CHAPTER 2

Some aspects of Harmonic Numbers which divide
the sum of its Positive divisors.

2.1 Introduction

Let $\mathcal{H}$ be a set of harmonic numbers and let $n$ be a natural number. Then we have two possibilities.

(i) $\mathcal{H}_1$ = the set of harmonic numbers such that $g.c.d \left( n, \sigma(n) \right) = n$

(ii) $\mathcal{H}_2$ = the set of harmonic numbers such that $g.c.d \left( n, \sigma(n) \right) \neq n$

In 1948, O Ore [55] introduced the concept of harmonic numbers for the first time. Let $H(n)$ denote the harmonic mean of all divisors of a natural number $n$ such that

$$H(n) = \frac{n \tau(n)}{\sigma(n)}$$

where $\sigma(n)$ and $\tau(n)$ denote the number of divisors of $n$, and their sum, respectively. If $H(n)$ is an integer then $n$ is a harmonic number.

In this Chapter, we consider some particular form of harmonic numbers $n$ such that $g.c.d \left( n, \sigma(n) \right) = n$. Three types of natural numbers which can be factorized as $p^m q$, $2^k p^m q$ and $2^k p_1 p_2 \ldots p_t$ ($p, q, \text{ and all } p_i's$ are distinct primes)

This chapter is the outcome of our paper “Some aspects of Harmonic Numbers which divide the sum of its Positive divisors” which is published in “IOSR Journal of Mathematics (IOSR-JM)”
have been discussed and some propositions have been developed to understand the properties of these kind of numbers. We have shown that harmonic numbers of prime numbers of the form $pq, p^2q$ and $p^mq$ does not exist if both $p$ and $q$ are odd primes. Further, some useful properties of the harmonic numbers of the form $2^kp^mq$, for some values of $m$ have been discussed with the help of some propositions.

### 2.2 Some properties of the harmonic numbers of the form $p^m q$.

In this section, we have produced some related properties of harmonic numbers of the form $p^m q$ such that $g.c.d. (n, \sigma(n)) = n$.

**Proposition 2.2.1:** 6 is the only harmonic number of the form $pq$, where $p$ and $q$ are distinct primes.

**Proof:** Let the number be $n = pq$. Then $\sigma(n) = 1 + p + q + pq$ and $\tau(n) = 4$.

If $n$ divides $\sigma(n)$, then there exist some positive integer $k$ such that $1 + p + q + pq = kpq$. Since $n$ is a harmonic number, $k|\tau(n) = 4$. Then the possible values of $k$ are 1, 2 and 4. But $k \neq 1$. If $k = 1$, $1 + p + q = 0$, which is not possible.

Then the other possibilities are

$$1 + p + q = pq$$  \hspace{1cm} (2.2.1)

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and \[1 + p + q = 3pq\] \hspace{1cm} (2.2.2)

From (2.2.1) \[1 + q = pq - p \text{ or } p(q - 1) = 1 + q \text{ or } p = \frac{1+q}{q-1}\]

From (2.2.2) \[3pq - p = 1 + q \text{ or } p = \frac{1+q}{3q-1}\]

Hence, either \(p = \frac{1+q}{q-1}\) or \(p = \frac{1+q}{3q-1}\)

If \((3q - 1)|(q + 1)\), then \(1 + q > 3q - 1 \implies 2q < 2\), which is not possible.

Therefore \(p\) must be equal to \(\frac{1+q}{q-1}\).

Also, we have \((q - 1)|(q + 1)\) and \((q - 1)|(q - 1)\)

\[\implies (q - 1)|(q + 1) + (q - 1)\]

\[\implies (q - 1)|2,\]

which is possible when \(q - 1 = 1\) and \(q - 1 = 2\), that is, \(q = 2\) and 3

If \(q = 2\) and 3, then the corresponding \(p = \frac{1+q}{q-1}\) are 3 and 2 respectively.

In any case, the only possible values of \(p\) and \(q\) are 2 and 3 or 3 and 2. So, \(n = 2 \times 3 = 6\) is the only harmonic number of the form \(pq\).

\(\Box\)

**Proposition 2.2.2:** 28 is the only harmonic number of the form \(p^2q\), where \(p\) and \(q\) are distinct primes.

**Proof:** Let the number be of the form \(n = p^2q\). Then \(\sigma(n) = 1 + p + p^2 + q + pq + p^2q\) and \(\tau(n) = 6\).
Let \( g. c. d. (n, \sigma(n)) = n \), then there exist some positive integer \( k \) such that

\[
\sigma(n) = nk ,
\]

Now \( \sigma(n) = 1 + p + p^2 + q + pq + p^2q \)

\[\Rightarrow nk = 1 + p + p^2 + q + pq + p^2q \]

\[\Rightarrow p^2qk = 1 + p + p^2 + q + pq + p^2 \]

\[\Rightarrow q(-1 - p - p^2 + p^2k) = 1 + p + p^2 \]

\[\Rightarrow q = \frac{1 + p + p^2}{-1 - p - p^2 + kp^2} . \]

\[\Rightarrow -(1 + p + p^2) + kp^2 = \frac{1 + p + p^2}{q} \] (2.2.3)

Since \( n \) is a harmonic number so \( k|\tau(n) = 6 \). Then the possible values of \( k \) are 1, 2, 3 and 6. \( k \neq 1 \). For \( k = 1 \) we have \( \sigma(n) = n \). This implies that

\[1 + p + p^2 + q + pq + p^2q = p^2q \]

\[\Rightarrow 1 + p + p^2 + q + pq = 0 , \]

which is absurd as \( n > 1 \). So assume \( k > 1 \)

Let \( kp^2 - (1 + p + p^2) = t \) (2.2.4)

Then (2.2.3) can be written as

\[1 + p + p^2 = qt \] (2.2.5)

for some positive integer \( t \). From (2.2.4) and (2.2.5), we have \( kp^2 = (q + 1)t \).
Case 2.2(a):

Consider the case when \( p \) and \( q \) are odd distinct primes. Thus \( 1 + p + p^2 \) is also odd. Therefore in (2.2.5), equality hold only when \( t \) is odd. i.e \( qt \) is an odd number. Thus, (2.2.4), we have seen that both \( t \) and \( 1 + p + p^2 \) are odd and the equality holds only for the even values of \( kp^2 \). But \( p \) is odd prime, so \( k \) must be even. That is, \( k \neq 3 \). So, only choice of \( k \) are 2 and 6.

