ is $h$-metrical i.e, $g_{ij}(k) = 0.$

In conditions of the Cartan’s connection $CT$ a Landsberg space is described by,

a) $P^h_{jk} = 0$
b) $P^h_{ijk} = 0.$
1.4.5 Douglas Space

An n-dimensional Finsler space $F^n$ is said to be a Douglas space if

$$D^{ij}(x, y) = G^i(x, y)(x, y)y^i - G^j(x, y)y^j.$$ 

all are “homogeneous polynomials” in $y^i$ of degree-3.

1.4.6 Finsler Space with $(\alpha, \beta)$-metric

The theory of a Finsler spaces equipped with $(\alpha, \beta)$-metric is quite old, but it is the most significant view of the Finsler geometry and its application.

Various Finsler spaces with special $(\alpha, \beta)$-metric have their particular names, some of which are given below:

**Definition 1.4.1.** “A Finsler metric $F(x, y)$ [52] is said to be an $(\alpha, \beta)$-metric, when $F$ is a positively homogeneous function $F(\alpha, \beta)$ of first degree in two variables $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$.”

It is important to make-believe that $\alpha$ is a Riemannian metric, i.e. non-degenerate (regular) and positive definite, but there are some cases for utilizations where these regulations are lenient. Further, we have to restrict our considerations to convenient domain of $(x, y)$ on explanation of fundamental from of the function $F(\alpha, \beta)$.

**Definition 1.4.2.** “The $(\alpha, \beta)$-metric $F = \alpha + \beta$ [52] is said to be a Randers metric and the Finsler space $F^n$ with this metric is said to be a Randers space.

In 1980, Hashiguchi and Ichijyo described the successive delightful observation on Randers metric”.

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**Proposition 1.4.1** Randers metric $F = \alpha + \beta$ [30] is positive valued, if and only if $a_{ij} - b_i b_j$ is positive-definite, given that $a_{ij}$ is positive-definite.

**Definition 1.4.3.** “The $(\alpha, \beta)$-metric $F = \frac{\alpha^2}{\beta}$ [52], is said to be a Kropina metric and the space $F^n$ with this metric is said to be a Kropina space”.

Wrona [81] has given the interesting example of Kropina metric for a Kropina space, the direction $y^i$ affinity to the hyperplane $\beta(x, y) = b_i(x)y^i = 0$ of the tangent space at any point $x$ must be definitely eliminated. The indicatrix is to develop asymptotically on this hyperplane. Thus, a Kropina metric is never positive definite.

It is believed that Kropina himself has played consideration to alike a metric from a purely mathematical viewpoint. There are closed relation among this kind of metric and Lagrangian function of analytic dynamics.

**Definition 1.4.4.** “The $(\alpha, \beta)$-metric $F = \alpha^{m+1} \beta^{-m} \ (m \neq 0, -1)$ [52] is said to be a generalized $m$-Kropina metric and the space associated with it is called generalized Kropina space”.

The identical establishment defined by Matsumoto is as follows.

**Proposition 1.4.2.** A slope, the graph of a function $z = f(x, y)$ [53] of the earth surface is regarded as a 2-dimensional Finsler space $F^2$ with basic function

$$L(x, y, \dot{x}, \dot{y}) = \frac{\alpha^2}{v\alpha - w\beta},$$

where $v$ and $w$ are non-zero constants and

$$\alpha^2 = \dot{x}^2 + \dot{y}^2 + (\dot{x}f_x + \dot{y}f_y)^2,$$

$$\beta = \dot{x}f_x + \dot{y}f_y.$$
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This $\alpha$ is the familiar convinced Riemannian metric and $\beta$ is a differential form

$$\beta(x, dx) = \alpha f(x, y).$$

The two constants $v$ and $w$ are such that one can stroll $v$ meters per minute on the horizontal plane and $2w$ is identical to the acceleration of descending. Aikou, Hashiguchi and Yamauchi generalized and conducted the aforesaid metric in the following manner:-

**Definition 1.4.5.** “The $(\alpha, \beta)$-metric $F = \frac{\alpha^2}{\alpha - \beta}$ [1] is called a slope metric or Matsumoto metric and the Finsler space associated with this metric is said to be a Matsumoto space”.

**Definition 1.4.6.** “The Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$ can be expressed as

$$L = \alpha \left[ \sum_{k=0}^{\infty} \left( \frac{\beta}{\alpha} \right)^k \right]$$

(1.4.1)

for $|\beta| < |\alpha|$. We regard $b_i(x)$ as very small numerically. If we neglect all the powers $\geq 2$ of $b_i(x)$ in (1.4.1), then $F = \alpha + \beta$, that is a Randers metric. If we neglect all the powers $\geq 3$ of $b_i(x)$ in (1.4.1), then $F$ is the first approximate Matsumoto metric [67]. If we neglect all the powers $\geq 4$ of $b_i(x)$ in (1.4.1) then $F$ is the second approximate Matsumoto metric [68]. If we neglect all the powers $\geq 5$ of $b_i(x)$ in (1.4.1), then $(\alpha, \beta)$-metric

$$F = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} + \frac{\beta^4}{\alpha^3}$$

(1.4.2)

is an approximate metric to the Matsumoto metric. We call the $(\alpha, \beta)$-metric (1.4.2) the third approximate Matsumoto metric”.

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1.5 Cartan space

A Cartan space is a pair $C^n = (M, H)$, where $H : T^*M \to \mathbb{R}$ is a function having the properties:

1. It is two homogeneous w.r.t. $p_i$.
2. It’s Hessian, particularly

$$g^{ij}(x, p) = \frac{1}{2}(\partial_i \partial_j H).$$