If \( k = 2 \)

(2.2.4) can be written as \( kp^2 = (q + 1)t \)

\[
\Rightarrow 2p^2 = (q + 1)t
\]

\[
\Rightarrow 2 \cdot p \cdot p = (q + 1)t \tag{2.2.6}
\]

The possible form of (2.2.6) are, either \( q + 1 = 2p^2, t = 1 \) or \( q + 1 = 2p, t = p \).

If \( q + 1 = 2p^2 \) and \( t = 1 \) then (2.2.4) becomes

\[
1 + p + p^2 = (2p^2 - 1) \cdot 1
\]

\[
\Rightarrow p^2 - p - 2 = 0 \tag{2.2.7}
\]

\[
\Rightarrow p = 2, -1 \text{, not possible}
\]

Similarly, if \( q + 1 = 2p \) and \( t = p \) then (2.2.5) is

\[
\Rightarrow 1 + p + p^2 = (2p - 1)p
\]

\[
\Rightarrow p^2 - p - 2 = 0 \tag{2.2.8}
\]
Thus, for $k = 2$ there is no odd prime $p$. Hence, $k \neq 2$

**If $k = 6$, from (2.2.4)**, we have

$$kp^2 = (q + 1)t$$

$$\Rightarrow 6p^2 = (q + 1)t$$

$$\Rightarrow 2.3.p.p = (q + 1)t \quad (2.2.9)$$

The possible form of $(q + 1, t)$ satisfying (2.2.9) are

$\{(2,3p^2), (6, p^2), (2p, 3p), (2p^2, 3), (6p, p), (6p^2, 1)\}$. We have seen that these order pair elements does not satisfy the equation (2.2.5).

Hence, there is no odd prime $p$ such that $k = 2$ and 6.

**Case 2.2(b):**

Next, we assume that $p$ is an even prime i.e. $p = 2$. Then (2.2.5) is

$$1 + p + p^2 = qt$$

$$\Rightarrow qt = 7 \quad (2.2.10)$$

So, $q = 7$ is the only choice.

**Case 2.2(c):**
If \( q \) is even, then \( p \) must be odd as \( p \) and \( q \) are distinct primes. Then (2.2.5) is not true for even value of \( q \) and odd value of \( p \). Therefore, \( q \) cannot be even.

Thus, from the above three cases, we have seen that, the only possible choice of \( p \) and \( q \) are: \( p = 2 \) and \( q = 7 \) and hence \( 28 = 2^2 \cdot 7 \) is the only form of \( p^2q \). ■

**Proposition 2.2.3:** For any natural number \( m \), if \( n \) is a harmonic number of the form \( p^m q \), where \( p \) is any odd prime and \( q \) is any prime, then \( H(n) = \frac{\tau(n)}{2} \).

**Proof:** Let \( n = p^m q \). If \( n \) is a harmonic number then

\[
H(n) = \frac{n \tau(n)}{\sigma(n)} \tag{2.2.11}
\]

Since \( n \mid \sigma(n) \), for some positive integer \( k \), we have \( \sigma(n) = nk \) \tag{2.2.12}

**Objective:** To find the possible values of \( k \).

Now \( nk = \sum_{i=0}^{m} p^i + q \sum_{i=0}^{m} p^i \)

\[
\Rightarrow p^m q k = \sum_{i=0}^{m} p^i + q \sum_{i=0}^{m} p^i \tag{2.2.13}
\]

\[
\Rightarrow q(p^m k - \sum_{i=0}^{m} p^i) = \sum_{i=0}^{m} p^i \]

Let \( kp^m - \sum_{i=0}^{m} p^i = t \) \tag{2.2.14}

and \( \sum_{i=0}^{m} p^i = qt \) \tag{2.2.15}

where \( t \) is some positive integer. Adding (2.2.14) and (2.2.15), we get \( kp^m = (q + 1)t \).
\[ q = \frac{\sum_{i=0}^{m} p^i}{kp^m - \sum_{i=0}^{m} p^i} > 1 \]

\[ \Rightarrow 2 \sum_{i=0}^{m} p^i > kp^m \]

\[ \Rightarrow 2 \frac{p^{m+1} - 1}{p - 1} > kp^m \]

\[ \Rightarrow k < \frac{2(p^{m+1} - 1)}{p^m(p-1)} \]

\[ \Rightarrow k < 2 \left( \frac{p}{p-1} - \frac{1}{p^m(p-1)} \right) \]

Now, \( \frac{p}{p-1} \) is the choice of \( \frac{p}{p-1} \) for the choice of \( \frac{p}{p-1} \) not possible.

If \( \frac{p}{p-1} > 2 \) then \( 2p - 2 < p \), which implies that \( p < 2 \), not possible.

If \( \frac{p}{p-1} = 2 \) then \( p = 2 \).

When \( p = 2 \), \( k < 4 \) (from 2.2.16) which is not possible, because \( p \) is an odd prime.

If \( \frac{p}{p-1} < 2 \), then again we have two possible choice \( \frac{p}{p-1} \leq 1.5 \)

If \( \frac{p}{p-1} > 1.5 \) then \( p < 3 \), it means that \( p = 2 \) can be considered.

If \( \frac{p}{p-1} = 1.5 \) then \( p = 3 \).
Thus, when \( \frac{p}{p-1} \geq 1.5 \), \( k < 3 \) (from 2.2.16) yields \( k \leq 2 \)

If \( \frac{p}{p-1} < 1.5 \), then \( p > 3 \), we have \( k < 3 \) yields \( k \leq 2 \)

Hence from (2.2.16) we have \( k \leq 2 \). Therefore, either \( k = 1 \) or \( k = 2 \)

If \( k = 1 \), (2.2.14) becomes

\[
p^m - \sum_{i=1}^{m} p^i = t
\]

\[
p^m - \sum_{i=0}^{m-1} p^i - p^m = t
\]

\[-\sum_{i=0}^{m} p^i = t, \text{ which is not possible.}
\]

So \( k \neq 1 \). Hence, \( k = 2 \) is the only choice such that

\[
H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{\tau(n)}{2}
\]

**Remark 2.2.1:** Pomerance [56] and Callan [5], proved that the only harmonic numbers of the form \( p^aq^b \) are perfect numbers.

The next two Propositions assert that there is no perfect numbers of the form \( p^mq \) for odd primes \( p, q \) and odd number \( m \) such that \( g.c.d. (n, \sigma(n)) = n \). Same is true for also the number of the form \( pq^m \).
**Proposition 2.2.4** If \( p \) and \( q \) are distinct odd primes, then there is no perfect number of the form \( p^3q \).

**Proof:** Let \( n = p^3q \) then \( \sigma(n) = (1 + p + p^2 + p^3)(1 + q) \).

If possible, let \( n \) be a perfect number. Then

\[
\sigma(n) = 2n.
\]

i.e.

\[
(1 + p + p^2 + p^3)(1 + q) = 2p^3q.
\]

i.e.

\[
\left(\frac{1+p+p^2+p^3}{p^3}\right)\left(\frac{1+q}{q}\right) = 2 \tag{2.2.17}
\]

Suppose, if possible, \( p | (1 + p + p^2 + p^3) \). Also \( p | (p + p^2 + p^3) \).

Now

\[
p | (1 + p + p^2 + p^3) \quad \text{and} \quad p | (p + p^2 + p^3)
\]

\[\Rightarrow \quad p | ((1 + p + p^2 + p^3) - (p + p^2 + p^3))
\]

\[\Rightarrow \quad p | 1, \quad \text{which is not possible.}
\]

Therefore

\[
p \not| (1 + p + p^2 + p^3) \quad \text{or} \quad p^3 \not| (1 + p + p^2 + p^3).
\]

Next, possible division either \( p^3 | (1 + q) \) or \( q | (1 + p + p^2 + p^3) \).

Now, (2.2.17) can be expressed in the following forms

(i) \( 2p^3 = 1 + q \) and \( q = 1 + p + p^2 + p^3 \)

(ii) \( 2q = 1 + p + p^2 + p^3 \) and \( p^3 = 1 + q \).

But \( q \neq 1 + p + p^2 + p^3 \) and \( p^3 \neq 1 + q \), as \( p \) and \( q \) both are odd primes.
Thus, we have seen that \( p^3 \nmid (1 + p + p^2 + p^3) \), \( p^3 \nmid (1 + q) \) and \( q \nmid (1 + p + p^2 + p^3) \).

This proves that \( \sigma(n) \neq 2n \). Hence \( n \) is not a perfect number. \( \blacksquare \)

The Prop. (2.3.4) may be generalized as follows:

**Proposition 2.2.5:** If \( m \) is an odd natural number and \( p, q \) are odd primes, then there is no perfect number of the form \( p^m q \).

**Proof:** Let \( n = p^m q \) then \( \sigma(n) = (\sum_{i=0}^{i=m} p^i)(1 + q) \).

Let us assume that \( n \) is a perfect number and so \( n | \sigma(n) \). Then

\[
\sigma(n) = 2n
\]

\[
\Rightarrow \quad (\sum_{i=0}^{i=m} p^i)(1 + q) = 2 p^m q
\]

\[
\Rightarrow \quad \left(\frac{\sum_{i=0}^{i=m} p^i(1+q)}{p^m q}\right) = 2 \quad \text{(2.2.18)}
\]

If possible, let \( p | (\sum_{i=0}^{i=m} p^i) \). Also, we have \( p | \sum_{i=0}^{i=m} p^i \)

\[
\Rightarrow \quad p | (\sum_{i=0}^{i=m} p^i) - (\sum_{i=1}^{i=m} p^i),
\]

\[
\Rightarrow \quad p | 1, \text{ not possible.}
\]

So \( p \nmid (\sum_{i=0}^{i=m} p^i) \). Then the other possibilities are, either \( p^m | (1 + q) \) or \( q | (\sum_{i=0}^{i=m} p^i) \).

The left side of (2.2.18) can be written as
either \[ 2p^m = 1 + q \] and \[ q = \sum_{i=0}^{m-1} p^i \]

or \[ 2q = \sum_{i=0}^{m} p^i \] and \[ p^m = 1 + q \]

But \( q \neq \sum_{i=0}^{m} p^i \) and \( p^m \neq 1 + q \), because left hand side of both parts is odd and right side of both parts is even.

– a contradiction.

Therefore, \( p^m \nmid (\sum_{i=0}^{m} p^i) \), \( p^m \nmid (1 + q) \), \( q \nmid (\sum_{i=0}^{m} p^i) \) and hence \( \sigma(n) \neq 2n \). i.e. \( n \) is not a perfect number.

\[ \blacksquare \]

**Remark 2.2.2:** We know that every perfect number is a harmonic number. As a consequence of Prop. (2.2.4) and Prop. (2.2.5), we can say that “There is no harmonic numbers of the form \( n = p^m q \), \( p \) and \( q \) are odd distinct primes and \( m \) is an odd natural number such that \( g.c.d. (n, \sigma(n)) = n \).”

### 2.3 Some properties of harmonic numbers of the form \( n = 2^kp^m q \)

such that \( g.c.d. (n, \sigma(n)) = n \).

In this section, we consider a number of the form \( 2^kp^m q \), \( p \) and \( q \) are primes and \( m, k \) are some natural numbers and produced some results. Here, in

some parts we used MATHEMATICA 7.0.1. for analysis the result.

**Proposition 2.3.1:** Let the number be of the form \( n = 2^k p \), \( p \) is any prime and \( k \) is positive integer, then there exists at least one \( n \) such that \( H(n) = \frac{\tau(n)}{2} \).
Proof: Given that \( n = 2^k p \). Then \( \sigma(n) = \sum_{i=0}^{k} 2^i \) \((1+p)\) and \( \tau(n) = 2(k+1) \)

Since \( \gcd(n, \sigma(n)) = n \), three exists a positive integer \( k_1 \) such that

\[
\sigma(n) = nk_1.
\]

\[
\Rightarrow nk_1 = \sum_{i=0}^{k} 2^i \) \((1+p)\)
\]

\[
\Rightarrow 2^k pk_1 = \sum_{i=0}^{k} 2^i \) \((1+p)\)
\]

\[
\Rightarrow k_1 = \frac{1+p}{p} \left( \frac{2^{k+1} - 1}{2^k} \right) \quad (2.3.1)
\]

Let \( H(n) \) denote the harmonic mean of positive divisors of \( n \). Then \( H(n) = \frac{n\tau(n)}{\sigma(n)} \)

\[
\Rightarrow \tau(n) = k_1 H(n)
\]

\[
\Rightarrow \tau(n) = H(n) \frac{1+p}{p} \left( \frac{2^{k+1} - 1}{2^k} \right)
\]

\[
\Rightarrow 2(1 + k) = H(n) \frac{1+p}{p} \left( \frac{2^{k+1} - 1}{2^k} \right)
\]

\[
\Rightarrow (1 + k) = H(n) \frac{1+p}{p} \left( \frac{2^{k+1} - 1}{2^{k+1}} \right) \quad (2.3.2)
\]

With the help of MATHEMATICA software, a search technique is done by choosing some values of \( k + 1 \) such that \( 2^{k+1} - 1 \) is a prime number. With that regards, \( k + 1 \) assumed to be a prime itself. Let the prime be \( p = 2^{k+1} - 1 \). Then (2.3.2) reduces to \( H(n) = k + 1 \) or \( H(n) = \frac{\tau(n)}{2} \).
Thus, if \( n \) is a number of the form \( 2^k p \), then there exists a harmonic number \( n \) such that \( H(n) = \frac{\tau(n)}{2} \).

**Example 2.3.1**: In Prop. 2.3.1, if we take \( k + 1 = 2 \) then the corresponding prime \( p = 2^{k+1} - 1 = 3 \) and hence \( H(n) = \frac{\tau(n)}{2} = k + 1 = 2 \), which is the harmonic mean of the harmonic number \( n = 6 \). Similarly, if \( k + 1 = 3 \), we have \( p = 7 \) and the corresponding harmonic number is 28 with harmonic mean 3 and so on.

**Remark 2.3.1**: In Prop.2.3.2, if we take the values of \( k + 1 \), such as 2,3,5,7,13,17,……..then the corresponding \( p = 2^{k+1} - 1 \) provides a prime number and these primes are Mersenne prime.

**Proposition 2.3.2**: If \( p \) is an odd prime, then there does not exist any harmonic number of the form \( 2^k p^2 \) for any natural number \( k \).

**Proof**: Let \( n = 2^k p^2 \).

Then \( \sigma(n) = (2^{k+1} - 1)(1 + p + p^2) \) and \( \tau(n) = 3(k + 1) \)

And \( H(n) = \frac{n \tau(n)}{\sigma(n)} = \frac{3n(k+1)}{\sigma(n)} \) \hspace{1cm} (2.3.3)

Now \( g. c. d \left( n, \sigma(n) \right) = n \)

\( \Rightarrow \) \( \sigma(n) = nk_1 \), for some positive integer \( k_1 \)

Therefore \( H(n) = \frac{3n(k+1)}{\sigma(n)} \).
\[
\frac{k_1}{k} = \frac{3(k+1)}{k_1}
\]

Now
\[
\frac{\sigma(n)}{n} = \frac{(2^{k+1} - 1)(1 + p + p^2)}{2^k p^2}
\]  \hspace{1cm} (2.3.5)

In (2.3.5), \(2^k \nmid (2^{k+1} - 1)\).

\(p\) is an odd prime, so \(1 + p + p^2\) is also odd.

Hence \(2^k \nmid (1 + p + p^2)\) and \(p^2 \nmid (2^{k+1} - 1)\).

Thus, we have seen that \(\frac{\sigma(n)}{n}\) is not an integer.

i.e.
\[\frac{\sigma(n)}{n} \neq k_1\]

and \[H(n) = \frac{3n(k+1)}{\sigma(n)} = \frac{3(k+1)}{k} \neq k_2\] and hence the result. \(\blacksquare\)

The proposition (2.3.2) may be generalized as follows:

**Proposition 2.3.3:** If \(p\) is an odd prime, then there is no harmonic number of the form \(2^k p^{2x}\), where \(k\) and \(x\) are some natural numbers.

**Proof:** Let \(n = 2^k p^{2x}\), then \(\sigma(n) = (2^{k+1} - 1)\sum_{i=0}^{2x} p^i\) and \(\tau(n) = (2x + 1)(k + 1)\).

\[H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{n(k+1)(2x+1)}{\sigma(n)}\]

The proof of this result is quite similar to Prop.(2.3.2), so we skip the details.

\[g.c.d\left(n, \sigma(n)\right) = n\] gives \(\frac{\sigma(n)}{n} = k_1\) (say), a positive integer.
Proceeding as in the Prop. (2.3.3), easily we can prove that $H(n)$ is not an integer, which means that the number $n$ will not form any harmonic numbers.

\textbf{Proposition 2.3.4:} There is no harmonic number of the form $2^k p^3$ in the interval $[1, 10^{76}]$ for any odd prime $p$ and a natural number $k$.

\textbf{Proof:} Let the number be $n = 2^k p^3$.

Then $\sigma(n) = (2^{k+1} - 1)(1 + p + p^2 + p^3)$ and $\tau(n) = 4(k + 1)$.

Since $g.c.d. (n, \sigma(n)) = n$, there exists a positive integer $k_1$ such that

$$\sigma(n) = nk_1$$

Or

$$k_1 = \frac{\sigma(n)}{n} = \frac{(2^{k+1}-1)(1+p+p^2+p^3)}{2^kp^3} \quad (2.3.6)$$

To prove that, $k_1$ is not an integer.

If possible, let $p|(1 + p + p^2 + p^3)$. Also $p|(p + p^2 + p^3)$.

Now

$p|(1 + p + p^2 + p^3)$ and $p|(p + p^2 + p^3)$

$\Rightarrow$

$p|((1 + p + p^2 + p^3) - (p + p^2 + p^3))$

$\Rightarrow$

$p|1$, which is not possible.

Therefore, $p \nmid (1 + p + p^2 + p^3)$ or $p^3 \nmid (1 + p + p^2 + p^3)$. Also $2^k \nmid (2^{k+1} - 1)$

Only possibilities are, either $p^3|(2^{k+1} - 1)$ or $2^k|(1 + p + p^2 + p^3)$.

Let $xp^3 = 2^{k+1} - 1 \quad (2.3.7)$
and \(1 + p + p^2 + p^3 = y2^k\) \hspace{1cm} (2.3.8)

for some positive integers \(x\) and \(y\).

With a suitable computer program performed in MATHEMATICA 7.0.1, we observed that, for \(k \in \mathbb{N}\) if \(2^{k+1} - 1\) is sufficiently large then the associate value of \(p\) is comparatively very small for which the equation (2.3.8) is not satisfied by the values of \(p\) and \(y \geq 1\). We have checked it for \(2^{k+1} - 1\), where \(1 \leq k \leq 251\) for which the values of \(p\) has been found so small and does not satisfy the equation (2.3.8), which shows that \(k_1\) is not an integer. Hence, there is no harmonic number of the form \(2^kp^3\) for \(k \leq 251\) or up to \(10^{76}\). However, it can be checked to take the higher range of interval also. \(\blacksquare\)

**Example 2.3.2:** Consider a number \(2^{35+1} - 1 = 3^3(5.7.13.19.37.73.109)\) which is a form of \(xp^3\), where \(x = 5.7.13.19.37.73.109\) and \(p = 3\) and \(k = 35\). For these values of \(x, k\) and some \(y \geq 1\) the relation (2.3.8) is not possible.

**Proposition 2.3.5:** Let the number be of the form \(n = 2^kpq\), \(p\) and \(q\) are odd distinct primes, then there does not exist any \(n\) such that \(H(n) = \frac{\tau(n)}{2^x}, x \geq 1\)

**Proof:** Given \(n = 2^kpq\) . Then

\[
\sigma(n) = (2^{k+1} - 1)(1 + p)(1 + q) \text{ and } \tau(n) = 4(k + 1).
\]

Now \(\text{g. c. d} (n, \sigma(n)) = n\)

For some positive integer \(k_1\) such that

\[
\sigma(n) = nk_1
\]
\[
k_1 = \frac{(2^{k+1-1})(1+p)(1+q)}{2^kpq}
\]

Suppose that \( n \) is a harmonic number. Then

\[
H(n) = \frac{n\tau(n)}{\sigma(n)}
\]

\[
\Rightarrow \tau(n) = k_1 H(n)
\]

\[
(1 + k)4 = H(n) \left( \frac{1+p}{p} \right) \left( \frac{1+q}{q} \right) \left( \frac{2^{k+1-1}}{2^k} \right)
\]

\[
(1 + k)2 = H(n) \left( \frac{1+p}{p} \right) \left( \frac{1+q}{q} \right) \left( \frac{2^{k+1-1}}{2^{k+1}} \right) \quad (2.3.9)
\]

Now

\[
H(n) = \frac{\tau(n)}{2^x}, x \geq 1
\]

Let us consider that \( x = 1 \). Then \( H(n) = \frac{\tau(n)}{2} \)

From (2.3.9), we have

\[
\left( \frac{1+p}{p} \right) \left( \frac{1+q}{q} \right) \left( \frac{2^{k+1-1}}{2^{k+1}} \right) = 1
\]

\[
\Rightarrow \left( \frac{1+p+q+pq}{pq} \right) = \left( \frac{2^{k+1}}{2^{k+1-1}} \right)
\]

\[
\Rightarrow (1 + p + q + pq)(2^{k+1} - 1) = pq \cdot 2^k
\]

\[
\Rightarrow (1 + p + q + pq)2^{k+1} - (1 + p + q + pq) - pq \cdot 2^k = 0
\]

\[
\Rightarrow 2^{k+1}(1 + p + q + pq - pq) = 1 + p + q + pq
\]

\[
\Rightarrow 2^{k+1}(1 + p + q) - (1 + p + q) - pq = 0
\]
\[ (2^{k+1} - 1)(1 + p + q) - pq = 0 \]
\[ (2^{k+1} - 1)(1 + p + q) = pq \quad (2.3.10) \]

If \( p = m (2^{k+1} - 1) \) or \( q = m (2^{k+1} - 1) \) for \( m > 1 \) then both \( p \) and \( q \) are in composite form, which is not possible. Therefore \( p \) and \( q \), both must be of the form \( 2^{k+1} - 1 \). But, if \( p = 2^{k+1} - 1 = q \) then (2.3.10) gives \( 1 + p + q = q \) implies \( p = -1 \) or \( 1 + p + q = p \) implies \( q = -1 \), which is not possible. Thus, if \( x = 1 \), \( H(n) \neq \frac{\tau(n)}{2^x} \).

Next, we consider that \( x > 1 \).

We have \( H(n) = \frac{\tau(n)}{2^x} \)

Or \( \frac{\sigma(n)}{2^x n} = 1 \).

\[ \Rightarrow \left( \frac{1+p}{p} \right) \left( \frac{1+q}{q} \right) \left( \frac{2^{k+1}-1}{2^{k+x}} \right) = 1 \]
\[ \Rightarrow \left( \frac{1+p+q}{pq} + 1 \right) \left( \frac{2^{k+1}-1}{2^{k+x}} \right) = 1 \]
\[ \Rightarrow \left( \frac{1+p+q}{pq} \right) \left( \frac{2^{k+1}-1}{2^{k+x}} \right) = 1 - \left( \frac{2^{k+1}-1}{2^{k+x}} \right) \]
\[ \Rightarrow (1 + p + q)(2^{k+1} - 1) = (2^{k+x} - 2^{k+1} + 1)pq \]
\[ \Rightarrow (1 + p + q)(2^{k+1} - 1) = \{2^{k+1}(2^{x-1} - 1) + 1\}pq \quad (2.3.11) \]

For \( x \geq 2 \), clearly \( 2^{k+1}(2^{x-1} - 1 + 1) > (2^{k+1} - 1) \) \( (2.3.12) \)

Also, \( pq > (1 + p + q) \) \( (2.3.13) \)
From (2.3.12) and (2.3.13), we have

\[ (2^{k+1}(2^{x-1} - 1) + 1)pq > (1 + p + q)(2^{k+1} - 1), \]

which is not possible.

Therefore, \( x \) cannot be greater than 1. Thus, in any case, for \( x \geq 1 \), \( H(n) \neq \frac{\tau(n)}{2^x} \).

Hence for any odd primes \( p \) and \( q \), there does not exist any harmonic number of the form \( n = 2^x pq \) such that \( H(n) = \frac{\tau(n)}{2^x}, x \geq 1 \).

\[ \blacktriangleleft \]

**Proposition 2.3.6:** There is no harmonic number of the form \( n = 2^k p^2 q \) such that \( H(n) = \frac{\tau(n)}{2} \) for any odd distinct primes \( p \) and \( q \).

**Proof:** If possible, let \( n = 2^k p^2 q \) be a harmonic number.

Then \( \sigma(n) = (2^{k+1} - 1)(1 + p + p^2)(1 + q) \)

and \( \tau(n) = 3(k + 1)2 \)

Now \( g.c.d. (n, \sigma(n)) = n \) implies that \( \sigma(n) = nk_1 \), where \( k_1 \in \mathbb{N} \)

\[ (2^{k+1} - 1)(1 + p + p^2)(1 + q) = 2^k p^2 q k_1 \]

\[ \Rightarrow \left( \frac{2^{k+1} - 1}{2^k} \right) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{1+q}{q} \right) = k_1 \quad (2.3.14) \]

and \( H(n) = \frac{n\tau(n)}{\sigma(n)} = \frac{\tau(n)}{k_1} \)

\[ \Rightarrow \quad k_1 = \frac{\tau(n)}{H(n)} \]

\[ \Rightarrow \quad \left( \frac{2^{k+1} - 1}{2^k} \right) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{1+q}{q} \right) = \frac{\tau(n)}{H(n)} \]
\[ H(n) \left( \frac{2^{k+1}-1}{2^k} \right) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{1+q}{q} \right) = 3(1 + k) \]

\[ \Rightarrow \]

\[ H(n) \left( \frac{2^{k+1}-1}{2^{k+1}} \right) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{1+q}{q} \right) = 3(1 + k) \quad (2.3.15) \]

Since \[ H(n) = \frac{r(n)}{2} = 3(1 + k) \]

Therefore \[ \left( \frac{2^{k+1}-1}{2^{k+1}} \right) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{1+q}{q} \right) = 1 \quad (2.3.16) \]

We shall show that left hand side of (2.3.16) is not integer.

Since \( p, q \) are odd primes and \( 2^{k+1} - 1 \) is an odd natural number, therefore either \( 2^{k+1} | 1 + q \) or \( q | (1 + p + p^2) \). If possible, let \( 2^{k+1} | 1 + q \).

Then for some positive integer \( x \),

we have \( q = x2^m - 1 \), where \( m \geq k + 1 \) \quad (2.3.17)

If \( m > k + 1 \), then (2.3.16)

give rise to \[ (2^{k+1} - 1) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{x2^y}{q} \right) = 1, \]

where \( y = m - k - 1 \geq 0 \) \quad (2.3.18)

But \( p \) and \( q \) are odd primes, so \( y = 0 \) i.e. \( m = k + 1 \)

Then (2.3.18) can be written as \[ (2^{k+1} - 1) \left( \frac{1+p+p^2}{p^2} \right) \left( \frac{x}{q} \right) = 1 \quad (2.3.19) \]

In (2.3.17), if \( x > 1 \) then \( q \) does not divide \( 2^{k+1} - 1 \). Hence, only the possibility is \( q | (1 + p + p^2) \)
Let \( qt = 1 + p + p^2 \) for some positive integer \( t \). \hspace{1cm} (2.3.20)

Then (2.3.19) is \( \left( 2^{k+1} - 1 \right) \frac{t}{p^2} = 1 \) \hspace{1cm} (2.3.21)

Since \( x > 1 \) and \( 2^{k+1} - 1 > 1 \), then \( t \) must be equal to 1 and \( x = 2^{k+1} - 1 = p \).

Now \( qt = 1 + p + p^2 \)

\[ \Rightarrow \quad q = 1 + p + p^2 \text{ for } t = 1 \]

\[ \Rightarrow \quad x2^{k+1} - 1 = 1 + p + p^2 \]

\[ \Rightarrow \quad p2^{k+1} - p = 2 + p^2 \]

\[ \Rightarrow \quad p(2^{k+1} - 1) = 2 + p^2 \]

\[ \Rightarrow \quad p^2 = 2 + p^2 , \text{ which is not possible.} \]

Next, we assume that \( x = 1 \).

Then (2.3.17) will be \( q = 2^{k+1} - 1 \)

and (2.3.21) can be written as \( \left( 2^{k+1} - 1 \right) \frac{t}{p^2} = 1 \) \hspace{1cm} (2.3.22)

If \( t = 1 \), (2.3.22) reduces to \( p^2 = 2^{k+1} - 1 = q \) and

the (2.3.20) reduces to \( q = 1 + p + p^2 \) or \( p^2 = 1 + p + p^2 \) which gives \( p = -1 \)

which is not possible again.

If \( t > 1 \), again there are two possibilities, either \( t = p \) or \( t = p^2 \).
If \( t = p^2 \), (2.3.22) will produce an absurd result, as \( 2^{k+1} - 1 = 1 \), which gives \( k = 0 \), not possible.

If \( t = p \), (2.3.22) becomes \( 2^{k+1} - 1 = p \) i.e. \( q = p \).

Hence, (2.3.20) can be written as \( p^2 = 1 + p + p^2 \) for \( q = p \), which implies that \( p = -1 \), which is not true. This shows that, (2.3.16) cannot be an integer.

Thus, there exist no natural number of the form \( n = 2^k p^2 q \) such that \( H(n) = 3(1 + k) = \tau(n) \). \( \blacksquare \)

**Proposition 2.3.7:** Let the number be of the form \( n = 2^k p^{2x} q \), where \( x \) is any positive integer and \( p, q \) are odd primes. If \( H(n) = \frac{\tau(n)}{2} \), then \( n \) is not a harmonic number.

**Proof:** If \( H(n) = \frac{\tau(n)}{2} \)

\[
\Rightarrow \quad \left( \frac{2^{k+1} - 1}{2^{k+1}} \right) \frac{\prod_{i=0}^{2x} p^i}{p^{2x}} \left( 1 + \frac{q}{q} \right) = 1 \quad (2.3.23)
\]

Being \( 2^{k+1} - 1 \) and \( \sum_{i=0}^{2x} p^i \) are odd primes, which are not divisible by any term of denominator. The only possibility is \( 2^{k+1} | 1 + q \).

Let \( q = x_1(2^m - 1), m \geq k + 1 \) and \( x_1 \) is a positive integer. \( (2.3.24) \)

If \( m > k + 1 \), then (2.3.23) becomes

\[
(2^{k+1} - 1) \left( \frac{\prod_{i=0}^{2x} p^i}{p^{2x}} \right) \frac{x_1^{2^y}}{q} = 1, \text{ for some } y \geq 1 \quad (2.3.25)
\]
Since \( y \geq 1 \) and for odd distinct primes \( p, q \), the left hand side of (2.3.25) is an even number, which is not possible. Therefore, \( m \) must be equal to \( k + 1 \).

For \( x_1 > 1 \), \( q \) does not divide \( 2^{k+1} - 1 \), so, the only possibility is \( q|\sum_{i=0}^{i=2x} p^i \).

Let \( q t = \sum_{i=0}^{i=2x} p^i \), for some natural number \( t \).

From (2.3.24) and (2.3.25) we have \( (2^{k+1} - 1) \frac{r_{x_1}}{p^{2x}} = 1 \)

The only possibility is \( 2^{k+1} - 1 = p^{y_1}, t = p^{y_2}, x_1 = p^{y_3} \), where \( y_1 + y_2 + y_3 = 2x \) and \( y_i \geq 0 \), \( i = 1,2,3 \)

Now \( q t = \sum_{i=0}^{i=2x} p^i \)

\[
(x_1 (2^{k+1} - 1)) p^{y_2} = \sum_{i=0}^{i=2x} p^i
\]

\[
p^{y_2+y_3}2^{k+1} - p^{y_2+y_3} = 1 + \sum_{i=1}^{i=2x} p^i
\]

\[
p^{y_2+y_3} (2^{k+1} - 1) = 1 + p^{2x} + \sum_{i=1}^{i=2x-1} p^i
\]

\[
p^{2x} = 1 + p^{2x} + \sum_{i=1}^{i=2x-1} p^i
\]

\[
0 = 1 + \sum_{i=1}^{i=2x-1} p^i \tag{2.3.26}
\]

which is not possible. Hence the result. \( \blacksquare \)
Remark 2.3.2: Prop. (2.3.7) assert that if \( n|\sigma(n) \), there is no harmonic number of the form \( n = 2^k p^2 q \). However, there are some harmonic numbers of the same form if \( n \) does not divide \( \sigma(n) \). Let \( q = 2^{k+1} - 1 \). Then

\[
H(n) = \frac{n\tau(n)}{\sigma(n)}
\]

\[
\frac{H(n)\sigma(n)}{n} = \tau(n)
\]

\[\Rightarrow \quad H(n) \left(\frac{2^{k+1+1}}{2^{k+1}}\right) \left(\frac{1+p+p^2}{p^2}\right) \left(\frac{1+q}{q}\right) = 3(1 + k)
\]

\[\Rightarrow \quad H(n) \left(\frac{1+p+p^2}{p^2}\right) = 3(1 + k) \quad (2.3.27)
\]

Now \( g.c.d.(p^2, 1+p + p^2) = 1 \), then \( p^2|H(n) \). Therefore for some positive integer \( y \), \( H(n) = yp^2 \). Then (2.3.27) becomes \( 3(1 + k) = y(1 + p + p^2) \), which is the only possible choice to form a harmonic number.

For example: If we consider \( y = 3 \) then some harmonic numbers of the form \( 2^k p^2 q \) are \( 2^{12} 3^2 (2^{13} - 1), 2^{30} 5^2 (2^{31} - 1) \) for some values of \( k \).

Similarly, if \( y = 1 \) some harmonic numbers of the said form are \( 2^{126} 19^2 (2^{127} - 1), 2^{60} 13^2 (2^{61} - 1), 2^{18} 7^2 (2^{19} - 1) \) for some values of \( k \).

2.4 Harmonic numbers of the form \( 2^k p_1 p_2 \cdots p_m \)

In this section, we have extended the number to the form of \( 2^k p_1 p_2 \cdots p_m \) for odd distinct primes \( p_i (i = 1, 2, \ldots, m), k \in \mathbb{N} \).
**Proposition 2.4.1:** Given a natural number $n$ of the form $p_1p_2 \ldots \ldots p_m$ where $p_i's$ are odd distinct primes. If one of the prime of $p_i \ (i = 1,2,\ldots,m)$ is $2^{k+1} - 1$, then $\sigma(n) \neq kn$ for $k = 2x \ , \ x \ \text{is \ odd}$. 

**Proof:** Given that $n = 2^k p_1p_2 \ldots \ldots p_m$. 

Then $\sigma(n) = (2^{k+1} - 1) \prod_{i=1}^{m}(1 + p_i)$. Let $p_1 = 2^{k+1} - 1$ be one of the prime of $p_i \ (i = 1,2,\ldots,m)$

If possible, let $\sigma(n) = 2xn$

$$\Rightarrow \ (2^{k+1} - 1) \prod_{i=1}^{m}(1 + p_i) = 2x2^k p_1p_2 \ldots \ldots p_m$$

$$\Rightarrow \ \frac{2^{k+1}-1}{2^{k+1}} \prod_{i=1}^{m} \frac{1+p_i}{p_i} = x$$

$$\Rightarrow \ \prod_{i=2}^{m} \frac{1+p_i}{p_i} = x, \quad (2.4.1)$$

which is not possible as the left hand side of (2.4.1) is even and the right hand side of (2.4.1) is odd. Therefore, we cannot assume $\sigma(n) = 2xn$. In other words, $\sigma(n) \neq kn$ for $k = 2x$. 

**Proposition 2.4.2:** Let $n = 2^k p_1p_2 \ldots \ldots p_m$, $k \in \mathbb{N}$ be a given natural number. Then $\sigma(n) \neq 2n$ for odd distinct primes $p_i \ (i = 1,2,\ldots,m)$

**Proof:** Without loss of generality, we may assume that $p_1 < p_2 \ldots < p_j < \ldots \ldots < p_m$. 

60
\[
\frac{\sigma(n)}{2n} = \frac{2^{k+1} - 1}{2^{k+1}} \prod_{i=1}^{m} \frac{1+p_i}{p_i} \quad (2.4.2)
\]

Clearly \( p_m \nmid (1 + p_i), \forall i = 1, 2, \ldots, m \) as \( p_m > p_i, \forall i = 1, 2, \ldots, m - 1 \).

Therefore, the only possible division is \( p_m | 2^{k+1} - 1 \). Let \( 2^{k+1} - 1 = r_m p_m, \ r_m \geq 1 \).

So

\[
\frac{\sigma(n)}{n} = \frac{2^{k+1} - 1}{2^k} \left( \frac{1+p_m}{p_m} \right) \prod_{i=1}^{m-1} \frac{1+p_i}{p_i}
\]

\[
= \left( \frac{r_m p_m}{2^k} \right) \left( \frac{2^{k+1} - 1 + r_m}{r_m p_m} \right) \prod_{i=1}^{m-1} \frac{1+p_i}{p_i}
\]

\[
= \frac{1}{2^k} (2^{k+1} - 1 + r_m) \left( \frac{1+p_{m-1}}{p_{m-1}} \right) \prod_{i=1}^{m-2} \frac{1+p_i}{p_i} \quad (2.4.2)
\]

Similarly, \( p_{m-1} \nmid (1 + p_i), \forall i = 1, 2, \ldots, m - 1 \) as \( p_{m-1} > p_i, \forall i = 1, 2, \ldots, m - 2 \).

Then the only possibility is \( p_{m-1} | 2^{k+1} - 1 + r_m \). Let \( 2^{k+1} - 1 + r_m = r_{m-1} p_{m-1}, r_{m-1} \geq 1 \).

Then (2.4.2) can be written as

\[
\frac{\sigma(n)}{n} = \frac{1}{2^k} (r_{m-1} p_{m-1}) \left( \frac{2^{k+1} - 1 + r_m + r_{m-1}}{r_{m-1} p_{m-1}} \right) \prod_{i=1}^{m-2} \frac{1+p_i}{p_i}
\]

Continuing in this way, at the \( j \)th stage (where the remaining primes are \( p_1, p_2, \ldots, p_j \))

We have
\[ \sigma(n) \frac{n}{n} = \frac{1}{2\pi} \left( \eta P_j \right) \left( \frac{2^{k+1} - 1 + \sum_{j=1}^{m} r_i}{r_j P_j} \right) \prod_{j=1}^{m} \frac{1 + p_j}{p_j} \]

\[ \frac{\sigma(n)}{n} = \frac{1}{2\pi} \left( 2^{k+1} - 1 + \sum_{i=1}^{m} r_i \right) \]  \hspace{1cm} (2.4.3)

If \( \frac{\sigma(n)}{n} = 2 \), then (2.4.3) becomes \( 2^{k+1} - 1 + \sum_{i=1}^{m} r_i = 2^{k+1} \) or \( \sum_{i=1}^{m} r_i = 1 \), which is not possible. Hence, \( \sigma(n) \neq 2n \) and the result follows. \( \blacksquare \)

**Remark 2.4.1:** The above result can be apply to search a harmonic number of the form \( 2^k p_1 p_2 \ldots \ldots p_m \) such that \( g.c.d. \left( n, \sigma(n) \right) = n \) such that \( p_1 < p_2 < \ldots \ldots < p_m \), where all \( p_i \)'s are odd primes. Briefly, we describe a procedure to search a harmonic number of the said form.

Step 1: Factorize the number \( 2^{k+1} - 1 \) as a product of primes and suppose that \( 2^{k+1} - 1 = (p_1 p_2 \ldots \ldots p_{m-1})p_m = r_m p_m \), where \( r_m = p_1 p_2 \ldots \ldots p_{m-1} \). We may allow, if the prime factors of \( 2^{k+1} - 1 \) contains 2 as a prime or a prime of the form say, \( q^m, m \geq 1 \). We consider the highest prime of the factorization of \( 2^{k+1} - 1 \) and suppose that, the prime \( p_m \) is the highest. Let \( 2^{k+1} - 1 = r_m p_m \).

Step 2: We repeat step 1, and the number \( 2^{k+1} - 1 + r_m \) as a product of primes. In fact, the prime factors of \( 2^{k+1} - 1 \) except the highest prime \( p_m \) will also be the prime factors of \( 2^{k+1} - 1 + r_m \), for which we never lost any primes at any stage of iteration. Repeating the above process and suppose, at the \( j \)-th stage, we consider the prime factors of \( 2^{k+1} - 1 + \sum_{i=1}^{m} r_i \). We choose the highest one prime provided in the factorization all the prime factors other than the prime factors of \( k + 1 \) are square free. We end the process when \( 2^{k+1} - 1 + \sum_{i=1}^{m} r_i \) has only prime factor 2.
Harmonic mean $H(n)$ of these numbers will be of the form $q_1q_2 \ldots q_i2^j$, for some $j$, where $q_1, q_2, \ldots, q_i$ are some of the prime factors of $k + 1$. Since the searching process is dependent only on the values of $k$, it takes less time to search the numbers of the said form in case of higher range of intervals. We have checked in the version of MATHEMATICA 7.0.1, that there is no harmonic numbers of the said form whose harmonic mean is $(k + 1)2^x$ for some values of $x$ in the interval $[1,10^{100}]$.

* * * * *